Algorithmic Resolution of Singularities

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Eidesstattliche Erklärung

Ich versichere, daß ich die Dissertation selbständig verfaßt habe, andere als die angegebenen Quellen und Hilfsmittel nicht verwendet und mich auch sonst keiner unerlaubten Hilfe bedient habe.

Gábor Bodnár
Hagenberg, am 18 Dezember 2000

Zusammenfassung

Das algorithmische Problem der Auflösung von Singularitäten ist, eine eigentliche birationale Abbildung von einer nichtsingulären Varietät zu einer gegebenen Varietät zu konstruieren. Das Problem ist lösbar, wenn der Grundkörper Charaketristik Null hat; in positiver Charakteristik ist es noch ungelöst. Die Abbildung und die nichtsinguläre Varietät werden durch Komposition von geeignet gewählten Transformationen (Aufblasungen) der gegebenen Varietät konstruiert.

In dieser Arbeit wird, ausgehend von konstruktiven Beweisen, die in der Literatur verfügbar sind, ein Algorithmus zur Auflösung von Singularitäten von Hyperflächen über Körpern von Charakteristik Null konstruiert. Weiters werden einige Verbesserungen dieses Algorithmus' ausgeführt, indem fortgeschrittene Techniken für Teilprobleme des Auflösungsprozesses entwickelt werden. Die Arbeit wird durch eine vollständige Implementierung des Algorithmus' ergänzt.

Schließlich wird ein verwandtes Problem gelöst, nämlich die Konstruktion einer Aufblasung die isomorph zu einer Komposition einer gegebenen Folge von Aufblasungen ist.

Abstract

The problem of algorithmic resolution of singularities is to construct a proper birational morphism from some nonsingular variety to a given variety. The problem is solvable if the base field is of characteristic zero; in positive characteristic it is still open. The morphism and the nonsingular variety is constructed by composing well chosen transformations (blowing ups) over the given variety.

In this thesis, relying on constructive proofs available in the literature, we develop an algorithm for the resolution of singularities of hypersurfaces over fields of zero characteristic. We also present several improvements of this algorithm, developing advanced computational techniques for the subtasks of the resolution process. The thesis is accompanied by a full implementation of the algorithm.

Finally, we solve a problem related to the main topic: the construction of a blowing up which is isomorphic to the composition of a given sequence of blowing ups.

Algorithmic Resolution of Singularities

(PhD thesis)

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Chapter 0

Introduction

Let X be a variety over a field k. Does there always exist a nonsingular variety Y together with a proper birational morphism $\pi : Y \to X$?

This question has been studied carefully in the last century, growing into a guiding problem in algebraic geometry due to its importance, difficulty and beauty. The resolution of singularities is a powerful tool in the analysis of varieties, as the structure of π and the regular preimages of the singularities provide additional information on X. It has many applications in several fields of mathematics like real algebraic geometry, algebraic number theory, differential equations, etc. There are also other important problems in algebraic geometry which rely on the existence of desingularizations: for instance the parametrization of algebraic varieties [AB87, SW91, SW97, Sch92, Sch95a, Sch95b].

The resolution problem above has several generalizations: for instance, embedding an algebraic set X into a nonsingular variety W defining π for W with respect to the singularities of X, with the additional requirement that the proper transforms of the irreducible components of X (the closure of the preimage of the intersection of X with the open subset of W over which π is an isomorphism) are nonsingular and the total transform $\pi^{-1}(X)$ has normal crossings. It has many specializations too: varying the characteristic of the base field or restricting the dimension of X.

The answer for its most general specialization, when $\mathrm{char}(k) = 0$ but there is no restriction on $\dim(X)$, is *yes* since H. Hironaka. Due to the work of authors who rethought Hironaka's approach—O. Villamayor, E. Bierstone and P. Milman, and others—we also have constructive proofs for this problem. On the other hand, the fact that the involved web of inductions in Hironaka's proof, which, the legend says, was once called "The most complicated mathematical work" by A. Grothendieck, could not be significantly simplified in the new approaches, shows the difficulty of the problem.

In this thesis we provide the description of an algorithm, accompanied by its full implementation, for the resolution of singularities of hypersurfaces over fields of zero characteristic. The basis of our work is the constructive proof of Hironaka's theorem by O. Villamayor. His results were studied, reformulated, improved, implemented and optimized, resulting several publications and a software package. The software is available at `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/`.

0.1 Brief historical overview

The existence of resolution of curve singularities when $k = \mathbb{C}$ was proved by Kronecker, Noether and Brill [NB93], and others in the late 19th century. For surfaces over $\mathbb{C}$ the first results are due to Italian mathematicians; a few references are [Alb24, Chi91, Lev97a, Lev97b]. Nevertheless the first rigorous proof was given by Walker in [Wal35], based on the "local-global principle" of Jung [Jun08] allowing local resolutions to be patched together forming global ones. This and other early approaches used also analytic arguments.

The first purely algebraic proofs of the resolution of surface singularities over fields of characteristic zero were given by Zariski in [Zar39, Zar42]. Later he also gave proofs for resolution of singularities of 3-folds in [Zar44].

The general case in characteristic zero was proved by Hironaka in his monumental paper [Hir64]. In positive characteristic the first works are the ones of Abhyankar [Abh56a, Abh65, Abh66], containing partial results with restrictions on the dimension of X and in some cases on the characteristic of the base field. There are also other contributions, for instance by Lipman [Lip78] or recently by de Jong [dJ96], but the general problem is still open. For an account on difficulties of resolution of singularities in arbitrary characteristic we refer to [Hau98]. As an interesting fact we can mention that both Hironaka, Abhyankar and Lipman were students of Zariski.

The first constructive proofs of the general case in zero characteristic, based on Hironaka results, are due to Villamayor [Vil89], and Bierstone and Milman [BM91, BM97]. Later Villamayor's results were published in a more general framework in [EV98, EV00] incorporating the "good points" theory of Abhyankar [Abh88].

0.2 The state of the art

Although already Hironaka has provided a constructive method for resolution of surface singularities in [Hir84] (which is actually a paper from 1967 whose results are generalized by Cossart in [Cos81]), the first results for arbitrary dimension came much later by Villamayor [Vil89], and Bierstone and Milman [BM91] with the elaborated proof in [BM97]. Nevertheless these constructive proofs and other recent results are rooted in the works of Zariski [Zar39, Zar40, Zar42, Zar44, Zar47, Zar65a, Zar65b, Zar67, Zar68, Zar75, Zar78], Abhyankar [Abh56a, Abh64, Abh65, Abh66, Abh68, Abh69, Abh82, Abh88] and Hironaka [Hir64, Hir67, Hir70a, Hir70b, Hir70c, Hir70d, Hir74, Hir77], which determined the direction of future research significantly, providing the classical approach to the problem.

The first papers of Zariski on resolution of singularities [Zar39, Zar40, Zar42, Zar44] contain not only stronger results than the ones of the earlier Italian geometers, but also open the algebraic approach to the problem. His student, Abhyankar, came up with the first results for resolution in positive characteristic p, in [Abh56a], where he applies Galois theory in function fields of surfaces to determine birational transformations of polynomials of degree p. His results are extended and further specialized in [Abh64, Abh66].

The breakthrough in characteristic zero was made by Hironaka in [Hir64], where he proves existence of resolution in any dimensions. He provides a clear presentation of concepts like blowing up, total and strict transform, and introduces notions like normal-flatness, distinguished data, standard basis. He presents sophisticated inductive arguments, using the singularities of the given variety in higher embedding codimensions to provide a resolution. He also developed several new techniques to examine singularities, like the notion of coordinate free Newton polygon in [Hir67]; additive group schemes in the tangent cones at infinitely near singular points with perfect residue fields in [Hir70a]; introduction of the notion of maximal contact and contact exponent in [Hir70c, Hir70d]; local idealistic presentation of schemes over perfect fields in [Hir77], where he provides local representation of schemes as intersection of hypersurfaces, which is "stable" under blowing ups; etc.

Other papers from the literature of the problem besides the previously cited ones are: Benett's study of the Hilbert-Samuel function under blowing ups [Ben70]. Laufer's monograph on explicit resolution of normal surface singularities [Lau71]. The theory of toric modifications by Kempf, Knudson, Mumford and Saint-Donat [KKMSD73]. The introduction of Tschirnhausen polynomial and transformation by Abhyankar and Moh [AM73a, AM73b]. Oda's study of the Hilbert-Samuel function and Hironaka's standard basis in context of resolution of singularities [Oda86]. A method for local analytic resolution of singularities by Bierstone and Milman [BM89]. Villamayor's equivariant resolution with respect to group actions [Vil92]. Toric modifications for resolution of plane curve singularities by Goldin and Teissier [GT95]. A complexity analysis for the resolution of plane curve singularities in the number of necessary blowing ups by Lê and Oka [LO95]. A combination of Zariski's valuation theory with Hironaka's approach by Spivakovsky [Spi96]. Uniformization and desingularization of surfaces by Cossart [Cos00]. Taut resolution of excellent hypersurfaces by Hauser [Hau00].

An introduction to basic notions for resolution of singularities can be found in [Gir72]. Lipman's survey [Lip75] provides an excellent introduction to the problem, reaching back to the work of Jung, Albanese and Walker. The collection [CGO84] by Cossart, Giraud and Orbanz presents different resolution methods for surfaces.

Further related monographs are [Abh56b, Abh83, AW97, AHV75, AHV77, BM87a, BM87b, BM88, CP87, Gir74, Gir75, Gir81, HIO88, Lip69, Lip88, Miz73, Moh92, Moh96, Oda73, Oda83, Oka96, Orb84, OW71, Sin74, Spi83a, Spi83b, Spi89, Spi90, Vil96, You90b, You90a].

A comprehensive study on algebraic curves can be found in [Wal50] (available in the edition of Springer from 1970) by Walker. Other monographs, that are related in the broad sense, are [Art69, GD71, LJ95, Mat70, Mat86, Nag62, Zar71].

It is worth mentioning particularly that there is a recent non-classical approach to the problem due to de Jong [dJ96] that works in any characteristic. He proves the existence of a regular alteration of X which can always be chosen to be isomorphic to the complement of a normal crossing divisor in some regular projective variety. This is actually a weaker result than the solution of the resolution problem, because it requires a finite extension of the function field of X, but it opens new ways to pursue the original goal. For instance, Abramovich and de Jong [AdJ97], and Bogomolov and Pantev [BP96] have already extended the original ideas of de Jong. For an account of applications of de Jong's results we refer to [Ber96].

There are several other theoretical approaches to resolution of singularities via valuation theory, normalization, uniformization, toroidal resolution, cohomology, etc., but we do not discuss them in this paper. We rather follow the classical approach, taking Villamayor's results as our starting point, where the desingularization $\pi : Y \to X$ is computed by composing a sequence of blowing ups along well chosen centers. The centers are determined by local invariants, functions on algebraic sets that take their values in a well-ordered set, and measure the "severeness" of the singularities at points of X. The next blowing up center in an intermediate stage of the process is defined as the set of points with maximal invariant value.

The other important requirement for such invariants is that they have to measure the progress the resolution makes with each blowing up, showing that the singularities are "less severe" after the transformation than before it. Moreover the invariant has to assign the minimal element of the well ordered set to every point of X iff X is regular. This way, by an inductive argument, one can attempt to show that it is enough to apply finitely many blowing ups to get the transform of X along them become regular, i.e. to show termination for the above sketched resolution strategy.

Local invariants defined so far that fulfill these requirements are not at all trivial; their codomain usually consists of tuples of natural numbers, nonnegative rationals, finite lists of geometric objects, and perhaps also functions from such lists to well ordered domains. The tuples are then compared in lexicographic order.

The local invariants sometimes might also be called "stratifying functions", because among the additional requirements one usually asks for upper-semicontinuity, i.e. that the set of points where the function attains values greater than or equal to a given threshold is closed for all threshold values. The invariants then give rise to a stratification of X, where the next blowing up center is defined by the maximum stratum.

In the sequence of papers [Hau96a, Hau96b, Hau96c, Hau96d] of Hauser, a comprehensive discussion on systematic construction of such invariants can be found, which resulted the invention of alternative invariants by Encinas and Hauser in [EH00].

May the stratifying function be defined, one is still far away from an implementation when by "pressing the button" the computer constructs the desingularization guided by the function. The naive algorithm we gave above by saying "blow up the maximum stratum till you get a resolution" in our case expanded into a few hundred kilobytes of Maple code. We have to face with the following problems:

- How to represent the stratifying function?
- How to compute the maximum stratum?
- How to represent algebraic sets so that all the necessary computations, like blowing ups, strict transforms, finding local generators of geometric objects with prescribed properties, etc., can be performed efficiently?
- At what costs can we use the standard methods, proposed by the authors of the stratifying function, in the computations?

– If the performance of the algorithm is catastrophic, how can we improve it?

The paper, among others, attempts to answer these questions. In brief, we dropped the idea of computing directly the stratifying function, we rather took the proofs of Villamayor, turning his inductive arguments into recursive algorithms. We chose to represent nonsingular quasi-projective varieties by atlases of affine charts, equipped with global system of local regular parameters. This way, having flexible differential frames over the charts, we can compute the components of the stratifying function "easily". The maximum strata then arise in a natural way at the deepest recursion levels, generated by regular parameters, which allows us to compute the blowing ups very efficiently.

In the actual computations we work in finitely generated k-algebras, thus the method of Gröbner basis is ubiquitous in the algorithm. For some of the standard results and computational techniques with Gröbner basis we refer to [Buc65, Buc76, Buc79, Buc83a, Buc83b, Buc85, BCK88, Win88, CLO92, BW93, AL94]

Unfortunately, the resolution algorithm obtained by the direct translation, using the "straightforward" computational methods had serious problems with the representation of resolution problems. Because of this, in order to improve the overall performance of the algorithm, we came up with several new computational techniques, and representation ideas.

Our results can be considered a partial success. Because the inherent complexity of the algorithm is very bad, meaning that the stratifying function (constructed for the general case) requires much more blowing ups than it would be necessary in an optimal situation (using all the available knowledge for a given example), the performance of the algorithm is still too bad to tackle middle sized and research problems. A theoretical obstacle in any complexity analysis of the algorithm is to determine the number of blowing ups the algorithm performs in the dimension and degree of the input. This is an open research problem whose hardest part is to find out how many blowing ups are necessary to solve the resolution problem arising in a recursive step, called coefficient object computation.

On the other hand, we have shown that, relying on recent results, it is possible to automatize one of the hardest problems in algebraic geometry. To our knowledge there exists no other complete implementation of a general resolution algorithm; except [ALR88, CCLM98], which treat particular classes of surface singularities, thus for a general resolution algorithm there exists no implementation even for surfaces.

Also, more importantly, our research provided several new ideas that are not strictly bound to the Villamayor algorithm, allowing their use, for instance, in further resolution algorithms, or in some cases even in other problems of algebraic geometry. A collection of papers on the results of the thesis is [BS00a, BS00b, BS00c].

0.3 Structure and originality of the thesis

We present the organization of the paper, pointing out what results are presented in what sections. Since most of the material arose from joint work with

J. Schicho, we indicate the role of the author in each achievements. The paper is supposed to be self contained, and in cases when proofs are omitted, they should be covered by references.

Chapter 1 The original algorithm.

- **Section 1.1** provides the necessary basic definitions, sets the context and presents the problem we are dealing with. This section is a summary of well known facts, no originality is claimed.
- **Section 1.2** describes the representation of quasi-projective algebraic sets used by the algorithm. The individual pieces of the representation are well known, but their cooperative application was the idea of J. Schicho. The author elaborated the formal description.
- **Section 1.3** presents the basic operations the resolution algorithm performs on algebraic sets in the representation of section 1.2. The correctness proofs are provided by the author.
- **Section 1.4** presents the overall structure of the resolution algorithm gotten by rewriting Villamayor's proofs. The achievements of this section belong to J. Schicho, it is included to make the paper self contained. The author had minor influence on the final version of the algorithm via personal discussions.
- **Section 1.5** provides some details on how certain nontrivial subtasks of the algorithm are performed. Subsection 1.5.1 presents an idea originally proposed by J. Schicho with a gap closed by the author. The proofs are provided by the author. Subsection 1.5.2 presents a gap in the original algorithm found by the author closed by the author in a joint effort with J. Schicho.
- **Section 1.6** includes a part of a resolution example to demonstrate the working algorithm.

Chapter 2 The improvements.

- **Section 2.1** presents an alternative way to compute local maxima of a component of the Villamayor stratifying function. The idea is due to J. Schicho, the proofs are by the author.
- **Section 2.2** discusses the possibility to skip certain levels of the recursion, jumping to a deeper level in one stroke. The idea is well known; its application in the resolution algorithm is initiated and the proofs are provided by the author.
- **Section 2.3** defines a modified resolution algorithm, using the ideas of the previous two sections, and shows termination for it.
- **Section 2.4** presents a major improvement in the representation of resolution problems by generalizing the notion of polynomial ideals. The original idea of using Hironaka's theory of idealistic exponents is due to J. Schicho. The definition of generalized ideals was established in a joint effort of J. Schicho and the author. The proofs are provided by the author.

Section 2.5 defines new data structures to represent resolution problems. The representation was invented in a joint effort of J. Schicho and the author.

Section 2.6 presents computational techniques with the new representation. The methods were elaborated and the proofs are provided by the author.

Section 2.7 shows that the optimized algorithm, using the results of the previous sections, computes the same sequence of basic operations for a given input as the original algorithm. The proof is provided by the author.

Section 2.8 provides a simple but powerful method to eliminate redundancies in the representation of algebraic sets with atlases of affine charts. The original idea is of J. Schicho. It was generalized and proved by the author.

Section 2.9 demonstrates the work of the old and the new algorithm on a resolution problem, following the algorithms step by step in parallel.

Chapter 3 Unification of blowing ups.

Section 3.1 presents the basic definitions and lemmas, together with the main theorem that allows to compute the blowing up which is isomorphic with the composition of two given blowing ups. According to our best knowledge, the constructive solution of the problem was previously open. The idea and the proofs are due to the author.

Section 3.2 provides a simple application of the results of the previous section: an algorithm that unifies a given sequence of blowing ups represented by a tree of charts. The proofs are provided by the author.

Section 3.3 demonstrates the results of section 3.1 by a fully extended example.

Chapter 3 might be a bit away from the main direction of the paper, but its achievements is closely related to the topic of resolution of singularities. It might also help to understand how sequences of simple blowing up transformations build up the proper birational morphism π (which is also just a blowing up) that resolves the singularities of X. An ultimate goal of the resolution of singularities would be to be able to define the ideal in the coordinate ring of X whose blowing up resolves the singularities of X. This can be the topic of future research, although the problem is clearly extremely difficult.

0.4 Acknowledgements

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I would like to express my gratitude to Josef Schicho, who helped me so much from my first steps in symbolic computation to my latest research papers to improve in every aspect a scientist can be judged by. He is with no doubt the teacher who influenced my progress in science the most.

I would also like to thank Franz Winkler for supervising my thesis and Herwig Hauser, who encouraged us in the project of developing the software package for the resolution of singularities, and who posed me the problem of unification of blowing ups.

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Chapter 1

An algorithmic approach

In this chapter we present the resolution algorithm in its original form, extracted from [Vil89, Vil96, EV98, EV00] by J. Schicho and the author. We are indebted to S. Encinas and O. Villamayor for their selfless help in the clarification of some details of their work.

Section 1.1 provides the necessary definitions and concepts, section 1.2 explains our representation of varieties, while section 1.3 presents the basic operations on them. Section 1.4 gives a detailed description of the original resolution algorithm, and section 1.5 sheds light on certain nontrivial computations the algorithm has to perform. Finally, in section 1.6, we demonstrate the working algorithm through an elaborated example.

1.1 Preliminary definitions

Henceforth k denotes a computable field of characteristic zero. The n-dimensional *affine space* over k, denoted as $\mathbb{A}^n$, is the set of all n-tuples $(a_1, \ldots, a_n) \in k^n$. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring; for any subset $F \subseteq R$ the *zero set* of F is defined as

$$\mathbf{Z}(F) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in F\}.$$

Let J be the ideal in R generated by F. We clearly have $\mathbf{Z}(F) = \mathbf{Z}(J)$. By Hilbert's basis theorem, since R is Noetherian, we have $J = \langle f_1, \ldots, f_k \rangle$ for some finite set of polynomials from R.

A subset $X \subseteq \mathbb{A}^n$ is called an *affine algebraic set* if there exists an ideal $J \subseteq R$ such that $X = \mathbf{Z}(J)$. We denote the unique radical vanishing ideal of X by $\mathbf{I}(X)$, and the affine coordinate ring of X by $k[X] = R/\mathbf{I}(X)$. There is one-to-one correspondence between algebraic sets of $\mathbb{A}^n$ and radical ideals of R. The *dimension* of an affine algebraic set, $\dim X$, is the *Krull dimension* of $k[X]$.

We consider the *Zariski topology* on $\mathbb{A}^n$, that is, the open subsets are the complements of the algebraic sets. A closed subset is said to be *irreducible* if it cannot be decomposed into the union of its two proper closed subsets. An irreducible affine algebraic set is called an *affine variety.* An open subset of an affine algebraic set is a *quasi-affine algebraic set.*

The n-dimensional *projective space* over k, denoted as $\mathbb{P}^n$, is the set of equivalence classes of $(n+1)$-tuples, $(a_0, \ldots, a_n) \in k^{n+1} \backslash (0, \ldots, 0)$, of the equivalence relation $(a_0, \ldots, a_n) \sim (\alpha a_0, \ldots, \alpha a_n)$ for $0 \neq \alpha \in k$. We denote the equivalence class of $(a_0, \ldots, a_n)$ by $(a_0 : \ldots : a_n)$. Let $S = k[x_0, \ldots, x_n]$ be a graded polynomial ring; the notions of zero set, algebraic set, variety, quasi-projective algebraic set can be defined analogously to the affine case, now considering homogeneous ideals in S. The largest homogeneous vanishing ideal of a projective algebraic set X is denoted as $\mathbf{I}(X)$, and its graded coordinate ring as $k[X] = S/\mathbf{I}(X)$.

We can represent the projective space $\mathbb{P}^n$ via its affine covering. Let $U_i = \mathbb{P}^n \setminus \mathbf{Z}(x_i)$ for $i = 0, \ldots, n$, also called as the ith principal open subset of $\mathbb{P}^n$. The finite set of maps

$$\{\varphi_i : U_i \to \mathbb{A}^n \,|\, \varphi_i : (a_0 : \ldots : a_n) \mapsto (a_j/a_i \,|\, j = 0, \ldots, n,\ j \neq i), i = 0, \ldots, n\}$$

establish one-to-one correspondence between points of the U_is and points of $\mathbb{A}^n$. We note that the maps are independent of the choice of the homogeneous coordinates, thus they are well defined, and that they preserve the topology, mapping closed sets to closed sets.

Let $U_{ij} = U_i \setminus \mathbf{Z}(x_j)$ for all possible i, j. From the construction we get $U_{ij} = U_{ji}$ immediately, so we can define *chart change maps*, establishing isomorphisms between open subsets of $\mathbb{A}^n$:

$$\begin{aligned}\{\varphi_{ij} : \varphi_i(U_{ij}) \to \varphi_j(U_{ji}) \,|\, \varphi_{ij} : (a_k/a_i \,|\, k = 0, \ldots, n,\ k \neq i) \mapsto& \\ (a_k/a_j \,|\, k = 0, \ldots, n,\ k \neq j) \text{ for } (a_0 : \ldots : a_n) \in U_{ij}\}.&\end{aligned}$$

We have $U_{ii} = U_i$, $\varphi_{ii} = \mathrm{id}_{\varphi_i(U_i)}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ over the open subset $\varphi_i(U_{ij}) \cap \varphi_i(U_{ik})$ for all $i, j, k \in \{0, \ldots, n\}$; therefore we say that the chart change maps *patch*.

Eventually, since the principal open subsets cover $\mathbb{P}^n$, the data $\{\varphi_i, \{\varphi_{ij}\}_j\}_i$, or often referred as $\{\mathbb{A}^n, \{\varphi_{ij}\}_j\}_i$, omitting the isomorphisms φ_i, represent the projective n-space by means of an *atlas* of affine charts. Projective points correspond to equivalence classes of points in the affine charts under the equivalence relation induced by the above isomorphisms. In a more general setup, a quasi-projective algebraic set can be represented via its open affine subsets (charts), where the chart change maps have to be defined over open subsets of the charts, fulfilling the patching condition above.

Justified by the previous observation, we continue the discussion by focusing on affine algebraic sets. Nevertheless, we provide representations of projective structures with affine covering explicitly whenever necessary.

Let $X \subset \mathbb{A}^n$ be an affine algebraic set, a function $f : X \to k$ is *regular at a point* $a \in X$, if there exists an open $U \subseteq X$ containing a, and $g, h \in R$, such that h is nowhere zero on U, and $f = g/h$ on U. If f is regular at every point of X then f is said to be *regular* on X. The definition of regularity can be analogously given for projective algebraic sets, where the polynomials $g, h \in S$ additionally have to be of the same degree.

Let X, Y be algebraic sets (affine or projective). A continuous map $\varphi : X \to Y$ is called a *morphism*, if for all open $V \subseteq Y$ and for all regular $f : V \to k$ the map $f \circ \varphi : \varphi^{-1}(V) \to k$ is also regular on $\varphi^{-1}(V)$. A morphism is basically a map between two varieties which can be given on open subsets by means of polynomials.

For the scheme theoretic definition of the properness of a morphism, which requires lengthy preparation, we refer to [Har77] page 100. We only say that a morphism is *proper* if it is not the restriction of a map from some larger domain to an open dense subset.

A morphism $\varphi : X \to Y$ of varieties is *birational* if there exists an open subset $V \subseteq Y$, with $U = \varphi^{-1}(V)$, and a map $\pi : V \to U$, which can be given by means of rational functions, such that $\pi \circ \varphi|_U = \mathrm{id}_U$ and $\varphi|_U \circ \pi = \mathrm{id}_V$.

Let X be an algebraic set of dimension r defined by $f_1, \ldots, f_k \in R$. It is *singular at a point* $a \in \mathbb{A}^n$ if the Jacobian matrix at a is of rank smaller than $n - r$. The set of singular points of X is called the *singular locus* of X. The singular locus is a closed algebraic set, which can be defined in X by $\partial_{x_i} f$, $i = 1, \ldots, n$, if X is the hypersurface defined by $f \in R$.

Definition 1.1. Given a variety Y, a *desingularization* of Y is a nonsingular variety X together with a proper, birational morphism $\pi : X \to Y$.

At a nonsingular point a of an algebraic set W of dimension n, $f_1, \ldots, f_n$ regular functions on some open $U \subseteq W$ containing a form a *regular system of parameters* iff they all vanish at a, and $df_1, \ldots, df_n$ generate the module of differential forms Ω_U, (or in other words they are linearly independent in the local ring $\mathcal{O}_{W,a}$ mod $\mathfrak{m}_a^2$, where $\mathfrak{m}_a$ is the maximal ideal of $\mathcal{O}_{W,a}$).

An algebraic set X contained in the nonsingular variety W has *normal crossings* iff for all points $a \in X$ there exists a regular system of parameters, such that each irreducible component of X is generated by a subset of this parameter system locally at a.

Let W be a nonsingular variety, and $X \subset W$ be an algebraic set of codimension one with respect to W. By abusing the language we call the pair (W, X) a *hypersurface*. If $W = \mathbb{A}^n$, (W, X) is an affine hypersurface, if $W = \mathbb{P}^n$, (W, X) is a projective hypersurface. Let $\pi : W' \to W$ be a proper, birational morphism which is an isomorphism over some open $U \subseteq W$, the algebraic set which is the closure of $\pi^{-1}(X \cap U)$ in W' is called the *proper transform* of X under π.

Definition 1.2. Given a hypersurface (W, X), an *embedded desingularization* is a nonsingular variety W' together with a proper, birational morphism $\pi : W' \to W$, such that the proper transform of X is nonsingular and the *total transform*, $\pi^{-1}(X)$, has normal crossings.

If X is a variety, an embedded desingularization of (W, X) provides a desingularization of X. The rational inverse is defined on an open subset of X and the closure of its image, say X', is nonsingular in W'. Now X' together with the map $\pi|_{X'} : X' \to X$ is a desingularization of X in the sense of definition 1.1.

Such a desingularization can be constructed using the technique of blowing ups, which we now describe.

Definition 1.3. Let W be an affine algebraic set with $\overline{x}_1, \ldots, \overline{x}_n$ generators for $k[W]$. Let $C \subset W$ be an algebraic set with $\mathbf{I}(C) = \langle g_0, \ldots, g_s \rangle$. The *blowing up* of W at C is defined as a proper, birational morphism $\pi : W' \to W$, where $W' \subset W \times \mathbb{P}^s$ with ideal

$$\mathbf{I}(W') = \{f \in k[x_1, \ldots, x_r, y_0, \ldots, y_s] \mid f(\overline{x}_1, \ldots, \overline{x}_n, tg_0, \ldots, tg_s) = 0 \text{ in } k[W][t]\}.$$

The morphism π is just the projection on the first n coordinates.

The ideal $\mathbf{I}(W')$ is homogeneous in the ys. The preimage of C in W', denoted by E, is called the *exceptional divisor* introduced by the blowing up.

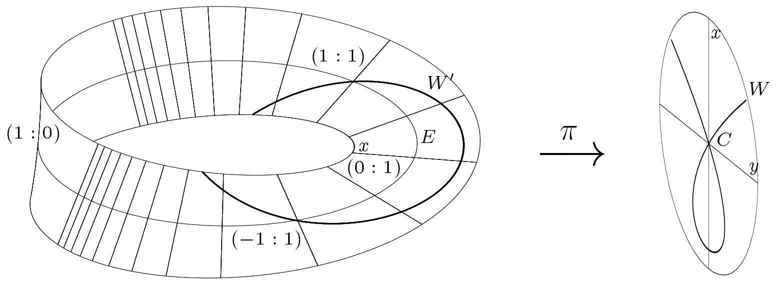


Figure 1.1: Blowing up $(\mathbb{A}^2, \mathbf{Z}(y^2 - x^3 - x^2))$ at the origin.

If in the above situation C and W are quasi-projective algebraic sets, we can compute the blowing up of W at C by using the construction of definition 1.3 in the charts of an atlas of W. In each chart we have to blow up the restriction of C to the chart. For more details on computation of atlases for algebraic sets we refer to section 1.2, and for the global structures of the blownups and the correspondence between representations with homogeneous vanishing ideals and with atlases, we refer to chapter 3.

Example 1.4. Let $W = \mathbb{A}^n$, $C = \mathbf{Z}(x_1, \ldots, x_r)$, $r \leq n$. The blowing up of $\mathbb{A}^n$ along C is $\pi : W' \to \mathbb{A}^n$, where $W' \subset \mathbb{A}^n \times \mathbb{P}^{r-1}$,

$$\mathbf{I}(W') = \langle x_i y_j - x_j y_i \,|\, i, j = 1, \ldots, r \rangle R[y_1, \ldots, y_r],$$

π is the projection on the x_is. The blownup can be covered by r affine charts of dimension n, covering $\mathbb{P}^{r-1}$ by its principal open subsets. So the first chart U_1 (with $y_1 \neq 0$) is generated by $x_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n$, where the x_js were eliminated by the equations $x_1 y_j = x_j$, $j = 2, \ldots, r$.

For a hypersurface S generated by $f \in R$, containing C in its singular locus with multiplicity d, the total transform in U_1, generated by the homomorphic image of f in $k[U_1]$, is of the form $x_1^d \cdot \overline{f}$, where $x_1 = 0$ is the local equation of E, the exceptional divisor, and $\overline{f} = 0$ is of the proper transform.

1.2 Representation of algebraic sets

In this section we present an efficient representation of algebraic sets, which extends the usual information provided by an atlas, on the level of charts. By this extension blowing ups, covers (see section 1.3), singular loci and operations like coefficient object computation (see section 1.4) can be computed easily.

In this representation we operate with quasi-projective algebraic sets and varieties by performing the computations on the local data of the charts of its atlas. Global operations are translated into several simpler local operations, deriving new charts from existing ones with the property that in the end the resulting charts form an atlas for the resulting algebraic set of the global operation. Due to this approach it is enough to define the operations used in the resolution for charts, and to provide consistency proofs for them.

We assume that the initial problem, the hypersurface (W_0, X) with $\dim W_0 = n$, is given by an atlas $\{V_i, \{\psi_{ij}\}\}$ of W_0 using the representation below, and X

is represented as a resolution problem for each chart of W_0 as in section 1.4, or after the introduction of the new computational methods, as in section 2.5.

An affine chart U of dimension n, derived from the the affine chart V of W_0 via operations in section 1.3, where the open subsets $U_V \subseteq U$ and $V_U \subseteq V$ are isomorphic, is described by the following data:

- A list of affine variables VAR $= (x_1, \ldots, x_m)$, for $m \geq n$, generating the polynomial ring $k[\text{VAR}]$.
- A list DEP $= (d_1, \ldots, d_s) \subset k[\text{VAR}]$. As a subset of the ambient space $\mathbb{A}^m$ the chart is $\mathbf{Z}(\text{DEP})$.
- A list IND $= (p_1, \ldots, p_n) \subset k[U]$.
- An $m \times n$ matrix PDER $= (p_{ij})$, with elements from $k[U]$.
- A list MAP $= (q_1, \ldots, q_n) \subset k[U]$.
- A list IMAP $= (r_1, \ldots, r_m) \subset \mathcal{O}(V_U)$ the ring of regular functions on V_U.

We postulate that:

1. U is a nonsingular algebraic set in $\mathbb{A}^m$ of dimension n.
2. The elements of IND form a *global regular system of parameters* over U, that is, at any point $a \in U$, $p_1 - p_1(a), \ldots, p_n - p_n(a)$ form a regular system of parameters in $\mathcal{O}_{U,a}$.
3. The matrix PDER is $p_{ij} = \partial_j x_i = \partial x_i / \partial p_j,\ i = 1, \ldots, m,\ j = 1, \ldots, n$.
4. The elements of MAP are the pullbacks of the generators of $k[V]$ (blowing up map), while the elements of IMAP are the pushouts of the generators of $k[U]$ (blowing down map) along the sequence of operations (see section 1.3) by which U was derived from V.

Remark 1.5. Because W_0 is a variety, all the charts in the representation of its blownups are varieties too. Nevertheless, as the resolution algorithm passes to lower dimensions (see subsection 1.4.2) we allow reducible ambient algebraic sets, taking special care of singularities lying in separate components (see subsection 1.5.2).

Example 1.6. If $W_0 = \mathbb{A}^n$, it can be described by the atlas consisting of a single affine chart with components: VAR $= (x_1, \ldots, x_n)$, DEP $= ()$, MAP $=$ IMAP $=$ IND $=$ VAR, PDER $= \text{id}_{n \times n}$. This chart obviously fulfills the requirements of the previous postulates.

We represent elements of $k[U]$ with polynomials from $k[\text{VAR}]$ and compute modulo $\langle \text{DEP} \rangle$ with them. When a new chart is generated via an operation like blowing up from an existing one, we overload variable names so that existing names become stable under further operations, while new names are created for new generators by continuing the variable numbering. On the other hand, for efficiency reasons, we wish to keep the number of variables in VAR minimal; therefore whenever $p - x_i \in \text{DEP}$, with $p \in k[\text{VAR} \setminus x_i]$, we drop x_i from VAR, $p - x_i$ from DEP and the IMAP entry for x_i, and substitute p for x_i in every algebraic structure over $k[U]$. Notice, that the elements of MAP may have to be changed in this operation, whereas, due to the stability of variable names, the initial values serve right if no variables are eliminated.

The partial derivative of $f \in k[U]$ with respect to the jth parameter can be computed as $\partial_j f = \sum_{i=1}^{m} p_{ij} \cdot \partial f / \partial x_i$, by the chain rule.

Remark 1.7. A necessary condition for a chart to satisfy the postulates, which is also easy to check, is $\partial_i d_k = 0$, $\partial_i p_j = \delta_i^j$ in $k[U]$, for all possible i, j, k, and regarding MAP and IMAP as ring homomorphisms $\mathcal{O}(V_U) \to k[U]$ and $k[U] \to \mathcal{O}(V_U)$ respectively, $\mathrm{IMAP} \circ \mathrm{MAP} = \mathrm{id}_{\mathcal{O}(V_U)}$, $\mathrm{MAP} \circ \mathrm{IMAP} = \mathrm{id}_{k[U]}$.

The following convention helps to describe the chart transition maps φ_{ij}: first, we overload the names in the VARs of different charts, such that U_i and U_j have common variables. Due to the nature of operations applied on charts in the resolution (see section 1.3), the remaining variables will always fulfill relations (in DEP) that allows to express them as rational functions in the common coordinates. Thus, there is a unique rational map from U_i to U_j respecting the common coordinates, i.e. the pullback of x under φ_{ij} is x if x is a common variable. Then U_{ij} is the domain of definition of this rational map, and φ_{ij} is just the restriction to U_{ij} (so that φ_{ij} is everywhere defined).

When no confusion can arise, we identify the corresponding points in the different charts. In this sense, we may write the blownup variety W as $\bigcup_{i=1}^{r} U_i$, where the U_i form the atlas describing W.

Regarding MAP and IMAP as in remark 1.7, the chart change maps can be given my means of rational functions. Counting with the chart change maps of W_0, assuming that U_i, U_j was derived from V_k, V_l respectively:

$$\varphi_{ij} : a \mapsto b \text{ iff } a \in \mathbf{Z}(\mathrm{MAP}_{U_i}((\mathrm{IMAP}_{U_j}(\mathfrak{m}_b))^{*\psi_{kl}})),$$
$$\varphi_{ji} : b \mapsto a \text{ iff } b \in \mathbf{Z}(\mathrm{MAP}_{U_j}((\mathrm{IMAP}_{U_i}(\mathfrak{m}_a))^{*\psi_{lk}})),$$

where $*_{\psi_{kl}}$ denotes pullback along ψ_{kl}, and the function $\mathbf{Z}$ is naturally extended to rational functions over their domains of definition.

To simplify the presentation considerably without the loss of generality, from now on we assume $W_0 = \mathbb{A}^n$.

1.3 Operations on algebraic sets

The fundamental operations of the resolution algorithm on charts are called *Cover*, *Exchange* and *BlowUp*. In the descriptions of the following algorithms we use the conventions and the notation for chart data as in section 1.2.

1.3.1 Cover

The *cover* operation takes as input a chart and a finite list $F \subset k[U]$, with the property $1 \in \langle F \rangle$. It returns a list of charts with the properties:

1. Each new chart is isomorphic to an affine open subset of the input chart.
2. The new charts cover the input chart.
3. The postulates are fulfilled by the new charts.

The operation is performed by the algorithm *Cover*.

Algorithm *Cover*((VAR, DEP, IND, PDER, MAP, IMAP), F);
  $L := ()$;
  for each $f \in F$ do
    VAR$'$:= append x_{m+1} to VAR;
    DEP$'$:= append $x_{m+1}f - 1$ to DEP;
    $R := (-x_{m+1}^2 \partial_k f \mid k = 1, \dots, n)$;
    PDER$'$:= append R as the $(m+1)$-st row to PDER;
    IMAP$'$:= attach IMAP$(1/f)$ to IMAP for x_{m+1};
    L := attach (VAR$'$, DEP$'$, IND, PDER$'$, MAP, IMAP$'$) to L;
  return L;

Theorem 1.8. *The algorithm* Cover *is correct.*

Proof. Let the input chart be denoted by U, $F = (f_1, \dots, f_r)$, and let the output charts be $U_1, \dots, U_r$ correspondingly.

(1): The isomorphisms are: $\psi_i : U \setminus \mathbf{Z}(f_i) \to U_i$, taking $(a_1, \dots, a_m) \mapsto (a_1, \dots, a_m, 1/f_i(a_1, \dots, a_m))$.

(2): Comes from the assumption $1 \in \langle F \rangle$, implying that the common zero set of the polynomials $f_1, \dots, f_r$ is empty, thus for every point of U there is an isomorphic point in some U_i.

(3): The first postulate is fulfilled for U_i because it is isomorphic to an open subset of the nonsingular U.

The second postulate is satisfied by U_i immediately, because IND has not changed and U_i is isomorphic to an open subset of U. This fact, and the rule for differentiating inverses ($\partial_k x_{m+1} = \partial_k(1/f_i) = -1/f_i^2 \partial_k f_i = -x_{m+1}^2 \partial_k f_i$) make U_i fulfilling the third postulate.

The fourth postulate is satisfied by U_i because the algebraic relation for the new variable, $x_{m+1}f_i - 1$, expresses $x_{m+1} = 1/f_i$ over $\mathcal{O}(U \setminus \mathbf{Z}(f_i))$ whose image can be determined via the IMAP entry of the input chart. The entries in MAP are not changed. ☐

1.3.2 Exchange

The *exchange* operation takes as input a chart, an element $q \in k[U]$, an index i to IND such that $\partial_i q$ is invertible in $k[U]$, and the inverse $a := (\partial_i q)^{-1}$. It returns a new chart being isomorphic to the input chart, having p_i replaced with q in IND, and fulfilling the requirements of the postulates.

This operation acts on the extended data of a chart, thus it does not change the atlas, i.e. the global algebraic set itself. It is typically used to prepare blowing ups (see next subsection).

Theorem 1.9. *The algorithm* Exchange *is correct.*

Proof. We assume that the input chart fulfills the requirements. The first and the fourth postulates are obviously satisfied because the chart, as an algebraic set in $\mathbb{A}^m$ and as a member of the atlas, hasn't changed.

Since $\Omega(U)$ is freely generated by the dp_ks, $dp_1, \dots, dp_{i-1}, dq, dp_{i+1}, \dots, dp_n$ also form a free basis because

$$dq = \sum_{j=1, j \neq i}^{n} \frac{\partial q}{\partial p_j} dp_j + \frac{\partial q}{\partial p_i} dp_i$$

Algorithm *Exchange*((VAR, DEP, IND, PDER, MAP, IMAP), q, i, a);
  IND$'$:= replace p_i by q in IND;
  for j from 1 to m do
    PDER$'[j,i] := ap_{ji}$;
    for k from 1 to m except i do
      PDER$'[j,k] := p_{jk} - ap_{ji}\partial_k q$;
  return (VAR, DEP, IND$'$, PDER$'$, MAP, IMAP);

and $\partial q/\partial p_i$ is invertible over U with inverse a, therefore

$$dp_i = a\big(dq - \sum_{j=1,j\neq i}^{n} \frac{\partial q}{\partial p_j} dp_j\big),$$

so the second postulate is fulfilled.

The third postulate is satisfied, because of the construction of the entries of PDER$'$. For any $f \in k[\mathrm{VAR}]$:

$$\frac{\partial f}{\partial p_i} = \sum_{j=1,j\neq i}^{n} \frac{\partial_{\text{new}} f}{\partial p_j} \cdot \frac{\partial p_j}{\partial p_i} + \frac{\partial_{\text{new}} f}{\partial q} \cdot \frac{\partial q}{\partial p_i}, \quad \text{so}$$

$$\frac{\partial_{\text{new}} f}{\partial q} = \frac{\partial f}{\partial p_i} \left(\frac{\partial q}{\partial p_i}\right)^{-1} = \frac{\partial f}{\partial p_i} \cdot a,$$

and

$$\frac{\partial f}{\partial p_k} = \sum_{j=1,j\neq i}^{n} \frac{\partial_{\text{new}} f}{\partial p_j} \cdot \frac{\partial p_j}{\partial p_k} + \frac{\partial_{\text{new}} f}{\partial q} \cdot \frac{\partial q}{\partial p_k} = \frac{\partial_{\text{new}} f}{\partial p_k} + \frac{\partial_{\text{new}} f}{\partial q} \cdot \frac{\partial q}{\partial p_k}, \quad \text{so}$$

$$\frac{\partial_{\text{new}} f}{\partial p_k} = \frac{\partial f}{\partial p_k} - \frac{\partial f}{\partial p_i} \cdot a \cdot \frac{\partial q}{\partial p_k},$$

where $k = 1, \ldots, n,\ k \neq i$, and $\partial_{\text{new}}/\partial$ denotes partial differentiation in the new differential frame. □

1.3.3 BlowUp

The most important operation performed on charts in the resolution is the blowing up along closed nonsingular centers. In the resolution algorithm the permissible centers are determined by intersections of hypersurfaces given by elements from IND (see section 1.4):

The *blowing up* operation takes as input a chart and a list of different elements of IND. For the sake of better readability we assume that the blowing up center is given by $(p_1, \ldots, p_r)$. The operation returns a list of charts with the following properties:

1. Each new chart is isomorphic to an affine open subset of the blownup.

2. The new charts cover the blownup.

3. The postulates are fulfilled by the new charts.

Algorithm *BlowUp*((VAR, DEP, IND, PDER, MAP, IMAP), $(p_1, \ldots, p_r)$);
  $L := ()$;
  for i from 1 to r do
    VAR$'$:= append $(x_{m+j} \,|\, j = 1, \ldots, r,\ j \neq i)$ to VAR;
    DEP$'$:= append $(x_{m+j}p_i - p_j \,|\, j = 1, \ldots, r,\ j \neq i)$ to DEP;
    IND$'$:= replace p_j by x_{m+j} in IND for $j = 1, \ldots, r,\ j \neq i$;
    for j from 1 to m do
      PDER$'[j, i] := p_{ji} + \sum_{k=1,k\neq i}^{r} x_{m+k}p_{jk}$;
      for k from 1 to r except i do
        PDER$'[j, k] := p_i p_{jk}$;
      for k from $r+1$ to n do
        PDER$'[j, k] := p_{jk}$;
    for j from 1 to r except i do
      R := a list of zeros of length n with a 1 at the jth position;
      PDER$'$:= append R as a new row to PDER$'$;
    IMAP$'$:= attach IMAP(p_j/p_i) to IMAP for x_{m+j}, $j = 1, \ldots, r, j \neq i$;
    L := append (VAR$'$, DEP$'$, IND$'$, PDER$'$, MAP, IMAP$'$) to L;
  return L;

The operation is performed by the algorithm *BlowUp*.

Theorem 1.10. *The algorithm* BlowUp *is correct.*

Proof. We assume that the input chart $U \subseteq \mathbb{A}^m$ whose coordinate ring is generated by $\overline{x}_1, \ldots, \overline{x}_m$, is a nonsingular algebraic set.

(1): Let Y be the blownup of U along $\mathbf{Z}(p_1, \ldots, p_r)$, with

$$\mathbf{I}(Y) = \{p \in k[x_1, \ldots, x_m, y_1, \ldots, y_r] \,|\, p(\overline{x}_1, \ldots, \overline{x}_m, tp_1, \ldots, tp_r) = 0 \text{ in } k[U][t]\}.$$

The dehomogenized ideal of $\mathbf{I}(Y)$ with respect to y_1 is

$$I_1 = \{p \,|\, \text{the homogenization } p' \text{ of } p \text{ with respect to } y_1 \text{ fulfills:}$$
$$p'(\overline{x}_1, \ldots, \overline{x}_m, tp_1, \ldots, tp_r) = 0 \quad \text{in } k[U][t]\}.$$

Here the homogenization of $p \in I_1$ with total degree d in the ys with respect to y_1 is $p(x_1, \ldots, x_m, y_2/y_1, \ldots, y_r/y_1)y_1^d$. Because of the definition of $\mathbf{I}(Y)$ this is equivalent to

$$p(\overline{x}_1, \ldots, \overline{x}_m, tp_2/tp_1, \ldots, tp_r/tp_1)p_1^d t^d = 0 \quad \text{in } k(U)[t],$$

which is equivalent to

$$p(\overline{x}_1, \ldots, \overline{x}_m, p_2/p_1, \ldots, p_r/p_1) = 0 \quad \text{in } k(U),$$

because $p_1, \ldots, p_r$ is a regular sequence for U (see [Eis94] Corollary 10.15, p 241). So I_1 is the kernel of the homomorphism sending $x_j \mapsto \overline{x}_j$, $y_i \mapsto p_i/p_1$, $i \neq 1$. We also know that this kernel is generated by the polynomials $d_1, \ldots, d_s, y_i p_1 - p_i$, $i \neq 1$ (see [Eis94] Exercise 17.14, p 441). Therefore the first chart is isomorphic to $Y \cap (\mathbb{A}^m \times \mathbb{A}_1^{r-1})$, where $\mathbb{A}^m \times \mathbb{P}^{r-1} = \bigcup_{i=1}^{r} (\mathbb{A}^m \times \mathbb{A}_i^{r-1})$ is the covering by principal open subsets of $\mathbb{P}^{r-1}$. This proves the case of the chart with exceptional divisor p_1, and the proof is analogous for the others.

(2): Holds, because the principal open subset decomposition covers $\mathbb{A}^m \times \mathbb{P}^{r-1}$, thus also the restrictions $Y \cap (\mathbb{A}^m \times \mathbb{A}_i^{r-1})$ cover Y.

(3): It has three subparts to show, each including the assumption that the requirements of the postulates are fulfilled by the input chart of the blowing up operation.

The first postulate is fulfilled because the blowing up of a nonsingular algebraic set along a regular center results a regular blownup, therefore the charts are also nonsingular. The new charts are of dimension n, because the dimension is a birational invariant.

The second postulate requires to show that

$$dp_1, dx_{m+2}, \dots, dx_{m+r}, dp_{r+1}, \dots, dp_n$$

generate $\Omega(Y \cap (\mathbb{A}^m \times \mathbb{A}_1^{r-1}))$ (taking only the first chart again). It suffices to show that $dp_2, \dots, dp_r$ is generated, and this is so because $p_i = x_{m+i}p_1$, by the new algebraic relations, thus $dp_i = dx_{m+i}p_1 + dp_1 x_{m+i}$, $i = 2, \dots, r$ everywhere. Since the cardinality of the IND list has not changed, $\Omega(Y \cap (\mathbb{A}^m \times \mathbb{A}_1^{r-1}))$ is generated freely in the new chart.

Compatibility with the third postulate can be checked by computation similar to the case of the exchange operation. We give the formulas for a chart with exceptional divisor given by p_i, so for the first chart one has to substitute $i = 1$. Let f depend only on the variables $x_1, \dots, x_m$, then for $k = 1, \dots, r,\ k \neq i$

$$\frac{\partial f}{\partial p_k} = \sum_{j=1, j\neq i}^{r} \frac{\partial_{\text{new}} f}{\partial x_{m+j}} \cdot \frac{\partial x_{m+j}}{\partial p_k} + \frac{\partial_{\text{new}} f}{\partial p_i} \cdot \frac{\partial p_i}{\partial p_k} + \sum_{j=r+1}^{n} \frac{\partial_{\text{new}} f}{\partial p_j} \cdot \frac{\partial p_j}{\partial p_k} =$$

the surviving summand is with index $j = k$ from the first sum

$$= \frac{\partial_{\text{new}} f}{\partial x_{m+k}} \cdot \frac{\partial (p_k/p_i)}{\partial p_k} = \frac{\partial_{\text{new}} f}{\partial x_{m+k}} \left(\frac{\partial p_k}{\partial p_k} p_i - \frac{\partial p_i}{\partial p_k} p_k \right) \frac{1}{p_i^2}, \quad \text{so}$$

$$\frac{\partial_{\text{new}} f}{\partial x_{m+k}} = p_i \cdot \frac{\partial f}{\partial p_k}.$$

For the case of the exceptional divisor we use the same decomposition together with the formula we just derived, and the surviving parts are:

$$\frac{\partial f}{\partial p_i} = \sum_{j=1, j\neq i}^{r} \frac{\partial_{\text{new}} f}{\partial x_{m+j}} \cdot \frac{\partial (p_j/p_i)}{\partial p_i} + \frac{\partial_{\text{new}} f}{\partial p_i} = \sum_{j=1, j\neq i}^{r} -x_{m+j} \cdot \frac{\partial f}{\partial p_j} + \frac{\partial_{\text{new}} f}{\partial p_i}, \quad \text{so}$$

$$\frac{\partial_{\text{new}} f}{\partial p_i} = \frac{\partial f}{\partial p_i} + \sum_{j=1, j\neq i}^{r} x_{m+j} \cdot \frac{\partial f}{\partial p_j}.$$

One gets analogously for $k = r+1, \dots, n,\quad \partial_{\text{new}} f/\partial p_k = \partial f/\partial p_k$. The values for the new variables come from the axiom $\partial_{\text{new}}^j x_{m+k} = \delta_k^j$, where $k = 1, \dots, r,\ k \neq i,\ j = 1, \dots, n$. Again, $\partial_{\text{new}}/\partial$ denoted partial differentiation in the new differential frame.

The fourth postulate is satisfied because the algebraic relations for the new generators in the ith new chart, $x_{m+j}p_i - p_j$, express $x_{m+j} = p_j/p_i$ over $\mathcal{O}(U \setminus \mathbf{Z}(p_i))$, whose image can be determined via the IMAP entry of the input chart. The entries in MAP are not changed. $\square$

Remark 1.11. Later, when basic objects are introduced, we may also use the procedure call *BlowUp*(Y), which means the blowing up of the ambient space of the basic object in the context, and passing to the transform (defined in section 1.4). We always have Y defined by global regular parameters from the IND entry of the underlying chart.

Example 1.12. We use the initial variables x, y, z, and v_1, v_2 as the introduced ones. Suppose we want to resolve the surface S in three-space with equation

$$z^2 + (x^2 + y^2 - 1)^2 y = 0.$$

The circle $x^2 + y^2 - 1 = z = 0$ is a double curve of S which has to be blown up. Actually, the resolution algorithm tells to first blow up the two points $(\pm 1, 0, 0)$. But for our purposes it suffices to consider what happens in the open chart $y \neq 0$. In this chart, Villamayor's algorithm will blow up the circle.

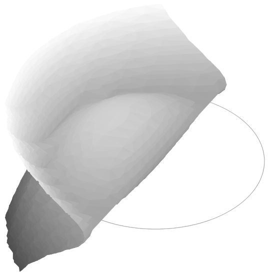


Figure 1.2: Outline of S in real space

So, we first do a cover operation, producing a chart C_1 isomorphic to the open set $y \neq 0$:

1. VAR $= [x, y, z, v_1]$,
2. DEP $= [yv_1 - 1]$,
3. IND $= [x, y, z]$,
4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -v_1^2 & 0 \end{pmatrix}$.

The entry of the last row of the matrix PDER is introduced by the cover operation and comes from the formula $\partial_y v_1 = \partial_y 1/y = -1/y^2 = -v_1^2$. The cover operation also produces another chart C_2, isomorphic to the open sets $y^2 - 1 \neq 0$, but here we only treat C_1.

Next, we exchange y by $x^2 + y^2 - 1$ in IND. This is possible because $\partial_y(x^2 + y^2 - 1)$ is invertible in C_1. We obtain the chart E:

1. VAR $= [x, y, z, v_1]$,
2. DEP $= [v_1 y - 1]$,

3. IND $= [x, x^2 + y^2 - 1, z]$,

4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2}v_1 & 0 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2}v_1^3 & 0 \end{pmatrix}$.

Now, we are ready for blowing up, and in this step we obtain two charts. The first, say, B_1 is given by

1. VAR $= [x, y, v_1, v_2]$,

2. DEP $= [v_1 y - 1]$,

3. IND $= [x, x^2 + y^2 - 1, v_2]$,

4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2}v_1 & 0 \\ v_1^3 x & -\frac{1}{2}v_1^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The variable z has been eliminated. The proper transform of S has equation $v_2^2 + y$, hence it is nonsingular. The exceptional divisor is $x^2 + y^2 - 1$, and one can show easily that the total transform is a normal crossing divisor.

The second chart B_2 is given by

1. VAR $= [x, y, z, v_1, v_2]$,

2. DEP $= [v_1 y - 1, v_2 z - x^2 - y^2 + 1]$,

3. IND $= [x, v_2, z]$,

4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & -\frac{1}{2}v_1 z & -\frac{1}{2}v_2 v_1 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2}v_1^3 z & -\frac{1}{2}v_2 v_1^3 \\ 0 & 1 & 0 \end{pmatrix}$.

Again the proper transform $v_2^2 y + 1$ is nonsingular and has normal crossing with the exceptional divisor z.

1.4 The resolution algorithm

In this section, we describe the overall structure of Villamayor's resolution algorithm. It has been obtained by rewriting the inductive definition of Villamayor's stratifying function [Vil89, Vil96, EV00] in an algorithmic way. The results of this section were achieved exclusively by J. Schicho; nevertheless it is included to make this document self contained.

The desingularization of the given hypersurface is obtained by repeatedly blowing up the ambient space, as described in section 1.1. The first restriction for the blowing up center is that it has to be a nonsingular variety. This ensures that the blownups are nonsingular at any stage.

The second restriction is that the center has normal crossing with respect to the exceptional divisors constructed so far. This restriction ensures that the exceptional divisors have normal crossing at any stage.

Villamayor's definition of the stratifying function is based on an intermixed induction on the order and the dimension, also involving the exceptional divisors that have arisen from previous blowing ups. He introduces *basic objects* of the kind (W, J, b, E), where

- W is a nonsingular algebraic set (when the dimension of W is the same as the dimension of the ambient space of the input problem, W is actually a variety),
- J is a nonzero ideal in $k[W]$ (initially the ideal of the hypersurface),
- b is a positive integer,
- E is a finite list of regular hypersurfaces of W with normal crossing (the exceptional divisors up to the current stage).

We sometimes omit components of basic objects when they are not needed.

In the following, we use "partial derivatives". If $W = \mathbb{A}^n$, these have the usual meaning. Otherwise, we can define the partial derivatives only locally with respect to a system of parameters. For charts, a precise definition is given in section 1.3.

The *singular locus* of a basic object (W, J, b, E) is defined by

$$\mathrm{Sing}(J, b) := \{x \mid \mathrm{ord}_x(J) \geq b\},$$

where

$$\mathrm{ord}_x(J) := \min\{\mathrm{ord}_x(f) \mid f \in J\},$$

and $\mathrm{ord}_x(f)$ is the smallest integer m such that there exists a partial derivative of f of order m not vanishing at x. The order of J at x is positive iff x is in the zero set of J.

A basic object is resolved iff its singular locus is empty. When trying to resolve a basic object (W, J, b, E), we do not want to change anything outside its singular locus. This motivates the third restriction: the blowing up center must be contained in the singular locus. It ensures that the total transform of J (i.e. the ideal that is generated by the pullbacks of elements in J) can be written as $\mathbf{I}(F)^b \cdot J'$, where F is the exceptional divisor.

Let W' be the blownup. Let E' be the list of all proper transforms of divisors in E, together with the new exceptional divisor. Let J' be as above. Then the basic object (W', J', b, E') is called the *transform* of (W, J, b, E). In the algorithm, this is going to be the next basic object to be resolved.

We say that a blowing up operation is *permissible* with respect to the basic object (W, J, b, E) iff the blowing up center fulfills the three restrictions above (nonsingular, normal crossing with E, contained in the singular locus).

Here is a technical lemma which is used several times in the correctness proof.

Lemma 1.13. *Let (W, J, b, E) and (W, I, b, E) be basic objects. Let Y be a permissible blowing up center, and let (W', J', b, E') and (W', I', b, E') be the transforms. Then the following hold.*

1. *Sing*$(J,b) =$ *Sing*(J^n, nb) *for* $n\geq 1$.
2. *Sing*$(J+I,b) =$ *Sing*$(J,b)\cap$*Sing*(I,b).
3. *The transform of* (W,J^n,nb,E) *is equal to* $(W',(J')^n,nb,E')$ *for* $n\geq 1$.
4. *The transform of* $(W,J+I,b,E)$ *is equal to* $(W',J'+I',b,E')$.

Proof. (1): follows immediately from the relation $\mathrm{ord}_x(f^n) = n\cdot\mathrm{ord}_x(f)$.

(2): follows from the fact that in the definition of $\mathrm{ord}_x(J)$, it suffices to take the minimum over a fixed set of generators.

(3): let $\pi : W'\to W$ be the blowing down map, and let π^* be the pullback homomorphism. Then both sides are generated by

$$\left\{\frac{(\pi^* f)^n}{h^{nb}}\ \Big|\ f\in J\right\} = \left\{\left(\frac{\pi^* f}{h^b}\right)^n\ \Big|\ f\in J\right\},$$

where h is a local equation of the exceptional divisor.

(4): both sides are generated by

$$\left\{\frac{\pi^* f}{h^b}\ \Big|\ f\in J \text{ or } f\in I\right\}.$$

□

Basic objects come in three variants: weighted objects, tagged objects, and balanced objects. We have resolution algorithms for these three types, which call each other recursively.

1.4.1 Weighted objects

A *weighted object* is a basic object (W,J,b,E), together with a function $a : E\to\mathbb{N}$, such that the following holds. Let $E=(H_1,\ldots,H_r)$, and let $(a_1,\ldots,a_r)$ be the list of values of a. Define

$$M = \mathbf{I}(H_1)^{a_1}\cdots\mathbf{I}(H_r)^{a_r}$$

(locally, this ideal is generated by a monomial in a system of parameters). Then the ideal J can be written as $J = M\cdot\overline{J}$. We call M the monomial part and $\overline{J}$ the proper part of J. The largest order of $\overline{J}$ at points within $\mathrm{Sing}(J,b)$ is called the *maxorder.*

We will blowup weighted objects only at centers which are contained in the subset of $\mathrm{Sing}(J,b)$ where the maximal order is achieved. The transform of the new component a is done as follows. For any hypersurface $H\in E$, the value of the proper transform of a is defined to be $a(H)$, i.e. old hypersurfaces remain with old values. The value of a at the new exceptional divisor F is defined to be

$$a(H_1)+\ldots+a(H_r)+\overline{b}-b,$$

where $\overline{b}$ is the maxorder, and $\{H_1,\ldots,H_r\}$ is the set of all hypersurfaces in E containing the blowing up center. To ensure that this assignment is valid, we have to show that the transformed ideal J' contains $\mathbf{I}(F)^{a(H_1)+\ldots+a(H_r)+\overline{b}-b}$. But this follows from the fact that the order of any point in the blowing up center is at least $a(H_1)+\ldots+a(H_r)+\overline{b}$.

Lemma 1.14. *Let $(\overline{J}', \overline{b})$ be the the transform of the basic object $(\overline{J}, \overline{b})$. Then $\overline{J}'$ is the proper part of the transformed weighted object.*

Proof. Assume that $E = \{H_1, \ldots, H_s\}$, $s \geq r$. For each H_i, let h_i be a local equation, and let h be a local equation of the new exceptional divisor. Then $h_1/h, \ldots, h_r/h, h_{r+1}, \ldots, h_s$ are local equations of the transformed exceptional divisors.

Let M' be the monomial part of the transformed weighted object. Locally, M' is generated by the element

$$(h_1/h)^{a(H_1)} \ldots (h_r/h)^{a(H_r)} h_{r+1}^{a(H_{r+1})} \ldots h_s^{a(H_s)} h^{a(H_1)+\ldots+a(H_r)+\overline{b}-b} = h_1^{a(H_1)} \ldots h_s^{a(H_s)} h^{\overline{b}-b}.$$

Thus, the proper part of the transformed weighted object is generated by

$$\left\{ \frac{f h_1^{a(H_1)} \ldots h_s^{a(H_s)}/h^b}{h_1^{a(H_1)} \ldots h_s^{a(H_s)} h^{\overline{b}-b}} \;\middle|\; f \in \overline{J} \right\} = \left\{ \frac{f}{h^{\overline{b}}} \;\middle|\; f \in \overline{J} \right\}.$$

But this set generates the transform of $(\overline{J}, \overline{b})$. □

A *simple object* is a basic object (W, J, b, E) satisfying $\mathrm{ord}_x(J) = b$ for all $x \in \mathrm{Sing}(J, b)$. One can easily show that the transform of a simple object is again simple (a stronger statement will also be shown in lemma 1.20).

A *tagged object* is a simple object (W, J, b, E) together with a subset E_- of E, satisfying an additional "condition T", which will be explained in subsection 1.4.3. For the moment, it suffices to know two things about condition T:

1. If (W, J, b, E) is simple, then $(W, J, b, E, E_- = E)$ is tagged.

2. The transform of a tagged object is tagged.

Here, the transform of the component E_- is done in the obvious way: E'_- is defined as the set of proper transforms of the hypersurfaces in E_-.

Two tagged objects are called *similar* iff their underlying simple objects coincide (in other words, they may only differ in the E_- component). Obviously, two similar tagged objects have the same singular locus. Moreover, the transforms of similar tagged objects are again similar. Therefore, any sequence of blowing ups which is permissible for one tagged object is also permissible for a similar tagged object.

For any weighted object (W, J, b, E, a) with nontrivial proper part, the *equiorder* is defined as the tagged object constructed by the algorithm *EquiOrder*.

The equiorder enjoys the following properties.

Lemma 1.15. *Let $\mathcal{W}$ be a weighted object with positive order, and let $\mathcal{T}$ be its equiorder. Then the following hold.*

1. *The singular locus of $\mathcal{T}$ is equal to the subset of the singular locus of $\mathcal{W}$ where the proper part has maximal order.*

2. *Let Y be permissible with respect to $\mathcal{T}$. Then Y is also permissible with respect to $\mathcal{W}$.*

Algorithm $EquiOrder(W, J, b, E, a)$;
  $M, \overline{J}$:= the monomial part and the proper part;
  $\overline{b}$:= the maxorder;
  if $\overline{b} \geq b$ then
    $I := \overline{J}$;
    $w := \overline{b}$;
  else
    $I := \overline{J}^{b-\overline{b}} + M^{\overline{b}}$;
    $w := (b - \overline{b})\overline{b}$;
  $E_- := E$;
  return (W, I, w, E, E_-);

3. *Let $\mathcal{W}'$ and $\mathcal{T}'$ be the transform of $\mathcal{W}$ and $\mathcal{T}$, respectively, along the blowing up of Y. If $\mathcal{T}'$ is not resolved, then maxorder($\mathcal{W}'$) = maxorder($\mathcal{W}$), and $\mathcal{T}'$ is similar to the equiorder of $\mathcal{W}'$.*

4. *If $\mathcal{T}'$ is resolved, then either maxorder($\mathcal{W}'$) $<$ maxorder($\mathcal{W}$), or $\mathcal{W}'$ is resolved.*

Proof. We use the same notation as in the description of the algorithm *EquiOrder*.

(1): We distinguish two cases.

If $\overline{b} \geq b$, then every point in the singular locus of $(\overline{J}, \overline{b})$ is also a singular point of $\mathcal{W}$, and it consists precisely of those points where the order is maximal (namely $\overline{b}$).

Now, suppose $\overline{b} < b$. By lemma 1.13, the singular locus of $\mathcal{T}$ is the intersection of the singular locus of $(\overline{J}, \overline{b})$ and the singular locus of $(M, b - \overline{b})$. But

$$\mathrm{ord}_x(\overline{J}) \geq \overline{b} \wedge \mathrm{ord}_x(M) \geq b - \overline{b}$$

is equivalent to

$$\mathrm{ord}_x(\overline{J}) \geq \overline{b} \wedge \mathrm{ord}_x(J) \geq b,$$

hence the singular locus of $\mathcal{T}$ consists of all the points in Sing($\mathcal{W}$) where $\overline{J}$ has maximal order.

(2): follows immediately from (1).

(3) and (4): we distinguish two cases again.

If $\overline{b} \geq b$, then $(\overline{J}, \overline{b})$ is simple. By lemma 1.14, the transform is $(\overline{J}', \overline{b})$, where $\overline{J}'$ is the proper part of $\mathcal{W}'$. This is again a simple object. If it is not resolved, then the maxorder of $\mathcal{W}'$ is $\overline{b}$, and $\mathcal{T}'$ is the equiorder of $\mathcal{W}'$. If it is resolved, then the maxorder is smaller than $\overline{b}$. Thus the maxorder has dropped—unless $\mathcal{W}'$ is resolved, in which case the maxorder is not defined.

Now, suppose $\overline{b} < b$. By lemma 1.13, $\mathcal{T}'$ is equal to $((\overline{J}')^{b-\overline{b}} + (M')^{\overline{b}}, (b - \overline{b})\overline{b})$, where $\overline{J}'$ is the proper part of $\mathcal{W}'$ (again by lemma 1.14), and M' is the transformed ideal of the basic object $(M, b - \overline{b})$. In the notation as in the proof of lemma 1.14, M' is generated by $h_1^{a(H_1)} \ldots h_s^{a(H_s)} h^{\overline{b}-b}$, hence M' is the monomial part of $\mathcal{W}'$. The object $(\overline{J}, \overline{b})$ is not necessarily simple, it could have points of higher order outside the singular locus of $\mathcal{W}$. But it behaves like a simple object because the blowing up center consists of points of order $\overline{b}$. If $\mathcal{T}'$ is not resolved, then the maximal order of $(\overline{J},\overline{b})$ within the singular locus of $(M',b-\overline{b})$ is $\overline{b}$, and $\mathcal{T}'$ is the equiorder of $\mathcal{W}'$. Otherwise, the maxorder has dropped—unless $\mathcal{W}'$ is resolved. □

Weighted objects with zero maxorder are also called *monomial objects*. One can easily show that the transform of a monomial object is again monomial. Monomial objects can be resolved by the algorithm *ResolveMonomial*. We say that the hypersurfaces $H_1,\ldots,H_r$ have *common singularities* iff the intersection of $H_1,\ldots,H_r$ and the singular locus of the basic object is non-empty.

Algorithm *ResolveMonomial*($\mathcal{T}$);
  while not resolved do
    $r :=$ the smallest number such that r hypersurfaces in E
      have common singularities;
    $\overline{b} :=$ the largest possible value of $a(H_1)+\ldots+a(H_r)$,
      where $H_1,\ldots,H_r\in E$ have common singularities;
    $H_1,\ldots,H_r :=$ the first r hypersurfaces for which the maximum $\overline{b}$ is
      achieved (with respect to lexicographical ordering);
    $Y :=$ the intersection of $H_1,\ldots,H_r$;
    BlowUp(Y);

Theorem 1.16. *The algorithm* ResolveMonomial *is correct.*

Proof. (This proof was communicated to us by S. Encinas.) It is obvious that the center is permissible and that the algorithm will not stop before the basic object is resolved. The difficult part is termination.

For any monomial object, we define the number m as cardinality of the set of all r-subsets $\{H_1,\ldots,H_r\}$ of E such that $a(H_1)+\ldots+a(H_r)=\overline{b}$, where r and $\overline{b}$ are defined as in the description of the algorithm *ResolveMonomial*. We claim that the triple $(\dim W-r,\overline{b},m)$ drops in every blowing up step in the algorithm. Here, triples are compared by lexicographical order. Obviously, this claim then implies termination.

To prove the claim, we assume indirectly that the claim does not hold. Then we have either $r'<r$, or ($r'=r$ and $\overline{b}'>\overline{b}$), or ($r'=r$ and $\overline{b}'=\overline{b}$ and $m'\geq m$), where the postfix $'$ denotes the corresponding value of the transform.

Assume that $r'<r$. By definition of r', there are elements $G_1',\ldots,G_{r'}'\in E'$ such that $a(G_1')+\ldots+a(G_{r'}')\geq b$. The new divisor F must be among the G_i, say $F=G_1'$. The other divisors are proper transforms of old divisors $G_2,\ldots,G_{r'}$. Among $H_1,\ldots,H_r$, there must be a divisor, say H_1, which is not in $\{G_2,\ldots,G_{r'}\}$. Then

$$b\leq a(G_1')+\ldots+a(G_{r'}')=a(H_1)+\ldots+a(H_r)-b+a(G_2)+\ldots+a(G_{r'})=$$

$$=(a(H_1)+a(G_2)+\ldots+a(G_{r'}))+(a(H_2)+\ldots+a(H_r))-b<b+b-b,$$

a contradiction.

Now, assume that $r'=r$ and $\overline{b}'>\overline{b}$. Then it follows that $a(G_1')+\ldots+a(G_r')>\overline{b}$ for some $G_1',\ldots,G_{r'}'\in E'$. Then the new divisor F must be among the G_i, say $F = G_1$. Among $H_1, \ldots, H_r$, there is again a divisor, say H_1, which is not in $\{G_2, \ldots, G_r\}$. Then

$$\overline{b} < a(G_1') + \ldots + a(G_r') = a(H_1) + \ldots + a(H_r) - b + a(G_2) + \ldots + a(G_r) =$$

$$= (a(H_1) + a(G_2) + \ldots + a(G_r)) + (a(H_2) + \ldots + a(H_r)) - b < \overline{b} + b - b,$$

again a contradiction.

Finally, assume that $r' = r$ and $\overline{b}' = \overline{b}$ and $m' \geq m$. By the inequality above, any r-subset containing F does not achieve the maximum value $\overline{b}$. All r-subsets achieving the maximum are old, i.e. consist of proper transforms. But the proper transforms of $H_1, \ldots, H_r$ do not have common singularities, because their intersection was blown up. Hence $m' < m$, which is again a contradiction. □

Remark 1.17. The statement *BlowUp*(Y) implies that we pass from the given basic object to its transform. Moreover, we will have in general a whole stack of "current objects" of all three types, due to the recursive calls. In this case, Y will be permissible with respect to all of them, and the statement *BlowUp*(Y) means that we pass to the proper transform for all basic objects in the stack.

The algorithm *ResolveWeighted* is for the resolution of weighted objects (it calls the resolution algorithm for tagged objects).

Algorithm *ResolveWeighted*($\mathcal{W}$);
    while not resolved and maxorder>0 do
        $\mathcal{T}$:= *EquiOrder*(current object);
        ResolveTagged($\mathcal{T}$);
    if not resolved then
        ResolveMonomial(current object);

Theorem 1.18. *The algorithm* ResolveWeighted *is correct (i.e. relative correct with respect to the subalgorithm* ResolveTagged*).*

Proof. Note that the algorithm *ResolveWeighted* does not depend on the E_- component (it is used inside *ResolveTagged*, but this is irrelevant to the relative correctness). Therefore, we do not need to distinguish between similar tagged objects. Especially, we may apply lemma 1.15 repeatedly.

By this repeated application, we see that resolving the equiorder either resolves the weighted object or it makes the maxorder drop. Repeating, we obtain a resolution or maxorder zero. In the latter case, *ResolveMonomial* resolves. □

1.4.2 Balanced objects

Let (W, J, b, E) be a simple object. The *delta-ideal* $\Delta(J, b)$ is defined as the ideal generated by all partial derivatives of elements in J of order less than b. Its zero set contains the singular locus. For every point $x \in \mathrm{Sing}(J, b)$, there exists a $z \in \Delta(J, b)$ which has order 1 at x (because for each point x in the singular locus, there is a partial derivative of order b not vanishing at x).

The simple object (W, J, b, E) is called *balanced* iff the following condition holds. For every point $x \in \mathrm{Sing}(J, b)$, there exists a $z \in \Delta(J, b)$ which has order 1 at x and $\mathbf{Z}(z)$ has normal crossing with E in a neighborhood of x.

Let (W, J, b, E) be a balanced simple object, and let $z \in \Delta(J, b)$ be an element which has order 1 and normal crossing with E at any point of its zero set. Then we say that this zero set (called V below) is a *hypersurface of maximal contact.*

Suppose that $W = \mathbb{A}^n$, and z is one of the coordinate functions. Then we define the *coefficient ideal* $C(J, b, V)$ as follows. Let

$$f_{0,1} + \ldots + f_{d_1,1} z^{d_1}, \ldots, f_{0,r} + \ldots + f_{d_r,r} z^{d_r}$$

be generators of J, where $f_{i,j}$ are polynomials in the remaining $n-1$ variables. The ideal C is generated by

$$f_{0,1}^{\frac{b!}{b}}, \ldots, f_{b-1,1}^{\frac{b!}{1}}, \ldots, f_{0,r}^{\frac{b!}{b}}, \ldots, f_{b-1,r}^{\frac{b!}{1}}.$$

In the general case, one can construct the coefficient ideal by partial differentiation.

The coefficient ideal is zero iff V is equal to the singular locus. In this case, V is permissible for blowing up. Since V is a hypersurface, the blowing up map is an isomorphism on W. It is easy to check that the transform of (W, J, b, E) is then resolved.

Remark 1.19. In general, V may be reducible, and there might be components of V where the coefficient ideal is zero, and other components where it is not. In this section, we assume for simplicity that V is irreducible. For the general case, refer to section 1.5.

If the coefficient ideal is not zero, then the *coefficient object* is defined as

$$(V, C(J, b, V), b!, E - V, 0),$$

where 0 is the constant zero function.

We must make an important remark here: The above construction of coefficient ideal is not well defined with respect to different sets of generators for J, i.e., different generating sets of J may result different C ideals. However, we have following properties for any sets of generators:

- $C = 0$ iff V is equal to the singular locus,
- when $C \neq 0$, $\mathrm{Sing}(C, b!) = \mathrm{Sing}(J, b)$ via the embedding $V \subseteq W$, and this remains true after any permissible sequence of blowing ups,
- for a fixed set of generators of J and a permissible blowing up that does not make the singular loci empty, we have $C(J', b, V) = C(J, b, V)'$, where primes denote transforms (see part 4 of lemma 1.20).

The consequence of these properties is that the resolution of the balanced object can be achieved via patching local resolutions of coefficient objects (see subsection 7.2 in [EV00]). These resolutions are invariant of the choice of generators of J.

In section 2.4 we explicitly deal with this problem by factoring the equivalence relation $\approx$ out from the set resolution problems (which was originally introduced by Hironaka on page 104 in [Hir77]).

The coefficient object enjoys the following properties.

Lemma 1.20. *Let $\mathcal{B}$ be a balanced object. Let V be a hypersurface of maximal contact. Suppose that the coefficient ideal of $\mathcal{B}$ in V is not zero. Let $\mathcal{W}$ be the coefficient object. Then the following hold.*

1. *The singular loci of $\mathcal{B}$ and $\mathcal{W}$ coincide.*
2. *Let $Y \subset V$ be permissible with respect to $\mathcal{W}$. Then Y is also permissible with respect to $\mathcal{B}$.*
3. *Let $\mathcal{B}'$ be the transform of $\mathcal{B}$ along the blowing up of Y, and let V' be the proper transform of V. Then V' has maximal contact to $\mathcal{B}'$. Especially, $\mathcal{B}'$ is again balanced.*
4. *The transform of $\mathcal{W}$ along the blowing up of Y is equal to the coefficient object of $\mathcal{B}'$ with respect to V', where the chosen set of generators for J' consists of the transforms of the fixed set of generators of J.*

Proof. We assume that $W = \mathbb{A}^n$, and z is one of the coordinate functions. We use the same notation as in the definition of the coefficient object.

(1): since V is a hypersurface of maximal contact, the singular locus of $\mathcal{B}$ is contained in V. Moreover, it is the zero set of all partial derivatives of $f_{m,n}$ of order less than $b - m$ with respect to the variables different from z. But this is precisely the singular locus of $\mathcal{W}$.

(2): follows immediately from (1).

(3): for any ideal I, let ∂I be the ideal generated by I and by all first order partial derivatives of elements of I (it suffices to take the partial derivatives of a generating set).

Let Y be permissible with respect to (I, m), $m \geq 2$, and let (I', m) be the transform. Obviously, Y is also permissible with respect to $(\partial I, m-1)$. Let $((\partial I)', m-1)$ be the transform. A straightforward calculation using the coordinates introduced in example 1.4 shows that $(\partial I)' \subset \partial(I')$. Applying this relation $b-1$ times to (J, b), we get that the transform of $\Delta(J, b)$—transformed with integer component 1—is contained in $\Delta(J', b)$. Especially, the local equation of V' is in $\Delta(J', b)$. Clearly V' has normal crossing with E', hence V' has maximal contact.

(4): assume that Y is the zero set of $(z, x_1, \ldots, x_{m-1})$. The chart corresponding to z is disjoint from V'. In the chart corresponding to x_1, $\mathcal{B}'$ is generated by the polynomials

$$f_{0,i}/x_1^b + f_{1,i}/x_1^{b-1}\tilde{z} + \ldots + f_{d_i,i}/x_1^{b-d_i}\tilde{z}^{d_i}, \; i = 1, \ldots, r.$$

The coefficient ideal is generated by

$$\left(f_{j,i}/x_1^{b-j}\right)^{b!/(b-j)}, \; i = 1, \ldots, r, \; j = 0, \ldots, b-1.$$

By lemma 1.13, this is precisely the transform of the coefficient ideal of $\mathcal{B}$. □

The algorithm *ResolveBalanced* is for the resolution of balanced objects. It calls a resolution algorithm for weighted objects of smaller dimension.

Theorem 1.21. *The algorithm* ResolveBalanced *is correct.*

Algorithm *ResolveBalanced*($\mathcal{B}$);
    V := a hypersurface of maximal contact;
    C := the coefficient ideal of $\mathcal{B}$ with respect to V;
    if C is the zero ideal then
        BlowUp(V);
    else
        $\mathcal{W}$:= the coefficient of object of $\mathcal{B}$ with respect to V;
        ResolveWeighted($\mathcal{W}$);

Proof. If the coefficient ideal is zero, we already mentioned that V is permissible and the transform is resolved.

Otherwise, repeated application of lemma 1.20 tells that by resolving $\mathcal{W}$, we also resolve $\mathcal{B}$. □

An obstacle is that the ideal $\Delta(J, b)$ may have no global element of order 1, even though it has elements of order 1 at any point by definition of being "simple". But then we may replace W by open subsets which cover it, and in each of the subsets we have a global element of order 1. In subsection 1.5.1, we will describe how this can be done.

1.4.3 Tagged objects

A tagged object is a simple object (W, J, b, E) together with a subset E_- of E, such that the simple object $(W, J, b, E - E_-)$—which we call the *shadow*—is a balanced object. (This condition was referred to as "property T" in subsection 1.4.1.)

Let $\mathcal{T}$ be a tagged basic object. Then the transform of $\mathcal{T}$ along some permissible blowing up is again tagged, because its shadow is the transform of the shadow of $\mathcal{T}$, which is balanced by lemma 1.20. This shows a fact that we have used in section 1.4.1: the transform of a tagged object is again tagged. The second statement that we have used in section 1.4.1 is that $E = E_-$ and being simple implies being tagged; but this is obvious.

Let r be the largest number such that r hypersurfaces in E_- have common singularities. If $r > 0$, let m be the number of such r-subsets of E_-. The pair (r, m) (or $(0, \text{"undefined"})$) is called the *balance* of (W, J, b, E, E_-). It is a measure of the severeness of the tagged object. Balances are compared by the lexicographic order.

The *support* is defined as the basic object constructed by the algorithm *Support*.

By enlarging the ideal, one does not destroy the property of being simple. Hence the support is a simple object. If $r > 0$, then the hypersurface H_1 has maximal contact and normal crossing with E_+, hence the support is balanced. If $r = 0$, then the support is balanced by property T.

We have the following lemma.

Lemma 1.22. *Let $\mathcal{T}$ be a tagged object. Let $\mathcal{B}$ be its support.*

1. *The singular locus of $\mathcal{B}$ is equal to the set of common singularities of a maximal number r of hypersurfaces in E_-.*

Algorithm *Support*(W, J, b, E, E_-);
  $r :=$ the largest number such that r hypersurfaces in E_-
    have common singularities;
  $H_1, \ldots, H_r :=$ the first r hypersurfaces in E_- having common
    singularities (with respect to lexicographical ordering);
  $I := J + \mathbf{I}(H_1)^b + \ldots + \mathbf{I}(H_r)^b$;
  $E_+ := E - E_-$;
  return (W, I, b, E_+);

2. *Let Y be permissible with respect to $\mathcal{B}$. Then Y is also permissible with respect to $\mathcal{T}$.*

3. *Let $\mathcal{T}'$ and $\mathcal{B}'$ be the transforms of $\mathcal{T}$ and $\mathcal{B}$, respectively. If $\mathcal{B}'$ is not resolved, then $\mathcal{B}'$ is the support of $\mathcal{T}'$, and balance$(\mathcal{T}')$ = balance$(\mathcal{T})$.*

4. *If $\mathcal{B}'$ is resolved, then balance$(\mathcal{T}') <$ balance$(\mathcal{T})$ or $\mathcal{T}'$ is resolved.*

Proof. We use the same notation as in the description of the algorithm *Support.*

(1): follows immediately from lemma 1.13.

(2): by (1), Y is contained in the singular locus of $\mathcal{T}$. Let X be the intersection of $H_1, \ldots, H_r$. Since Y is a nonsingular algebraic subset of X that has normal crossing with $E_- \cap X$, we can extend the system of parameters defined by $E_- \cap X$ by $p_1, \ldots, p_m$, such that Y is defined by a subset of the extended system. Let $P_1, \ldots, P_m$ be elements in $k[W]$ that restrict to $p_1, \ldots, p_m$ on X. Then for any $x \in Y$, there is a neighborhood such that the elements $P_1, \ldots, P_m$ together with E_- and $H_1, \ldots, H_r$ form a system of parameters. Thus Y has normal crossing with E.

(3) and (4): Let $H_1', \ldots, H_r'$ be the proper transforms of $H_1, \ldots, H_r$. By lemma 1.13, $\mathcal{B}'$ is $(W', J' + \mathbf{I}(H_1')^b + \ldots + \mathbf{I}(H_r')^b, b, E' - E_-')$.

If $\mathcal{B}'$ is not resolved, then we still have the r hypersurfaces $H_1', \ldots, H_r' \in E_-'$ having common singularities. Note that Y is disjoint from all hypersurfaces in $E_- - \{H_1, \ldots, H_r\}$, so that the intersection relations between the other hypersurfaces in E_- are not changed. Hence balance$(\mathcal{T})$ = balance$(\mathcal{T}')$, and $\mathcal{B}'$ is the support of $\mathcal{T}'$.

If $\mathcal{B}'$ is resolved, then the number of r-subsets of E_- with common singularities has dropped, which implies that the balance has dropped, or the singular locus of $\mathcal{T}$ is empty. □

The algorithm *ResolveTagged* is for the resolution of tagged objects. It calls the algorithm for resolution of balanced objects.

Algorithm *ResolveTagged*$(\mathcal{T})$;
  while not resolved do
    $\mathcal{B} :=$ *Support*(current object);
    ResolveBalanced$(\mathcal{B})$;

Theorem 1.23. *The algorithm* ResolveTagged *is correct.*

Proof. By repeated application of lemma 1.22, we see that resolving the equiorder either resolves the weighted object or it makes the maxorder drop. Since the balance can only drop finitely many times, we obtain a resolution. □

1.4.4 Desingularization

The problem of embedded desingularization of hypersurfaces can be reduced to the resolution of basic objects in the following way. Given a nonsingular affine variety U and a hypersurface X in it, we just have to resolve the weighted basic object $(U, \mathbf{I}(X), 1, (), \text{empty\_function})$. Then the final transform has ideal $\langle 1 \rangle$ and therefore the total transform is a product of ideals of exceptional divisors (i.e. elements in E), which always have normal crossing.

A desingularization (not necessarily embedded) is already obtained as soon as the maxorder has become less than 2. This follows from the fact that the proper part is nothing but the equation of the proper transform.

We try to outline the work of the resolution algorithm in the diagram of figure 1.3. The objects of U_i form a finite sequence which starts with the first weighted object, $\mathcal{W}_1^i$, being the transform of the initial weighted object, $\mathcal{W}_1^1$, along the sequence of blowing ups made so far, on the branch of the resolution tree with the given list of exceptional divisors. The "descendant" object of $\mathcal{W}_1^i$ is $\mathcal{T}_1^i$ via equiorder computation, followed by $\mathcal{B}_1^i$ via support computation, followed by $\mathcal{W}_2^i$ via coefficient object computation, and so on.

The maximum length of such an object list (it is depth in the diagram) can be $3n$, where n is the dimension of U.

The progress of the resolution along the branch of the resolution tree corresponding to the exceptional divisor list $E^{(1)}, \ldots, E^{(N-1)}$ of consecutive blowing ups, appears in the horizontal direction in the figure. We also assume that over the charts $U_1, \ldots, U_{N-1}$ there exist global order 1 elements generating hypersurfaces of maximal contacts in each balanced objects. That is, we need not to pass to local subcases.

The double crosses denote commutative subdiagrams, where similar tagged objects are considered to be equivalent. It has to be noted, that basic objects don't get transformed: they are recomputed from the tagged objects of the same dimension. Moreover, when a tagged object gets resolved, and the weighted object of the same dimension is not resolved yet, a new tagged object is created which is not the transform of the previous tagged object along the last blowing up; therefore the subdiagram fails to commute (e.g. the situation of $\mathcal{T}_1^j$ and $\mathcal{T}_1^i$).

1.5 Computational details

In this section, we describe the computational subtasks of the desingularization algorithm, and how they are solved by the Gröbner bases method.

The computational subtasks in the desingularization algorithm are the following:

1. polynomial arithmetic and partial differentiation;

2. testing whether an ideal contains 1 (to find the maxorder, for instance);

$$\begin{tikzcd}
U_1 = U & & & & & U_i & & \\
\mathcal{W}_1^1 \arrow[r, "E^{(1)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] \arrow[dr, phantom, "\#"] & \mathcal{W}_1^2 \arrow[r, "E^{(2)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] \arrow[drr, phantom, "\#"] & \cdots \arrow[r, "E^{(j-1)}", "\text{tr.}"'] & \mathcal{W}_1^j \arrow[r, "E^{(j)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] & \cdots \arrow[r, "E^{(i-1)}", "\text{tr.}"'] & \mathcal{W}_1^i \arrow[r, "E^{(i)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] & \cdots \arrow[r, "E^{(N-1)}", "\text{tr.}"'] & \mathcal{W}_1^N \\
\mathcal{T}_1^1 \arrow[r, "E^{(1)}", "\text{tr.}"'] \arrow[d, "\text{supp.}"] & \mathcal{T}_1^2 \arrow[r, "E^{(2)}", "\text{tr.}"'] \arrow[d, "\text{supp.}"] & \cdots \arrow[r, "E^{(j-1)}", "\text{tr.}"'] & \mathcal{T}_1^j \arrow[d, "\text{supp.}"] & & \mathcal{T}_1^i \arrow[d, "\text{supp.}"] & & \text{resolved} \\
\mathcal{B}_1^1 \arrow[r, phantom, "\#"] \arrow[d, "\text{coeff.}"] & \mathcal{B}_1^2 \arrow[d, "\text{coeff.}"] & & \mathcal{B}_1^j & & \mathcal{B}_1^i \arrow[d, "\text{coeff.}"] & & \\
\mathcal{W}_2^1 \arrow[r, "E^{(1)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] & \mathcal{W}_2^2 \arrow[d, "\text{eqo.}"] & & \text{blowup}_j & & \mathcal{W}_2^i \arrow[d, "\text{eqo.}"] & & \\
\vdots & \vdots & & & & \vdots \arrow[d, "\text{supp.}"] & & \\
 & & & & & \mathcal{B}_k^i & \text{blowup}_i &
\end{tikzcd}$$

Figure 1.3: A local branch of the resolution of the weighted object $\mathcal{W}_1^1$.

3. computing the transform (with respect to the value b) of a polynomial;
4. finding an element of order 1 in the delta-ideal;
5. separating the components of V where the coefficient ideal is zero from the others.

Task 1 is easy.

To perform task 2, we compute a Gröbner basis and check whether it contains a constant.

To perform task 3, we need to divide an element in $k[U]$ by some power of an element in IND. If $U = \mathbb{A}^n$, then we can simply use polynomial division. In general, we have a set $\{d_1, \ldots, d_s\}$ (arising by DEP and N) of polynomials, and two polynomials f, g, and we want to compute a q such that $f - qg$ is in the ideal $\langle d_1, \ldots, d_s \rangle$. We do this by solving the linear equation

$$x_1 d_1 + \ldots + x_s d_s + qg = f$$

by the method of extended Gröbner bases (see [BW93], section 6.1).

The remaining tasks are now being discussed in some detail.

The following tasks seem to be necessary to deal with, but their solutions are automatically provided by our way of representing charts and basic objects.

1. Computing equations for the blowing up center. Note that the blowing up center arises as an intersection of hypersurfaces of maximal contact and exceptional divisors. We always keep these hypersurfaces in IND. Thus, we already have the center generated by elements in IND when it comes to blowing up. (Especially, no radical computation and no primary decomposition is necessary.)

2. Computing the proper part of a weighted object. Checking carefully the algorithms *ResolveWeighted* and *EquiOrder*, we see that the proper part is equal to the ideal component of the just resolved tagged object, except in the beginning, where it is equal to J.

1.5.1 Finding an element of order one

In the subalgorithm *ResolveBalanced*, we have to construct a hypersurface of maximal contact for a given balanced object $\mathcal{B} = (W, J, b, E)$ over a chart U. Let I be the delta-ideal, $\Delta(J, b)$, of the balanced object, defined in subsection 1.4.2. An element $z \in I$ is a local equation of a hypersurface of maximal contact iff $\mathrm{ord}(z) = 1$, i.e. $\mathbf{Z}(\langle z\rangle + \langle \partial_p z \mid p \in \mathrm{IND}\rangle) = \emptyset$.

We do not just want to compute an element of order 1, but we also want to be able to perform an *Exchange* operation that brings this element into the IND entry of a (not necessarily proper) subchart of U. In general, there is no global solution, so the task has to be prepared by a *Cover* operation.

In each subchart, U_i, we want to find an element $z \in I$ and a parameter p such that $\partial_p z$ is invertible (actually we need to know the inverse), where no exceptional divisor is defined by p. This last condition is important because exceptional divisors are defined by IND elements to ensure normal crossing at each stage, thus a careless exchange operation could lead to information loss by replacing such distinguished regular parameters. Incidentally, the algorithm also produces charts in this preparatory process which are disjoint from the zero set of I; these charts will be reported as being outside the singular locus.

Let us describe the problem in full detail, using the notation as above, and introducing e the list of IND elements that generate the exceptional divisors in E. Examining $\mathcal{B}$, we distinct the following cases in U:

1. We have a $z \in I$ such that $z \in \mathrm{IND}$.
2. We do not have 1, but we have an order one element $z \in I$ such that $\partial_p z$ is invertible, with inverse a, for some $p \in \mathrm{IND} \setminus e$.
3. We do not have 1 or 2, but we have $\mathbf{Z}(I + \langle \partial_p f \mid f \in I,\ p \in \mathrm{IND} \setminus e\rangle) = \emptyset$.
4. We do not have 1 or 2 or 3, but we have an order one element $z \in I$ such that $\partial_p z$ is invertible for all $p \in e$.
5. We do not have 1 or 2 or 3 or 4, but we have $\mathbf{Z}(I + \langle \partial_p f \mid f \in I, p \in \mathrm{IND} \setminus e\rangle + \langle p \cdot \partial_p f \mid f \in I,\ p \in e\rangle) = \emptyset$.

If we consider only the last formulae in the above enumeration, cutting off the negation of the disjunctions from each, we do not get independent conditions. For instance, $\partial_p z$ is invertible for some $p \in \mathrm{IND} \setminus e$ implies $\mathbf{Z}(I + \langle \partial_p f \mid f \in I,\ p \in \mathrm{IND} \setminus e\rangle) = \emptyset$, which implies the last formula of 5. However, we separated these cases because they represent more difficulties to handle as their number increases.

The input output specifications of the preparation algorithm, *MaximalContact*, is as follows:

Input:

a chart $U = (\mathrm{VAR}, \mathrm{DEP}, \mathrm{IND}, \mathrm{PDER}, \mathrm{MAP}, \mathrm{IMAP})$;

a balanced object $\mathcal{B}$ over U;

Output:

a list $(U_1, \ldots, U_s)$ of subcharts of U, such that we have for each U_i either case 1 of the above distinction or $\mathcal{B}$ has empty singular locus over U_i.

In the algorithm we use the notation for the input chart U as in section 1.2 (especially $\mathrm{VAR} = (x_1, \ldots, x_m)$, $\mathrm{IND} = (p_1, \ldots, p_n)$), and for balanced objects as in subsection 1.4.2.

Algorithm $\mathit{MaximalContact}(U, \mathcal{B})$;
 let $\langle f_1, \ldots, f_r \rangle = I = \Delta(J, b)$;
 w.l.o.g. let $e = (p_1, \ldots, p_t)$;
 if for some i we have $p_i \in I$ then
  return (U);
 if for some $z \in I$ and $i > t$, $\mathrm{ord}(z) = 1$ and $a = (\partial_{p_i} z)^{-1} \in k[U]$ exists then
  return $(\mathit{Exchange}(U, z, i, a))$;
 if $\mathbf{Z}(I + \langle \partial_{p_j} f_i \mid i = 1, \ldots, r,\ j = t+1, \ldots, n) = \emptyset$ then
  $(U_1, \ldots, U_r, U_1^{t+1}, \ldots, U_r^n) := \mathit{Cover}(U, (f_1, \ldots, f_r, \partial_{p_{t+1}} f_1, \ldots, \partial_{p_n} f_r)$;
  return $(U_1, \ldots, U_r) + (\mathit{Exchange}(U_i^j, f_i, j, x_{m+1}) \mid i = 1, \ldots, r,$
   $j = t+1, \ldots, n)$;
 if for some $z \in I$, $\mathrm{ord}(z) = 1$ and $a_i = (\partial_{p_i} z)^{-1}$ exists in $k[U]$
   for $i = 1, \ldots, t$ then
  $(U', U_1, \ldots, U_t) := \mathit{Cover}(U, (z, p_1, \ldots, p_t))$;
  return $(U') + (\mathit{Exchange}(U_i, z, i, a_i) \mid i = 1, \ldots, t)$;
 $(U_1, \ldots, U_r, U_1^1, \ldots, U_r^n) :=$
  $\mathit{Cover}(U, (f_1, \ldots, f_r, p_1 \partial_{p_1} f_1, \ldots, p_t \partial_{p_t} f_r, \partial_{p_{t+1}} f_1, \ldots, \partial_{p_n} f_r)$;
 return $(U_1, \ldots, U_r) + (\mathit{Exchange}(U_i^j, f_i, j, x_{m+1}) \mid i = 1, \ldots, r,\ j = 1, \ldots, n)$;

Notice that in the branches of the algorithm that correspond to case 3 and case 5, over the output charts $(U_1, \ldots, U_r)$, the singular locus of $\mathcal{B}$ is empty. The same is true for U' in the branch of case 4.

The procedure in the first three cases is straightforward, but in case 4 and case 5 we have to show that the indicated *Cover* operations are valid. Therefore the correctness of the *MaximalContact* algorithm needs the following lemmas proved.

Lemma 1.24. *Using the notation introduced so far, in case 4 we have* $\mathbf{Z}(\langle z \rangle + \langle p \mid p \in e \rangle) = \emptyset$.

Proof. We assume that the given zero set is not empty. Locally at a point a of this zero set z is generated exactly by the parameters in e, while it is not equal to any of them, because of the condition formulated by case 4. Notice that the parameters of e are local regular parameters at a, because $a \in \mathbf{Z}(e)$. This results that z, the generator of a hypersurface of maximal contact, and the exceptional divisors cannot have normal crossings; which is a contradiction to $\mathcal{B}$ being balanced. $\square$

Lemma 1.25. *Using the notation introduced so far, in case 5 we have*

$$\mathbf{Z}(I + \langle \partial_p f \mid f \in I, p \in \mathrm{IND} \setminus e \rangle + \langle p \cdot \partial_p f \mid f \in I, p \in e \rangle) = \emptyset.$$

Proof. We know that $\mathbf{Z}(I+\langle \partial_p f \mid f \in I, p \in \mathrm{IND}\rangle) = \emptyset$, because b is the maximal order of J in $\mathcal{B}$; therefore we can reformulate the statement as

$$\mathbf{Z}(\langle p \mid p \in e\rangle) \cap \mathbf{Z}(\langle \partial_p f \mid f \in I, p \in \mathrm{IND} \setminus e\rangle + I) = \emptyset.$$

Then we argue indirectly, as in lemma 1.24. We assume that the above zero set is not empty. Let z generate a hypersurface of maximal contact, i.e. z is an order one element in I. Locally at a point of the intersection, z is generated only by exceptional divisors, because we are not in case 3. This leads to contradiction considering that it should have normal crossings with them, because $\mathcal{B}$ is balanced, while it coincides none of them (since we are not in case 1). □

It is not hard to see that this algorithm would work in principle. However, it pays off to spend some more effort in order to reduce subsequent computations in the charts produced by subsequent *Cover* and *Exchange* operations. In our implementation, we first compute a reduced Gröbner basis for G with respect to a total degree order and then apply the algorithm above. Since elements of the ideal with minimal degree appear in the Gröbner basis, this has the effect that the degree of the hypersurfaces of maximal contact is as small as possible.

1.5.2 Separating components

Look at the algorithm *ResolveBalanced* (subsection 1.4.2). The case where the coefficient ideal (let us call it I) is zero has to be treated in a special way: if we tried to call *EquiOrder* for a coefficient object with a zero ideal, then the maxorder would be infinity, and the algorithm would not terminate.

Now, recall that V is not necessarily irreducible, it may be a disjoint union of nonsingular varieties (the union is disjoint because intersections would be singular points). Suppose that I restricts to the zero ideal (in the sense of sheaf theory) with respect to one component, but to a nonzero ideal on another component. Then the maxorder of I would still be infinity, but we cannot perform the "if"-branch because the second component is not contained in the singular locus. Therefore, we have to cover V by open subsets that separate these two components.

Let V_1 be a component of V where I is zero (type 1). Then the restriction of the annihilator $\mathrm{Ann}(I)$ is $\langle 1\rangle$, because 1 annihilates I inside V_1. On the other hand, the ideal $\partial^\infty I$ generated by I and its partial derivatives of any order is zero.

Now, let V_2 be a component of V where I is nonzero (type 2). Then the restriction of the annihilator is zero, because $k[V_2]$ does not have zero divisors. On the other hand, one can show that $\partial^\infty I = \langle 1\rangle$, because the maxorder of I is finite.

By combining these facts, we see that

$$\mathrm{Ann}(I) + \partial^\infty I = \langle 1\rangle.$$

As an input to *Cover*, we take a finite generating set A for $\mathrm{Ann}(I)$ together with a finite generating set B for $\partial^\infty I$. For any $f \in A$, the open subset $f \neq 0$ contains only components of type 1, whereas for any $g \in B$, the open subset $g \neq 0$ contains only components of type 2.

In terms of polynomials, the annihilator is just the quotient ideal of I and $\mathbf{I}(V)$, and this can be computed by Gröbner bases (see [BW93]).

Algorithm *Separate*(I):
    $b := 1; D := I$;
    $A :=$ a generating set for $\mathrm{Ann}(I)$;
    $B := D\cup$ the partial derivatives of D of order 1;
    while $\langle A, B\rangle$ does not contain 1 do
        $D := B$;
        $B := D\cup$ the partial derivatives of D of order 1;
        $b := b + 1$;
    return A (for creating components of type 1);
    return B (for creating components of type 2);
    return b (order of coefficient object);
    return D (delta-ideal of coefficient object);

The algorithm *Separate* computes the maxorder and the delta-ideal of the coefficient object at no extra cost.

1.6 An extended example

We take the hypersurface defined by $x_1^6 + x_2^6 - x_1x_2x_3$ in $\mathbb{A}^3$, and follow its resolution by the algorithm presented in this chapter. The data of the entire resolution is far too much to be presented here and, of course, contains a lot of uninteresting repetitions of the same operations in different situations. Nevertheless, the interested reader can look at the main stages of the full resolution under the URL `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/gb/e1/tree.html`.

Here we present one branch of the resolution moving from one chart to another along the operations performed by the algorithm. To make the correspondence of the charts and the basic objects clear we use a double indexing. Upper indices count charts, while lower indices count the basic objects over the same chart. The path we chose along the resolution is shown in figure 1.4.

The initial setup consists of the chart U^1: $\mathrm{VAR}^1 = (x_1, x_2, x_3)$, $\mathrm{DEP}^1 = ()$, $\mathrm{IND}^1 = (x_1, x_2, x_3)$, $\mathrm{PDER}^1 = \mathrm{id}_{3\times 3}$, $\mathrm{MAP}^1 = (x_1 \mapsto x_1, x_2 \mapsto x_2, x_3 \mapsto x_3)$, $\mathrm{IMAP}^1 = (x_1 \mapsto x_1, x_2 \mapsto x_2, x_3 \mapsto x_3)$, and the weighted object $\mathcal{W}_1^1$: $W_1^1 = U^1$, $J_1^1 = \langle x_1^6 + x_2^6 - x_1x_2x_3\rangle$, $b_1^1 = 1$, $E_1^1 = ()$, $a_1^1 = 0_{E_1^1}$, where $0_{E_1^1}$ denotes the constant zero function on E_1^1. This object is not resolved and not monomial, so the algorithm sets up a tagged object through the *EquiOrder* algorithm, $\mathcal{T}_2^1$: $W_2^1 = U^1$, $J_2^1 = \langle x_1^6 + x_2^6 - x_1x_2x_3\rangle$, $b_2^1 = 3$, $E_2^1 = ()$, $E_{2-}^1 = ()$ and continues the resolution of this. In the *ResolveTagged* algorithm a new balanced object is created through the *Support* algorithm, $\mathcal{B}_3^1$: $W_3^1 = U^1$, $J_3^1 = \langle x_1^6 + x_2^6 - x_1x_2x_3\rangle$, $b_3^1 = 3$, $E_3^1 = ()$ and *ResolveBalanced* is called to resolve it. The hypersurface of maximal contact found by the *ResolveBalanced* algorithm is $V_3^1 = \mathbf{Z}(x_1)$, and because the ideal $C(J_3^1, b_3^1, V_3^1)$ is not zero, a new weighted object of dimension 2 is created, $\mathcal{W}_4^1$: $W_4^1 = V_3^1$, $J_4^1 = \langle x_2^{12}, x_2^3x_3^3\rangle$, $b_4^1 = 6$, $E_4^1 = ()$, $a_4^1 = 0_{E_4^1}$.

This weighted object is not resolved, because $\mathcal{B}_3^1$ is not resolved and it is also not monomial. The procedure for $\mathcal{W}_4^1$ happens to be fairly similar to the one for $\mathcal{W}_4^1$; a tagged object is created, $\mathcal{T}_5^1$: $W_5^1 = V_3^1$, $J_5^1 = \langle x_2^{12}, x_2^3x_3^3\rangle$, $b_5^1 = 6$, $E_5^1 = ()$, $E_{5-}^1 = ()$, then a balanced object is created $\mathcal{B}_6^1$: $W_6^1 = V_3^1$,

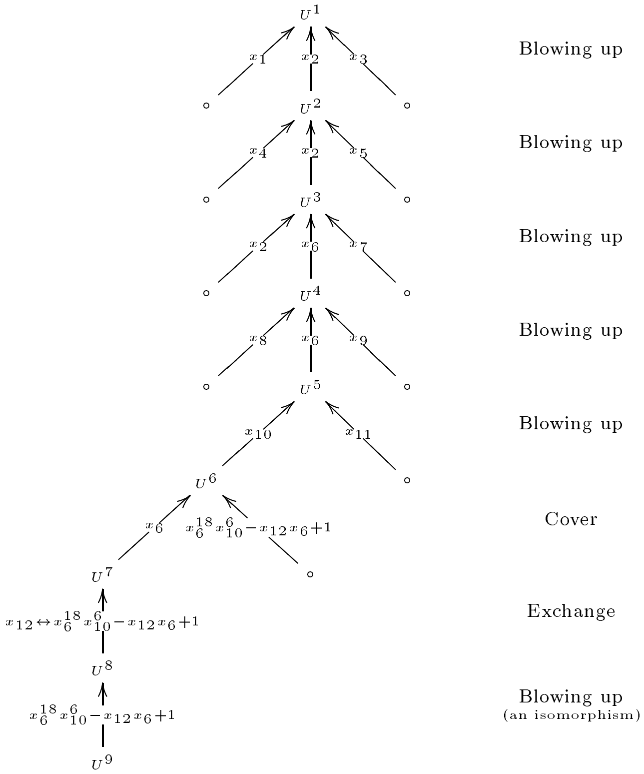


Figure 1.4: The presented path of the resolution

$J_6^1 = \langle x_2^{12}, x_2^3 x_3^3\rangle$, $b_6^1 = 6$, $E_6^1 = ()$. The hypersurface of maximal contact found by the algorithm is $V_6^1 = \mathbf{Z}(x_2)$, and the coefficient ideal $C(J_6^1, b_6^1, V_6^1)$ is not zero, so the coefficient object to be resolved is, $\mathcal{W}_7^1$: $W_7^1 = V_6^1$, $J_7^1 = \langle x_3^{720}\rangle$, $b_7^1 = 720$, $E_7^1 = ()$, $a_7^1 = 0_{E_7^1}$.

The weighted object $\mathcal{W}_7^1$ is not resolved and not monomial so a tagged object is created, $\mathcal{T}_8^1$: $W_8^1 = V_6^1$, $J_8^1 = \langle x_3^{720}\rangle$, $b_8^1 = 720$, $E_8^1 = ()$, $E_{8-}^1 = ()$, and a balanced object follows, $\mathcal{B}_9^1$: $W_9^1 = V_6^1$, $J_9^1 = \langle x_3^{720}\rangle$, $b_9^1 = 720$, $E_9^1 = ()$. The hypersurface of maximal contact is $V_9^1 = \mathbf{Z}(x_3)$ and $C(J_9^1, b_9^1, V_9^1)$ is now zero; therefore a blowing up operation is performed on U^1 and all the basic objects are pulled back, more precisely properly transformed along the blowing up.

The blowing up center is $\mathbf{Z}(x_1, x_2, x_3)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_2, U^2: $\mathrm{VAR}^2 = (x_2, x_4, x_5)$, $\mathrm{DEP}^2 = ()$, $\mathrm{IND}^2 = (x_4, x_2, x_5)$,

$$\mathrm{PDER}^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathrm{MAP}^2 = (x_1 \mapsto x_4 x_2, x_2 \mapsto x_2, x_3 \mapsto x_5 x_2)$, $\mathrm{IMAP}^2 = (x_2 \mapsto x_2, x_4 \mapsto x_1/x_2, x_5 \mapsto x_3/x_2)$. Notice that the generators x_1, x_3 were eliminated by the algebraic dependencies $x_4x_2 - x_1, x_5x_2 - x_3$ using the technique described in section 1.2.

The transform of $\mathcal{W}_1^1$ is $\mathcal{W}_1^2$: $W_1^2 = U^2$, $J_1^2 = \langle x_4^6x_2^3 + x_2^3 - x_4x_5\rangle$, $b_1^2 = 1$, $E_1^2 = (\mathbf{Z}(x_2))$, $a_1^2 = (\mathbf{Z}(x_2) \mapsto 2)$. The other objects got resolved therefore they are dropped and the recursion is returned to the resolution of $\mathcal{W}_1^2$, which is neither resolved nor monomial, so the algorithm creates new objects to improve the resolution of $\mathcal{W}_1^2$. We present the next operations in a more concise way:

- $\mathcal{T}_2^2$: $W_2^2 = U^2$, $J_2^2 = \langle x_4^6x_2^3 + x_2^3 - x_4x_5\rangle$, $b_2^2 = 2$, $E_2^2 = (\mathbf{Z}(x_2))$, $E_{2-}^2 = (\mathbf{Z}(x_2))$,
- $\mathcal{B}_3^2$: $W_3^2 = U^2$, $J_3^2 = \langle x_4^6x_2^3 + x_2^3 - x_4x_5, x_2^2\rangle$, $b_3^2 = 2$, $E_3^2 = ()$, hypersurface of maximal contact: $V_3^2 = \mathbf{Z}(x_2)$,
- $\mathcal{W}_4^2$: $W_4^2 = V_3^2$, $J_4^2 = \langle x_4x_5\rangle$, $b_4^2 = 2$, $E_4^2 = ()$, $a_4^2 = 0_{E_4^2}$,
- $\mathcal{T}_5^2$: $W_5^2 = V_3^2$, $J_5^2 = \langle x_4x_5\rangle$, $b_5^2 = 2$, $E_5^2 = ()$, $E_{5-}^2 = ()$,
- $\mathcal{B}_6^2$: $W_6^2 = V_3^2$, $J_6^2 = \langle x_4x_5\rangle$, $b_6^2 = 2$, $E_6^2 = ()$, hypersurface of maximal contact: $V_6^2 = \mathbf{Z}(x_4)$,
- $\mathcal{W}_7^2$: $W_7^2 = V_6^2$, $J_7^2 = \langle x_5^2\rangle$, $b_7^2 = 2$, $E_7^2 = ()$, $a_7^2 = 0_{E_7^2}$,
- $\mathcal{T}_8^2$: $W_8^2 = V_6^2$, $J_8^2 = \langle x_5^2\rangle$, $b_8^2 = 2$, $E_8^2 = ()$, $E_{8-}^2 = ()$,
- $\mathcal{B}_9^2$: $W_9^2 = V_6^2$, $J_9^2 = \langle x_5^2\rangle$, $b_9^2 = 2$, $E_9^2 = ()$, hypersurface of maximal contact: $V_9^2 = \mathbf{Z}(x_5)$, which leads to a blowing up.

The blowing up center is $\mathbf{Z}(x_2, x_4, x_5)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_2, U^3: $\mathrm{VAR}^3 = (x_2, x_6, x_7)$, $\mathrm{DEP}^3 = ()$, $\mathrm{IND}^3 = (x_6, x_2, x_7)$,

$$\mathrm{PDER}^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathrm{MAP}^3 = (x_1 \mapsto x_6x_2^2, x_2 \mapsto x_2, x_3 \mapsto x_7x_2^2)$, $\mathrm{IMAP}^3 = (x_2 \mapsto x_2, x_6 \mapsto x_1/x_2^2, x_7 \mapsto x_3/x_2^2)$.

The transform of $\mathcal{W}_1^2$ is $\mathcal{W}_1^3$: $W_1^3 = U^3$, $J_1^3 = \langle x_6^6x_2^7 - x_6x_7 + x_2\rangle$, $b_1^3 = 1$, $E_1^3 = (\mathbf{Z}(x_2))$, $a_1^3 = (\mathbf{Z}(x_2) \mapsto 3)$. The other objects got resolved therefore they are dropped and the recursion is returned to the resolution of $\mathcal{W}_1^3$, which is neither resolved nor monomial, so the algorithm continues creating the following new objects:

- $\mathcal{T}_2^3$: $W_2^3 = U^3$, $J_2^3 = \langle x_6^6x_2^7 - x_6x_7 + x_2\rangle$, $b_2^3 = 1$, $E_2^3 = (\mathbf{Z}(x_2))$, $E_{2-}^3 = (\mathbf{Z}(x_2))$,
- $\mathcal{B}_3^3$: $W_3^3 = U^3$, $J_3^3 = \langle x_6^6x_2^7 - x_6x_7 + x_2, x_2\rangle$, $b_3^3 = 1$, $E_3^3 = ()$, hypersurface of maximal contact: $V_3^3 = \mathbf{Z}(x_2)$,
- $\mathcal{W}_4^3$: $W_4^3 = V_3^3$, $J_4^3 = \langle x_6x_7\rangle$, $b_4^3 = 1$, $E_4^3 = ()$, $a_4^3 = 0_{E_4^3}$,
- $\mathcal{T}_5^3$: $W_5^3 = V_3^3$, $J_5^3 = \langle x_6x_7\rangle$, $b_5^3 = 2$, $E_5^3 = ()$, $E_{5-}^3 = ()$,
- $\mathcal{B}_6^3$: $W_6^3 = V_3^3$, $J_6^3 = \langle x_6x_7\rangle$, $b_6^3 = 2$, $E_6^3 = ()$, hypersurface of maximal contact: $V_6^3 = \mathbf{Z}(x_6)$,

- $\mathcal{W}_7^3$: $W_7^3 = V_6^3$, $J_7^3 = \langle x_7^2 \rangle$, $b_7^3 = 2$, $E_7^3 = ()$, $a_7^3 = 0_{E_7^3}$,
- $\mathcal{T}_8^3$: $W_8^3 = V_6^3$, $J_8^3 = \langle x_7^2 \rangle$, $b_8^3 = 2$, $E_8^3 = ()$, $E_{8-}^3 = ()$,
- $\mathcal{B}_9^3$: $W_9^3 = V_6^3$, $J_9^3 = \langle x_7^2 \rangle$, $b_9^3 = 2$, $E_9^3 = ()$, hypersurface of maximal contact: $V_9^3 = \mathbf{Z}(x_7)$, which leads to a blowing up.

The blowing up center is $\mathbf{Z}(x_2, x_6, x_7)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_6, U^4: $\mathrm{VAR}^4 = (x_6, x_8, x_9)$, $\mathrm{DEP}^4 = ()$, $\mathrm{IND}^4 = (x_6, x_8, x_9)$,

$$\mathrm{PDER}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathrm{MAP}^4 = (x_1 \mapsto x_6^3 x_8^2, x_2 \mapsto x_8 x_6, x_3 \mapsto x_9 x_6^3 x_8^2)$, $\mathrm{IMAP}^4 = (x_6 \mapsto x_1/x_2^2, x_8 \mapsto x_2^3/x_1, x_9 \mapsto x_3/x_1)$.

The non-resolved basic objects of U^3 are transformed in U^4 as follows:

- $\mathcal{W}_1^4$: $W_1^4 = U^4$, $J_1^4 = \langle x_8^7 x_6^{12} - x_9 x_6 + x_8 \rangle$, $b_1^4 = 1$, $E_1^4 = (\mathbf{Z}(x_8), \mathbf{Z}(x_6))$, $a_1^4 = (\mathbf{Z}(x_8) \mapsto 3, \mathbf{Z}(x_6) \mapsto 3)$,
- $\mathcal{T}_2^4$: $W_2^4 = U^4$, $J_2^4 = \langle x_8^7 x_6^{12} - x_9 x_6 + x_8 \rangle$, $b_2^4 = 1$, $E_2^4 = (\mathbf{Z}(x_8), \mathbf{Z}(x_6))$, $E_{2-}^4 = (\mathbf{Z}(x_8))$,
- $\mathcal{B}_3^4$: because balanced objects are resolved iff their coefficient objects are, we do not need to compute the transform for them, the preimage of V_3^3 is $V_3^4 = \mathbf{Z}(x_8)$,
- $\mathcal{W}_4^4$: $W_4^4 = V_3^4$, $J_4^4 = \langle x_9 \rangle$, $b_4^4 = 1$, $E_4^4 = (\mathbf{Z}(x_6))$, $a_4^4 = (\mathbf{Z}(x_6) \mapsto 1)$.

The weighted object $\mathcal{W}_4^4$ is not resolved and is not monomial, thus the following objects are created:

- $\mathcal{T}_5^4$: $W_5^4 = V_3^4$, $J_5^4 = \langle x_9 \rangle$, $b_5^4 = 1$, $E_5^4 = (\mathbf{Z}(x_6))$, $E_{5-}^4 = (\mathbf{Z}(x_6))$,
- $\mathcal{B}_6^4$: $W_6^4 = V_3^4$, $J_6^4 = \langle x_9, x_6 \rangle$, $b_6^4 = 1$, $E_6^4 = ()$, hypersurface of maximal contact: $V_6^4 = \mathbf{Z}(x_6)$,
- $\mathcal{W}_7^4$: $W_7^4 = V_6^4$, $J_7^4 = \langle x_9 \rangle$, $b_7^4 = 1$, $E_7^4 = ()$, $a_7^4 = 0_{E_7^4}$,
- $\mathcal{T}_8^4$: $W_8^4 = V_6^4$, $J_8^4 = \langle x_9 \rangle$, $b_8^4 = 1$, $E_8^4 = ()$, $E_{8-}^4 = ()$,
- $\mathcal{B}_9^4$: $W_9^4 = V_6^4$, $J_9^4 = \langle x_9 \rangle$, $b_9^4 = 1$, $E_9^4 = ()$, hypersurface of maximal contact: $V_9^4 = \mathbf{Z}(x_9)$, which leads to a blowing up.

The blowing up center is $\mathbf{Z}(x_8, x_6, x_9)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_6, U^5: $\mathrm{VAR}^5 = (x_6, x_{10}, x_{11})$, $\mathrm{DEP}^5 = ()$, $\mathrm{IND}^5 = (x_6, x_{10}, x_{11})$,

$$\mathrm{PDER}^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathrm{MAP}^5 = (x_1 \mapsto x_6^5 x_{10}^2, x_2 \mapsto x_{10} x_6^2, x_3 \mapsto x_{11} x_6^6 x_{10}^2)$, $\mathrm{IMAP}^5 = (x_6 \mapsto x_1/x_2^2, x_{10} \mapsto x_2^5/x_1^2, x_{11} \mapsto x_2^2 x_3/x_1^2)$.

The non-resolved basic objects of U^4 are transformed in U^5 as follows:

- $\mathcal{W}_1^5$: $W_1^5 = U^5$, $J_1^5 = \langle x_{10}^7 x_6^{18} - x_1 1 x_6 + x_{10}\rangle$, $b_1^5 = 1$, $E_1^5 = (\mathbf{Z}(x_{10}), \mathbf{Z}(x_6))$, $a_1^5 = (\mathbf{Z}(x_{10}) \mapsto 3, \mathbf{Z}(x_6) \mapsto 6)$,
- $\mathcal{T}_2^5$: $W_2^5 = U^5$, $J_2^5 = \langle x_{10}^7 x_6^{18} - x_1 1 x_6 + x_{10}\rangle$, $b_2^5 = 1$, $E_2^5 = (\mathbf{Z}(x_{10}), \mathbf{Z}(x_6))$, $E_{2-}^5 = (\mathbf{Z}(x_{10}))$,
- $\mathcal{B}_3^5$: because balanced objects are resolved iff their coefficient objects are, we do not need to compute the transform for them, the preimage of V_3^4 is $V_3^5 = \mathbf{Z}(x_{10})$,
- $\mathcal{W}_4^5$: $W_4^5 = V_3^5$, $J_4^5 = \langle x_{11}\rangle$, $b_4^5 = 1$, $E_4^5 = (\mathbf{Z}(x_6))$, $a_4^5 = (\mathbf{Z}(x_6) \mapsto 1)$,
- $\mathcal{T}_5^5$: $W_5^5 = V_3^5$, $J_5^5 = \langle x_{11}\rangle$, $b_5^5 = 1$, $E_5^5 = (\mathbf{Z}(x_6))$, $E_{5-}^5 = ()$.

The tagged object $\mathcal{T}_5^5$ is not resolved; therefore the computations are continued with the resolution of its support:

- $\mathcal{B}_6^5$: $W_6^5 = V_3^5$, $J_6^5 = \langle x_{11}\rangle$, $b_6^5 = 1$, $E_6^5 = ()$, hypersurface of maximal contact: $V_6^5 = \mathbf{Z}(x_{11})$,

where we have $C(J_6^5, b_6^5, V_6^5)$ zero, so a blowing up is performed.

The blowing up center is $\mathbf{Z}(x_{10}, x_{11})$, and we follow the course of the resolution along the chart with exceptional divisor given by x_{10}, U^6: $\mathrm{VAR}^6 = (x_6, x_{10}, x_{12})$, $\mathrm{DEP}^6 = ()$, $\mathrm{IND}^6 = (x_6, x_{10}, x_{12})$,

$$\mathrm{PDER}^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathrm{MAP}^6 = (x_1 \mapsto x_6^5 x_{10}^2, x_2 \mapsto x_{10} x_6^2, x_3 \mapsto x_{12} x_{10}^3 x_6^6)$, $\mathrm{IMAP}^6 = (x_6 \mapsto x_1/x_2^2, x_{10} \mapsto x_2^5/x_1^2, x_{12} \mapsto x_3/x_2^3)$.

The non resolved basic objects of U^5 are transformed in U^6 as follows:

- $\mathcal{W}_1^6$: $W_1^6 = U^6$, $J_1^6 = \langle x_6^{18} x_{10}^6 - x_{12} x_6 + 1\rangle$, $b_1^6 = 1$, $E_1^6 = (\mathbf{Z}(x_{10}), \mathbf{Z}(x_6))$, $a_1^6 = (\mathbf{Z}(x_{10}) \mapsto 3, \mathbf{Z}(x_6) \mapsto 6)$,
- $\mathcal{T}_2^6$: $W_2^6 = U^6$, $J_2^6 = \langle x_6^{18} x_{10}^6 - x_{12} x_6 + 1\rangle$, $b_2^6 = 1$, $E_2^6 = (\mathbf{Z}(x_{10}), \mathbf{Z}(x_6))$, $E_{2-}^6 = ()$.

The *ResolveTagged* algorithm computes the support of $\mathcal{T}_2^6$:

- $\mathcal{B}_3^6$: $W_3^6 = U^6$, $J_3^6 = \langle x_6^{18} x_{10}^6 - x_{12} x_6 + 1\rangle$, $b_3^6 = 1$, $E_3^6 = ()$.

This time the hypersurface of maximal contact, $V_3^6 = \mathbf{Z}(x_6^{18} x_{10}^6 - x_{12} x_6 + 1)$, is not a member of $\mathbf{IND}^6$, thus we have to perform an exchange operation which has to be prepared by a cover, since no partial derivatives of $x_6^{18} x_{10}^6 - x_{12} x_6 + 1$ is invertible over U^6. We are in case 3 of the case distinction of subsection 1.5.1, and, using the notation introduced there, we have $I\langle x_6^{18} x_{10}^6 - x_{12} x_6 + 1\rangle$, $\langle \partial_p f \mid f \in I,\ p \in \mathrm{IND} \setminus e\rangle = \langle x_6\rangle$.

The cover operation generates two subcharts of U^6: in the one where $x_6^{18} x_{10}^6 - x_{12} x_6 + 1$ is invertible no singularity is present thus the resolution is over. We follow the resolution process over the chart where x_6 is invertible, U^7: $\mathrm{VAR}^7 = (x_6, x_{10}, x_{12}, x_{13})$, $\mathrm{DEP}^7 = (x_{13} x_6 - 1)$, $\mathrm{IND}^7 = (x_6, x_{10}, x_{12})$,

$$\mathrm{PDER}^7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_{13}^2 & 0 & 0 \end{pmatrix},$$

$\mathrm{MAP}^7 = (x_1 \mapsto x_6^5 x_{10}^2, x_2 \mapsto x_{10}x_6^2, x_3 \mapsto x_{12}x_{10}^3x_6^6)$, $\mathrm{IMAP}^7 = (x_6 \mapsto x_1/x_2^2, x_{10} \mapsto x_2^5/x_1^2, x_{12} \mapsto x_3/x_2^3, x_{13} \mapsto -x_2^2/x_1)$.

The basic objects has not changed: $\mathcal{W}_1^7 = \mathcal{W}_1^6$, $\mathcal{T}_2^7 = \mathcal{T}_2^6$, $\mathcal{B}_3^7 = \mathcal{B}_3^6$. Now the partial derivative above is invertible over U^7 with inverse x_{13}, so we can perform the exchange operation.

The output chart of the *Exchange* algorithm is not really new because only the extended data, IND and PDER, is changed, but for the clearer presentation we assign a new index to it: U_8 with data $\mathrm{VAR}^8 = (x_6, x_{10}, x_{12}, x_{13})$, $\mathrm{DEP}^8 = (x_{13}x_6 - 1)$, $\mathrm{IND}^8 = (x_6, x_{10}, x_6^{18}x_{10}^6 - x_{12}x_6 + 1)$,

$$\mathrm{PDER}^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_{13}(18x_6^{17}x_{10}^6 - x_{12}) & -6x_{10}^5x_{13}x_6^{18} & x_{13} \\ x_{13}^2 & 0 & 0 \end{pmatrix},$$

$\mathrm{MAP}^8 = (x_1 \mapsto x_6^5 x_{10}^2, x_2 \mapsto x_{10}x_6^2, x_3 \mapsto x_{12}x_{10}^3x_6^6)$, $\mathrm{IMAP}^8 = (x_6 \mapsto x_1/x_2^2, x_{10} \mapsto x_2^5/x_1^2, x_{12} \mapsto x_3/x_2^3, x_{13} \mapsto -x_2^2/x_1)$.

Of course, the basic objects has also not changed: $\mathcal{W}_1^8 = \mathcal{W}_1^7$, $\mathcal{T}_2^8 = \mathcal{T}_2^7$, $\mathcal{B}_3^8 = \mathcal{B}_3^7$.

The coefficient ideal $C(J_3^8, b_4^8, V_3^8)$, is zero, where V_3^8 is just the preimage of V_3^6 along the open immersion of $U^8 (= U^7)$ into U^6. Hence a blowing up operation is to be performed, which is in this case an isomorphism, because the center V_3^8 is a nonsingular hypersurface in U^8. We get $U^9 = U^8$, and the weighted basic object $\mathcal{W}_1^9$ is

- $\mathcal{W}_1^9$: $W_1^6 = U^9$, $J_1^9 = \langle 1 \rangle$, $b_1^6 = 1$, $E_1^6 = (\mathbf{Z}(x_{10}), \mathbf{Z}(x_6^{18}x_{10}^6 - x_{12}x_6 + 1))$, $a_1^6 = (\mathbf{Z}(x_{10}) \mapsto 2, \mathbf{Z}(x_6^{18}x_{10}^6 - x_{12}x_6 + 1) \mapsto 0)$.

This object is monomial and it takes three more blowing ups to complete its resolution, but since these blowing ups are also isomorphisms (their only effect is to reduce the values of a_1^9 one by one), we omit to present them.

Eventually we arrive to a chart, isomorphic to U^9, which is the member of the atlas of the nonsingular variety of the embedded resolution of $(\mathbb{A}^3, \mathbf{Z}(x_1^6 + x_2^6 - x_1x_2x_3))$.

Chapter 2

Improvements

The original algorithm, presented in chapter 1, is not at all optimal from the viewpoint of computational complexity. Certain operations, like generating the coefficient ideal (see subsection 1.4.2) or creating the balanced sum of the monomial and the proper part in the *EquiOrder* algorithm (see subsection 1.4.1) introduce generators with extremely high powers to compute with. Experiments showed that the algorithm has to apply *Cover* operation quite often to achieve a state in which it can further the resolution (see for instance subsection 1.5.1).

In this chapter we overcome the these difficulties by introducing a generalized version of polynomial ideals, with which the cumbersome powering can be avoided, and by constructing new, optimized data structures to represent basic objects, in which we carefully separate algebraic and combinatorial data on the singularities.

In section 2.1 a simple and straightforward method is given to find local maxima of the exceptional divisor selector function in algorithm *Support*. In section 2.2 a speedup idea is presented to make the dimension reduction in coefficient ideal computation faster. Section 2.3 contains a modified version of the original algorithm, using the ideas from sections 2.1, 2.2, and proves termination for the modified resolution algorithm.

In section 2.4 we present the first stages of the theory of generalized ideals, which was developed to efficiently describe classes of resolution problems that can be solved by the same sequence of blowing ups. Section 2.5 contains descriptions of basic objects using generalized ideals, while taking care of optimal representation of components of the algebraic set that have special structure. Then in section 2.6 we develop computational techniques for this new representation, and finally, in section 2.7 we show that the optimized algorithm, using the new representation of basic objects, computes equivalent resolution to the one defined in section 2.3.

In section 2.8 a simple but powerful method is presented that allows the resolution algorithm to detect redundancy in the representation of the blownup manifolds, or to detect whenever a chart contains local copies of singularities whose isomorphic counterparts have been treated in another chart. The method allows the algorithm to cut down on redundant computations in a significant measure.

Finally, in section 2.9, we present an elaborated example following the original and the optimized algorithms in parallel.

2.1 Computing common singularities

In algorithm *Support* on page 32 we have to compute the balance of a tagged object $\mathcal{T} = (W, J, b, E, E_-)$. This involves the computation of the largest natural number r such that r exceptional divisors from E_- have common singularities (for the notion look up subsection 1.4.1). Actually, we have to determine the first r exceptional divisors $H_1, \ldots, H_r$ from E_- with respect to lexicographic ordering.

Here we present another approach that replaces the exhaustive search by a simpler subdivision of the underlying chart. Eventually we get in each new subchart either $\mathcal{T}$ resolved or $r = |E_-|$.

Theorem 2.1. *Given a tagged object (W, J, b, E, E_-) in a chart U, then there always exists an open covering of U, where in each open subset either the object is resolved or all the exceptional divisors from E_- have common singularities.*

Proof. We construct the open covering: If the singular locus and E_- has no common singularities, we perform a *cover* operation (see section 1.3) with E_- and a finite set of generators of $\mathbf{I}(\mathrm{Sing}(J, b))$.

In the new charts either one of the exceptional divisors from E_- is absent, or the singular locus is empty. This way we reduced the situation to a simpler one in each of the charts, and since there are always finitely many members of E_-, repeating the previous step finitely many times, we reach a state that fulfills the claim of the lemma. □

The result of the lemma is used in the *Support2* algorithm. The instruction "fork" means that the computation continues for each new chart independently and that the current *Support2* routine and its caller (a *ResolveTagged* procedure) is linked out from the call tree of each new thread. Actually, they get replaced by the branch started at the fork command.

Algorithm *Support2*(W, J, b, E, E_-);
    drop all elements from E and E_- which are outside the current chart;
    let $H_1, \ldots, H_s$ be the remaining exceptional divisors in E_-;
    if $|E_-| > 0$ and E_- has no common singularities then
        let $f_1, \ldots, f_r$ be generators of $\mathbf{I}(\mathrm{Sing}(J, b))$;
        let $g_1, \ldots, g_s$ generate $\mathbf{I}(H_1), \ldots, \mathbf{I}(H_s)$ respectively;
        New_charts := *Cover*(current_chart, $(f_1, \ldots, f_r, g_1, \ldots g_s)$);
        fork *ResolveTagged*(W, J, b, E, E_-) for all the charts in *New_charts*;
    else
        $I := J + \mathbf{I}(H_1)^b + \ldots + \mathbf{I}(H_s)^b$;
        $E_+ := E - E_-$;
        return (W, I, b, E_+);

2.2 Efficient coefficient object computation

Careful examination of the resolution algorithm unveils that in the case when a tagged object $\mathcal{T} = (W, J, b, E, E_-)$ has $E = E_-$ having common singularities, there is a strong relationship between derived basic objects if we use the natural choices of E-hypersurfaces for hypersurfaces of maximal contact in the upcoming balanced objects.

In the following we define the coefficient ideal with respect to finitely many hypersurfaces of maximal contact, and we present a technique which uses this new definition to “skip” redundant weighted, tagged and balanced objects during the resolution.

Definition 2.2. Let $V_1, \ldots, V_r$ be hypersurfaces of maximal contact for $\mathcal{B} = (W, J, b, E)$. The coefficient ideal of (J, b) with respect to all of them is

$$\mathrm{Coeff}(J, b, (V_1, \ldots, V_r)) = C(C(\ldots C(J, b, V_1)\ldots), b(!)^{r-1}, V_r).$$

The corresponding coefficient object is

$$(W \cap V_1 \cap \cdots \cap V_r, \mathrm{Coeff}(J, b, (V_1, \ldots, V_r)), b(!)^r, E - (V_1, \ldots, V_r), 0_E),$$

where $b(!)^r$ denotes the computation of r-accumulated factorials $((b!)!...)!$, and 0_E is the zero function on E.

Consider the following situation in the resolution process: Given a tagged object $\mathcal{T}_0 = (W_0, J_{0(t)}, b_{0(t)}, E_{0(t)}, E_{0-})$ with $E_- = E = \{H_1, \ldots H_r\}$. Without the loss of generality, because of theorem 2.1, we can assume that all the members of E_- have common singularities. The support of $\mathcal{T}_0$ is the balanced object $\mathcal{B}_0 = (W_0, J_{0(b)} = J_{0(t)} + \mathbf{I}(H_1)^{b_{0(t)}} + \cdots + \mathbf{I}(H_r)^{b_{0(t)}}, b_{0(b)} = b_{0(t)}, E_{0(b)} = \emptyset)$. By some bookkeeping, i.e. storing the H_i hypersurfaces, we can achieve that in this situation the algorithm (by a systematic choice of hypersurfaces of maximal contact) computes the following sequence of objects:

$$\begin{aligned}
\mathcal{W}_1 &= (W_1 = W_0 \cap H_1, J_{1(w)} = \mathrm{Coeff}(J_{0(b)}, b_{0(b)}, H_1),\\
&\qquad\qquad\qquad b_{1(w)} = b_0!, E_{1(w)} = \emptyset, a_1 = \emptyset),\\
\mathcal{T}_1 &= (W_1, J_{1(t)} = J_{1(w)}, b_{1(t)} = b_{1(w)}, E_{1(t)} = \emptyset, E_{1-} = \emptyset),\\
\mathcal{B}_1 &= (W_1, J_{1(b)} = J_{1(t)}, b_{1(b)} = b_{1(t)}, E_{1(b)} = \emptyset),\\
\mathcal{W}_2 &= (W_2 = W_1 \cap H_2, J_{2(w)} = \mathrm{Coeff}(J_{1(b)}, b_{1(b)}, H_2),\\
&\qquad\qquad\qquad b_{2(w)} = b_{1(b)}!, E_{2(w)} = \emptyset, a_2 = \emptyset),\\
&\vdots\\
\mathcal{B}_{r-1} &= (W_{r-1}, J_{r-1(b)} = J_{r-1(t)}, b_{r-1(b)} = b_{r-1(t)}, E_{r-1(b)} = \emptyset),\\
\mathcal{W}_r &= (W_r = W_{r-1} \cap H_r, J_{r(w)} = \mathrm{Coeff}(J_{r-1(b)}, b_{r-1(b)}, H_r),\\
&\qquad\qquad\qquad b_{r(w)} = b_{r-1(b)}!, E_{r(w)} = \emptyset, a_r = \emptyset).
\end{aligned}$$

This is possible because all the objects with index less than r are simple, and the list of exceptional divisors are empty; moreover for $i = 0, \ldots, r-1$, H_{i+1} is a hypersurface of maximal contact in $\mathcal{B}_i$.

Lemma 2.3. *In the above situation we have*

$$\begin{aligned}
\mathcal{W}_r = (W_r = W_0 \cap H_1 \cap \cdots \cap H_r, J_{r(w)} &= \mathrm{Coeff}(J_{0(b)}, b, (H_1, \ldots, H_r)),\\
b_{r(w)} &= b(!)^r, E_{r(w)} = \emptyset, a_r = \emptyset).
\end{aligned}$$

Proof. By the definition 2.2, the statement is obvious. □

In the next lemma we mark the transforms of objects along a finite sequence of blowing ups in a chart U of the affine covering of the blownup with an asterisk.

Lemma 2.4. *Consider the situation as in lemma 2.3. The transform of the first coefficient object $\mathcal{W}_{1*}$ is resolved iff $\mathcal{W}_{r*}$ is resolved.*

Proof. For $r = 1$ the statement is trivial. Assume now that the lemma is true for $r - 1$ $(< \dim(U))$.

We have that $\mathbf{I}(H_r^*)^{b_{r-1}}$ is a subideal of $J^*_{r-1(w)}$, and since $\mathcal{W}_{r-1}$ is simple $\mathcal{W}^*_{r-1}$ is also simple. It follows that $J^*_{r-1(t)} = J^*_{r-1(w)}$ and $b_{r-1(t)} = b_{r-1(w)}$ in $\mathcal{T}^*_{r-1}$, whence $\mathcal{W}^*_{r-1}$ is resolved iff $\mathcal{T}^*_{r-1}$ is resolved.

This also means that whenever $\mathcal{W}^*_{r-1}$ is not resolved, its maxorder hasn't drop, so E_- hasn't changed in $\mathcal{T}^*_{r-1}$; therefore $J^*_{r-1(b)} = J^*_{r-1(t)}$ in $\mathcal{B}^*_{r-1}$. Thus $\mathcal{B}^*_{r-1}$ is resolved iff $\mathcal{T}^*_{r-1}$ is resolved.

The singular loci of $\mathcal{B}^*_{r-1}$ is the same as of $\mathcal{W}^*_r$, because coefficient ideal computation and transform along a chain of blowing ups are commutative operations. Finally, because for all $x \in \mathrm{Sing}(J^*_{r-1(w)}, b_{r-1(w)})$ we must have $x \in H^*_r$, we have $x \in \mathrm{Sing}(J^*_{r-1(w)}, b_{r-1(w)})$ iff $x \in \mathrm{Sing}(J^*_{r(w)}, b_{r(w)})$. $\square$

Remark 2.5. According to lemma 2.4, the claim of lemma 2.3 also holds after finitely many blowing up operations, provided that the sequence of objects is not resolved. In this case E_- does not change in the tagged objects, but E gets extended with new exceptional divisors.

Now we have, by lemma 2.4 and lemma 2.3, that whenever we create the support of a tagged object with all members of E_- having common singularities, we can compute the next coefficient object with respect to all the exceptional divisors of E_-. This way we save all the computations needed in the intermediate objects. The results are used in the *ResolveTagged2* and *ResolveBalanced2* algorithms.

Algorithm *ResolveTagged2*($\mathcal{T}$);
    while not resolved do
        $\mathcal{B}$:= *Support*(current object);
            ResolveBalanced2($\mathcal{B}, E_-$);

Algorithm *ResolveBalanced2*($\mathcal{B} = (W, J, b, E), H$);
    if $H = \emptyset$ or $E \neq \emptyset$ then
        V := a one-element list of a hypersurface of maximal contact;
    else
        $V := H$;
    $C := \mathrm{Coeff}(J, b, V)$;
    if C is the zero ideal then
        BlowUp($\bigcap_{H_i \in V} H_i$);
    else
        $\mathcal{W}$:= the coefficient of object of $\mathcal{B}$ with respect to V;
        ResolveWeighted($\mathcal{W}$);

2.3 A modified version of the resolution algorithm

We consider a revised version of the algorithm presented in section 1.4 to be our reference algorithm; let us call it *Villamayor_resolution2*. This modified version uses the ideas and algorithms (*Support2*, *ResolveTagged2*, *ResolveBalanced2*) presented in sections 2.1, 2.2, instead of the old counterparts.

Theorem 2.6. *The* Villamayor_resolution2 *algorithm computes an embedded desingularization of the input hypersurface within finitely many steps.*

Proof. We give a relative argument relying on the termination of the original algorithm. By lemma 2.3 and lemma 2.4 the replacement of *ResolveTagged2* and *ResolveBalanced2* does not alter the termination of the original algorithm.

By theorem 2.1, in the resolution of a tagged object with $r = |E_-|$, the *Support2* algorithm may be called at most r times before the execution goes to the 'else' branch, where termination is ensured by the original proofs given in section 1.4, regarding that the termination does not depend on the representation of the algebraic sets. □

Let us also remark that the new ways of computation does not interfere with the patching of local resolutions (in the case of *ResolveTagged2*, *ResolveBalanced2* it is obvious, while in the case of *Support2* it comes from the fact that all we change is the representation of the atlas of charts).

2.4 Generalized ideals

The purpose of the introduction of generalized ideals is to simplify the representation of vanishing ideals of algebraic sets in the resolution process. This section presents the first stages of the theory of generalized ideals.

In this section $F = \{(f_1, b_1), \ldots, (f_k, b_k)\}$, $G = \{(g_1, c_1), \ldots, (g_m, c_m)\}$ are finite sets of ordered pairs from $k[U] \times \mathbb{Q}_+$, where U is a affine chart given as in section 1.2.

Definition 2.7. (cf. Definition 1 on page 54 in [Hir77]) A *generalized ideal* is an equivalence class of finite sets of pairs from $k[U] \times \mathbb{Q}_+$ under the equivalence relation $\sim$ defined as: $F \sim G$ if and only if for any ring homomorphism $\varphi : k[U] \to A$ we have $v_A(F) = v_A(G)$ with $v_A(F) := \min_i(v(\varphi(f_i))/b_i)$, where A is a valuation ring and v is its valuation.

Remark 2.8. The relation $\sim$ of definition 2.7 is indeed an equivalence, because it is based on equality.

We denote the generalized ideal of the representative F by $\langle F \rangle$ or simply by the representative F itself.

Definition 2.9. The *order* of F at $a \in U$, is $\mathrm{ord}_a(F) := \min_i(\mathrm{ord}_a(f_i)/b_i)$, where $\mathrm{ord}_a(f_i)$ is the maximal d such that $f_i \in \mathfrak{m}^d$, $\mathfrak{m}$ being the maximal ideal of $\mathcal{O}_{U,a}$; in case $f_i = 0$, $\mathrm{ord}_a(f_i) = \infty$.

Lemma 2.10. *The* $\mathrm{ord}_a : k[U] \times \mathbb{Q}_+ \to \mathbb{Q}_{\geq 0} \cup \infty$ *function is compatible with* $\sim$.

Proof. The $\mathrm{ord}_a : k[U] \to \mathbb{Z}_{\geq 0} \cup \infty$ function gives rise to a valuation of the field of fractions of $\mathcal{O}_{U,a}$, hence the ord_a function takes the same value on equivalent representatives. □

Definition 2.11. The *zero set* of F is defined as $\mathbf{Z} = \{a \in U \mid \mathrm{ord}_a(F) > 0\}$. A point $a \in U$ is a *singularity* of F if $\mathrm{ord}_a(F) \geq 1$. The *singular locus* of F is defined as $\mathrm{Sing}(F) = \{a \in U \mid \mathrm{ord}_a(F) \geq 1\}$. The *update* of F with $c \in \mathbb{Q}_+$ is $F_c := \{(f_i, b_i c)\}_i$.

Remark 2.12. (i) The $\mathbf{Z}$, Sing and update functions of definition 2.11 are compatible with $\sim$, because of lemma 2.10. (ii) We can define the *maximal order* of F over $X \subseteq U$, denoted by $\mathrm{ord}_X(F)$, as the maximal c such that $\mathrm{Sing}(F_c) \cap X \neq \emptyset$ ($\mathrm{ord}_X(F) = \infty$ if all $f_i = 0$), or 0 is no such c exists. (iii) The Sing, ord_a, ord_X functions are effectively computable using partial differentiation with respect to a global system of regular parameters over U, and the method of Gröbner bases.

Lemma 2.13. *We have* $\{(f, b)\} \sim \{(f^l, bl)\}$ *for* $l \in \mathbb{N}_+$.

Proof. Apply the property $v(xy) = v(x) + v(y)$ of valuations to the situation of definition 2.7, getting $v(\varphi(f))/b = v(\varphi(f))l/bl$. □

Definition 2.14. The *sum* of F and G, denoted as $F + G$, is defined as the set theoretic union of F and G.

Lemma 2.15. *If* $F \sim G$ *and* $F' \sim G'$, *we also have* $F + F' \sim G + G'$.

Proof. We have for any ring homomorphism $\varphi : k[U] \to A$ as in definition 2.7, $v_A(F) = v_A(G)$ and $v_A(F') = v_A(G')$, but then we also have $v_A(F + F') = \min(v_A(F), v_A(F')) = \min(v_A(G), v_A(G')) = v_A(G + G')$. □

Remark 2.16. (i) For any $a \in U$, $\mathrm{ord}_a(F + G) = \min(\mathrm{ord}_a(F), \mathrm{ord}_a(G))$. (ii) $\mathrm{Sing}(F + G) = \mathrm{Sing}(F) \cap \mathrm{Sing}(G)$.

Definition 2.17. The blowing up $\pi : W \to U$ with center Z is *permissible* for the generalized ideal F if Z is regular and $Z \subseteq \mathrm{Sing}(F)$.

In the classical approach to the resolution of singularities it is crucial to have coefficient ideal computation and transformation of ideals along blowing ups commute. To achieve this we need to compute weak transform (see Section 2 on page 209 in [Hir64]) for the d-dimensional objects and controlled transform (see Definition 4.7 in [EV00] or the Concepts section of [EH00]) for the $d-1$-dimensional objects. Due to the structure of the algorithm we always compute controlled transform (we call it shortly "transform") which, of course, coincides with the weak transform whenever it is necessary.

Definition 2.18. Let $\pi : W \to U$ be a permissible blowing up for F with center Z, let $\langle h \rangle := \mathbf{I}(\pi^{-1}(Z))$ and let us assume for F, $b_i = r_i/q_i$, $r_i, q_i \in \mathbb{Z}_+$. The (controlled) *transform* of F along π is $\mathrm{Transform}(F) := \{(h^{-r_i} f_i^{q_i} k[W], r_i)\}_i$.

Remark 2.19. (i) The Transform operation first cancels all the denominators of the rational components of the generators, moving to another representative of the generalized ideal using lemma 2.13 and lemma 2.15. (ii) To avoid ambiguity, especially when we consider the transform over a chart with exceptional divisor h, we may denote $\mathrm{Transform}(F)$ as $\mathrm{Transform}(F, h)$.

Theorem 2.20. *For $F \sim G$ and a permissible blowing up as in definition 2.18, we have* $\mathrm{Transform}(F) \sim \mathrm{Transform}(G)$.

Proof. Let $b_i = r_i/q_i$, $c_j = p_j/o_j$, $r_i, q_i, p_j, o_j \in \mathbb{Z}_+$. Let $\psi : k[U] \to k[W]$ be the ring inclusion associated with π, but then for any ring homomorphism $\varphi : k[W] \to A$ as in definition 2.7 we have

$$\min_i(v(\varphi(\psi(f_i^{q_i})))/r_i) = \min_i(v(\varphi(\psi(f_i)))/b_i) = \\ \min_j(v(\varphi(\psi(g_j)))/c_j) = \min_j(v(\varphi(\psi(g_j^{o_j})))/p_j),$$

because $\varphi \circ \psi : k[U] \to A$ is also a homomorphism as in definition 2.7. Since Z was permissible h^{r_i} is a factor in $\psi(f_i^{q_i})$, so let $\bar{f}_i$ such that $\psi(f_i^{q_i}) = \bar{f}_i h^{r_i}$. Then we have $v(\varphi(\psi(f_i^{q_i}))) = v(\varphi(\bar{f}_i)) + v(\varphi(h^{r_i}))$, thus

$$v_A(\mathrm{Transform}(F)) = \min_i(v(\varphi(\bar{f}_i))/r_i) = \min_i(v(\varphi(\psi(f_i)))/b_i) - v(\varphi(h));$$

therefore $v_A(\mathrm{Transform}(F)) = v_A(\mathrm{Transform}(G))$. ☐

Definition 2.21. Let $J := \langle f_1, \ldots, f_k \rangle \subseteq k[U]$, $b \in \mathbb{Z}_+$. The *translation* of the pair (J, b) is $\mathrm{Tr}(J, b) := \{(f_i, b)\}_i$.

Lemma 2.22. *The* Tr *function is well defined with respect to $\sim$ for any $b \in \mathbb{Z}_+$.*

Proof. Let $\langle f_1, \ldots, f_k \rangle = \langle f_1', \ldots, f_l' \rangle$, and consider a ring homomorphism $\varphi : k[U] \to A$ as in definition 2.7. We have $f_i = \sum_{j=1}^{l} h_j' f_j'$ and $f_j' = \sum_{i=1}^{k} h_i f_i$, ($h_j', h_i \in k[U]$), so $v(\varphi(f_i)) \geq \min_j(v(\varphi(h_j')) + v(\varphi(f_j')))$. Because A is a valuation ring $v(f) \geq 0$ for all $f \in A$, thus $v(\varphi(f_i)) \geq \min_j(v(\varphi(f_j')))$ and by the same token $v(\varphi(f_j')) \geq \min_i(v(\varphi(f_i)))$; therefore $v_A(\mathrm{Tr}(\langle f_1, \ldots, f_k \rangle, b)) = v_A(\mathrm{Tr}(\langle f_1', \ldots, f_l' \rangle, b))$. ☐

Remark 2.23. In context of definition 2.21, we have: (i) The pair (J, b) is a representative of an idealistic exponent on U in the sense of Definition 3 on page 56 in [Hir77]. (ii) $\mathrm{ord}_a(J) = \mathrm{ord}_a(\mathrm{Tr}(J, b)) \cdot b$ at any $a \in U$, for any $b \in \mathbb{Z}_+$. (iii) We have $\mathrm{Sing}(J, b) = \mathrm{Sing}(\mathrm{Tr}(J, b))$. (iv) A blowing up $\pi : W \to U$ with center Z is permissible for (J, b) if Z is regular and $Z \subseteq \mathrm{Sing}(J, b)$. (v) In the above blowing up, with $H = \langle h \rangle$ being the ideal for $\pi^{-1}(Z)$, we define the (controlled) transform: $\mathrm{Transform}(J, b) = (H^{-b} J k[W], b)$.

Theorem 2.24. *For (J, b) as in definition 2.21, a blowing up is permissible for* $\mathrm{Tr}(J, b)$ *iff it is permissible for (J, b), and we have* $\mathrm{Transform}(\mathrm{Tr}(J, b)) \sim \mathrm{Tr}(\mathrm{Transform}(J, b))$.

Proof. Because $\mathrm{Sing}(J, b) = \mathrm{Sing}(\mathrm{Tr}(J, b))$ (see remark 2.23(iii)) the first part is immediate.

Because of permissibility, h^b is a factor of every element of $Jk[W]$; therefore $Jk[W] \subseteq H^b$. Knowing this, by Theorem 11 on page 195 in [CLO92], we may compute $H^{-b} J k[W]$ by division on the generators as in definition 2.18. ☐

Lemma 2.25. *For an ordinary ideal J and $b \in \mathbb{Z}_+$ we have for any $n \in \mathbb{Z}_+$,* $\mathrm{Tr}(J, b) \sim \mathrm{Tr}(J^n, bn)$.

Proof. Let $J = \langle f_1, \ldots, f_k \rangle$ and $J^n = \langle \{f_{i_1} \cdots f_{i_n}\}_{i_1,\ldots i_n=1,\ldots k} \rangle$. For any $\varphi : k[U] \to A$, with valuation v as in definition 2.7 we have

$$v_A(\mathrm{Tr}(J^n, bn)) = \min_{i_1,\ldots i_n=1,\ldots k} (v(\varphi(\prod_{i_j} f_{i_j}))/nb) =$$
$$\min_{i_1,\ldots i_n=1,\ldots k} (\sum_{i_j} v(\varphi(f_{i_j})))/nb = \min_i (v(\varphi(f_i)))n/bn = v_A(\mathrm{Tr}(J, b)).$$

□

Definition 2.26. A sequence of blowing ups $\pi_1, \ldots, \pi_l$ is permissible for F, if π_i is permissible for F_{i-1}, where $F_0 = F$, $F_i = \mathrm{Transform}(F_{i-1})$ along π_i, $i = 1, \ldots, l$.

Definition 2.27. (cf. the definition of $\approx$ on page 104 in [Hir77]) Two generalized ideals generated by F, G are *strongly equivalent*, denoted as $\langle F \rangle \approx \langle G \rangle$ or sometimes simply $F \approx G$, if for any $d \geq 0$ we have for F and G considered in $k[U \times \mathbb{A}^d] \times \mathbb{Q}_+$ that $\mathrm{Sing}(F) = \mathrm{Sing}(G)$ and if a sequence of blowing ups is permissible for F iff it is permissible for G.

Remark 2.28. (i) Note that Hironaka's definition of strong local equivalence on page 67 in [Hir77] slightly differs from his definition of strong equivalence, which is the analogue of definition 2.27. (ii) When $d = 0$ in definition 2.27 the product $\times\mathbb{A}^d$ can be omitted. (iii) The relation $\approx$ is indeed an equivalence. (iv) For $F \approx G$ we have $\mathrm{Sing}(F) = \mathrm{Sing}(G)$ by definition. (v) If $F \approx G$ then for a permissible blowing up $\mathrm{Transform}(F) \approx \mathrm{Transform}(G)$. (vi) If $F \approx G$ we have $F_c \approx G_c$ for $c \geq 1$. (vii) As the property $F \sim G$ implies equality of orders of transforms of F and G along any blowing up at any point of the blownup (which yields identical singular loci at any stage) we have $F \approx G$.

Lemma 2.29. *If $F \approx G$ and $F' \approx G'$, we also have $F + F' \approx G + G'$.*

Proof. Using the notation of definition 2.27, for any d, a sequence of blowing ups is permissible for $F + F'$ iff it is permissible for F and for F', because $\mathrm{Sing}(F + F') = \mathrm{Sing}(F) \cap \mathrm{Sing}(F')$. This is true also for the transforms of F and F' after any permissible blowing up sequence (this fact comes easily from $\mathrm{Transform}(F + F') = \mathrm{Transform}(F) + \mathrm{Transform}(F')$). Then by the assumption we get $\mathrm{Sing}(F + F') = \mathrm{Sing}(G + G')$ and blowing up sequences are permissible for $F + F'$ iff they are permissible for $G + G'$. □

Definition 2.30. We say that a hypersurface with equation p has maximal contact to a generalized ideal F if $F \approx F + \{(p, 1)\}$.

Remark 2.31. (i) Obviously, the property of maximal contact depends only on the strong equivalence class of the generalized ideal. (ii) The existence of hypersurface of maximal contact implies that the maximal order $\mathrm{ord}_U(F) = 1$.

Theorem 2.32. *For $F \approx G$ we have $\mathrm{ord}_a(F) = \mathrm{ord}_a(G)$ for any $a \in \mathrm{Sing}(F)$.*

Proof. By lemma 2.13 and lemma 2.15 we can choose a representative $F' := \{(f_i^{q_i b/r_i}, b)\}_i$ for $\langle F \rangle$ (likewise a G' for $\langle G \rangle$), where we assume $b_i = r_i/q_i$, $b := \mathrm{lcm}_i(r_i)$. Such representatives are in one to one correspondence with

Hironaka's representatives of idealistic exponents via $I : F' \mapsto (\langle\{f_i^{q_i b/r_i}\}_i\rangle, b)$, with the property $\mathrm{ord}_a(F') = \mathrm{ord}_a(I(F'))$ for any $a \in U$. We also have by remark 2.28(vii), $F' \approx G'$.

Now we can apply Proposition 8 on page 68 in [Hir77] for our particular case, where, using Hironaka's terminology, $I(F')$ has strong local equivalence to $I(G')$ and vice versa via the identity on U. We get that the order of F' and G', hence of F and G, is the same at any point of the common singular locus of F and G. □

Theorem 2.33. *For a pair (J, b), if a hypersurface has maximal contact to (J, b) (in the sense of [BS00a]) then it also has maximal contact to $\mathrm{Tr}(J, b)$ (in the sense of definition 2.30).*

Proof. Suppose that p has maximal contact to (J, b). Then p is contained in the ideal generated by the partial derivatives of elements in J of order up to $b-1$, hence it is contained in $\mathbf{I}(\mathrm{Sing}(\mathrm{Tr}(J, b)))$, so it does not change the strong equivalence class. □

Definition 2.34. Let V be a hypersurface of maximal contact to F, defined by a global regular parameter p over U. The *coefficient ideal* of F with respect to V, is

$$\mathrm{Coeff}(F, V) = \{(\partial_p^j f_i, b_i - j) \mid j = 0, \dots, \lceil b_i \rceil - 1,\ i = 1, \dots, k\} \ \subset k[V] \times \mathbb{Q}_+,$$

where ∂_p denotes the partial derivative into the direction of p in a system of regular parameters containing p.

Theorem 2.35. *In the context of definition 2.34, if $F \approx G$,* $\mathrm{Coeff}(F, V) \approx \mathrm{Coeff}(G, V)$.

Proof. Suppose that F and G are translations, i.e., $b_1 = \cdots = b_k \in \mathbb{Z}_+$ and $c_1 = \cdots = c_m \in \mathbb{Z}_+$. By the proof of theorem 5 on page 111 in [Hir77], $\mathrm{Coeff}(F, V)$ has strong local equivalence with F via the embedding $V \subset U$, in the sense of Hironaka. The same holds for G. It follows that $\mathrm{Coeff}(F, V)$ and $\mathrm{Coeff}(G, V)$ are in strong local equivalence via the identity on V, which implies strong equivalence in the sense of definition 2.27.

In the general case, an obvious modification in the proof of Hironaka's Lemma 5.4 on page 112 in [Hir77] shows equality of singular loci, considering that the submodules of derivations generated by the powers of ∂_p generate modules of derivations which are, using Hironaka's terminology, normally full to V.

We also have to show equality of singular loci after permissible transformations. Unfortunately, we cannot apply the method of Note 5.5 on page 113 directly, because the transformations of F and G may introduce powers of the polynomials of the generators. However, one can show for a permissible blowing up $\pi : W \to U$ for F and G with center Z (inducing a permissible blowing up $\pi_V : \overline{V} \to V$ for $\mathrm{Coeff}(F, V)$ and $\mathrm{Coeff}(G, V)$, where $\overline{V} \subset W$ is the proper transform of V along π), with $h \in k[W]$ defining the exceptional divisor $\pi^{-1}(Z)$, assuming $b_i = r_i/q_i$, that $\mathrm{ord}_a(h^{-r_i+q_i j}(\partial_p^j f_i)^{*q_i}) \geq r_i - q_i j$ for $j = 0, \dots, \lceil b_i \rceil - 1$ iff $\mathrm{ord}_a(h^{-r_i} f_i^{*q_i}) \geq r_i$ for all $a \in \mathrm{Sing}(\mathrm{Transform}(F))$ (which is contained in $\overline{V}$ since controlled transform coincides with the weak transform in this case).

The key to this is to observe that $\mathrm{ord}_a(h^{-r_i+q_ij}(\partial_p^j f_i)^{*q_i}) \geq r_i - q_ij$ iff $\mathrm{ord}_a(h^{q_ij}(\partial_p^j f_i)^{*q_i}) \geq r_i - q_ij + r_i\mathrm{ord}_a(h)$ iff $\mathrm{ord}_a(h^j(\partial_p^j f_i)^*) \geq b_i - j + b_i\mathrm{ord}_a(h)$, and that locally, over open neighborhoods (charts) of a where p^* does not define the exceptional divisor (assuming p is a generator of the ideal of the regular center Z) $h\partial_p^* = \partial_{\overline{p}}$, where $\overline{p} := h^{-1}p^*$ is the equation of $\overline{V}$. Moreover the powers of $\partial_{\overline{p}}$ generate submodules of derivations which are normally full to $\overline{V}$ (see Lemma 5.6 on page 113 in [Hir77]). Thus by Hironaka's Lemma 5.4 we have $\mathrm{ord}_a(f_i^*) \geq b_i + b_i\mathrm{ord}_a(h)$ iff $\mathrm{ord}_a(\partial_{\overline{p}}^j f_i^*) \geq b_i - j + b_i\mathrm{ord}_a(h)$ for $j = 0, \ldots \lceil b_i \rceil - 1$, because h and $\overline{p}$ are independent at a.

This way we can show equality of the singular loci of F and $\mathrm{Coeff}(F, V)$ (and of G and $\mathrm{Coeff}(G, V)$) after any sequence of permissible transformations. □

Remark 2.36. If (J, b) be a pair as in definition 2.21, V a regular hypersurface for $\mathrm{Tr}(J, b)$ as in definition 2.34, we have $\mathrm{Coeff}(\mathrm{Tr}(J, b), V) \approx \mathrm{Tr}(\mathrm{Coeff}(J, b, V), b!)$.

If we are given a list of IND elements $p_1, \ldots, p_r$ generating hypersurfaces of maximal contact $V_1, \ldots, V_r$ respectively, the coefficient ideal of F with respect to all of them is just

$$\mathrm{Coeff}(F, (V_1, \ldots, V_r)) = \mathrm{Coeff}(\mathrm{Coeff}((\ldots \mathrm{Coeff}(F, V_1) \ldots), V_{r-1}), V_r).$$

2.5 New representation of objects

We define the new representations of weighted, tagged and balanced objects, which rely on the notion of generalized ideals, presented in section 2.4. Exceptional divisors can always be represented locally by regular parameters in the algorithm; therefore we assume that they are generated by elements from IND of the underlying chart U.

Definition 2.37. A *weighted object*, $\mathcal{W} = (W, J, C, c, E, a)$ contains the following data:

- W is a nonsingular algebraic set (when the dimension of W is the same as the dimension of the ambient space of the input problem, W is actually a variety),
- J is a nonzero generalized ideal over $k[W] \times \mathbb{Q}_+$,
- C is list of functions, $C = (a_i : \mathrm{IND} \to \mathbb{Q}_{\geq 0} \mid i = 1, \ldots, m)$,
- $c \in \mathbb{Q}_{\geq 0}$, the maxorder (see section 2.6.2),
- E is a list of different IND elements (the list of parameters that define exceptional divisors),
- a is a function, $a : E \to \mathbb{Q}_{\geq 0}$.

Let us construct the vanishing generalized ideal of such a weighted object from the data of J, C, E, a in several steps. In the sequel, the function list C_c denotes the list of the functions of C with all the values multiplied by $c \in \mathbb{Q}_{\geq 0}$.

Definition 2.38. Use the notation of definition 2.37, let $E = (p_{i_1}, \ldots, p_{i_r})$, $(i_1 < \cdots < i_r)$, $a = (d_{i_1}/d, \ldots, d_{i_r}/d)$, and let $M = \prod_{j=1}^r p_{i_j}^{d_{i_j}}$. The *monomial part* of $\mathcal{W}$, denoted as E^a, is $\langle (M, d) \rangle$.

If $c = 0$, the weighted object is either monomial or resolved; in none of the cases we need to consider the proper part (which we could denote in this case as an exceptional generalized ideal with the symbol $\langle(1,0)\rangle$). Because of this, in the next definition we assume $c > 0$.

Definition 2.39. Use the notation of definition 2.37, apply the translation of definition 2.38 for all $a_i \in C_c$ getting $\langle(g_1, c_1), \dots, (g_m, c_m)\rangle$ and let $J_{1/c} = \langle(f_1, b_1), \dots, (f_k, b_k)\rangle$.

The *proper part* of $\mathcal{W}$ is $\langle(f_1, b_1), \dots, (f_k, b_k), (g_1, c_1), \dots, (g_m, c_m)\rangle$.

Remark 2.40. In the algorithm we use the convention that the rational components of generalized ideals and C-functions get updated at each order computation, where the update is recorded in c (see subsection 2.6.2). Therefore, in order to restore the original components we have to take $J_{1/c}$, C_c.

Definition 2.41. *The generalized ideal* of the weighted object $\mathcal{W}$ in the case $c > 0$, using the notation introduced in definitions 2.38, 2.39 is

$$\langle(f_1^{l/b_1} M^{l/d}, l), \dots, (f_k^{l/b_k} M^{l/d}, l), (g_1^{l/c_1} M^{l/d}, l), \dots, (g_m^{l/c_m} M^{l/d}, l)\rangle,$$

where $l \in \mathbb{N}$ is such that l/d, l/b_i, l/c_j are all natural numbers.

If $c = 0$ the generalized ideal of $\mathcal{W}$ is $\langle(M, d)\rangle$.

Actually, we will never have to compute this ideal in the algorithm explicitly.

Definition 2.42. A *tagged object*, $\mathcal{T} = (W, J, C, E, E_-)$, contains the data: W, J, C, E with types as in a weighted object, and

- E_- a sublist of E.

Definition 2.43. *The generalized* ideal of a tagged object $\mathcal{T}$ is computed from J and C in the same way as the proper part of a weighted object in definition 2.39, considering that there is no need to update the rational numbers in J and C.

Definition 2.44. A *balanced object*, $\mathcal{B} = (W, J, C, E, H)$, contains the data: W, J, C, E with types as in a weighted object, and

- H a sublist of IND.

Definition 2.45. Let $\mathcal{B}$ be a balanced object as in definition 2.44, let the generalized ideal computed from J and C as in definition 2.39 be $\langle(f_1, b_1), \dots, (f_k, b_k)\rangle$, considering that no update of rational components of J and C is needed, and let $H = (p_1, \dots, p_r)$. Then *the generalized ideal* of the balanced object is $\langle(f_1, b_1), \dots, (f_k, b_k), (p_1, 1), \dots, (p_r, 1)\rangle$.

Let $L = (f_1, \dots, f_k)$, we introduce the notation $\mathbf{Z}\{L\} := (\mathbf{Z}(f_1), \dots, \mathbf{Z}(f_k))$.

Definition 2.46. Let $\mathcal{W} = (W, J, C, c, E, a)$ and $\mathcal{W}' = (W', J', b', E', a')$ be weighted objects in the new and in the old representations respectively, with I_w the generalized ideal of $\mathcal{W}$ as in definition 2.41, $\overline{I}_w$ the proper part as in definition 2.39 and $\overline{J}'$ the proper part of J' as in subsection 1.4.1. Set $\overline{c} := \mathrm{ord}_{\mathrm{Sing}(I_w)}(\overline{I}_w)$ and $\bar{b}' := \mathrm{ord}_{\mathrm{Sing}(J',b')}(\overline{J}')$. We say that $\mathcal{W}$ and $\mathcal{W}'$ represent the *same resolution problem* iff $W' = W$, $\mathrm{Tr}(J', b') \approx I_w$, $E' = \mathbf{Z}\{E\}$, $a'(H) = a(p)b'$ for all $H = \mathbf{Z}(p)$, $p \in E$, and whenever $\bar{b} > 0$ (equivalently $\overline{c} > 0$) $\mathrm{Tr}(\overline{J}', \bar{b}') \approx (\overline{I}_w)_{\overline{c}}$.

Let $\mathcal{T} = (W, J, C, E, E_-)$, $\mathcal{T}' = (W', J', b', E', E'_-)$ be tagged objects, with I_t the generalized ideal of $\mathcal{T}$ as in definition 2.43. The objects $\mathcal{T}$ and $\mathcal{T}'$ represent the same resolution problem if $W' = W$, $\mathrm{Tr}(J', b') \approx I_t$, $E' = \mathbf{Z}\{E\}$, $E'_- = \mathbf{Z}\{E_-\}$.

Let $\mathcal{B} = (W, J, C, E, H)$, $\mathcal{B}' = (W', J', b', E')$ be balanced objects, with I_b the generalized ideal of $\mathcal{B}$ as in definition 2.45. The objects $\mathcal{B}$ and $\mathcal{B}'$ represent the same resolution problem if $W' = W$, $\mathrm{Tr}(J', b') \approx I_b$, $E' = \mathbf{Z}\{E\}$.

Theorem 2.47. *If the weighted objects $\mathcal{W}$, $\mathcal{W}'$ (tagged objects $\mathcal{T}$, $\mathcal{T}'$, balanced objects $\mathcal{B}$, $\mathcal{B}'$ respectively) represent the same resolution problem, the one is resolved iff the other is resolved.*

Proof. In case of $\mathcal{W}$ and $\mathcal{W}'$, because the components W and W', E and E', a and a' correspond to each other directly via definition 2.46, and by remark 2.23(iii) and remark 2.28(iv) we have $\mathrm{Sing}(J', b') = \mathrm{Sing}(I_w)$, so the claim is immediate.

The argument is analogous for tagged and balanced objects. □

2.6 Computations using the new representation

We define the operations of the resolution algorithm on the new data structures given in the previous section. We overload function names like Sing, Coeff or Transform.

The point in using the complicated formulae presented in this section instead of computing the ideal of definition 2.41 and using the corresponding functions for generalized ideals, is to avoid the powering of the generators prescribed in definition 2.41. Although in general we cannot avoid Gröbner basis computation, the new data structure allows us, for instance, to detect resolvedness by means of combinatorial methods in certain cases. It is also beneficial to keep the generators of the input ideals of Gröbner basis computations simple, which is not the case in the construction of definition 2.41.

We would like to bring to the reader's attention a simple but useful way to check the correctness of the operations defined in this section. The sometimes seemingly complicated formulae of certain operations were gotten by constructing the generalized ideal of the given object using definition 2.41, 2.43 or 2.45 then applying the corresponding generalized ideal operation and reverse engineering what happens with the function values of C or a.

2.6.1 Singular locus

The singular locus of generalized ideals given in definition 2.11 can also be defined in terms of partial derivatives: $\mathrm{Sing}(\langle(f_1, b_1), \ldots, (f_k, b_k)\rangle) =$

$$\mathbf{Z}\Big(\Big\langle \frac{\partial^j f_i}{\partial p_1^{j_1} \cdots \partial p_n^{j_n}} \,\Big|\, i = 1, \ldots, k,\ j = 0, \ldots, \lceil b_i \rceil - 1,\ j_l \in \mathbb{N},\ \sum\nolimits_{l=1}^{n} j_l = j \Big\rangle\Big),$$

where the derivations are performed with respect to the global regular parameters of the underlying chart.

Consider the monomial part E^a of a weighted object with M and d as in definition 2.38. We define $\mathrm{Sing}(E^a) = \mathrm{Sing}(\langle(M, d)\rangle)$.

Lemma 2.48. *With the above notation we have*

$$\mathrm{Sing}(E^a) = \mathbf{Z}\Big(\bigcap_{\substack{\{p_{i_1},\dots,p_{i_l}\}\subseteq E \\ \sum_{j=1}^{l} a(p_{i_j})\geq 1}} \langle p_{i_1},\dots,p_{i_l}\rangle\Big).$$

Proof. Take M and d as in definition 2.38. By definition 2.11 $x \in \mathrm{Sing}(\langle (M,d)\rangle)$ iff $\mathrm{ord}_x(M) \geq d$, and for all $x \in \mathbf{Z}(p_i)$ we have $\mathrm{ord}_x(p_i) = 1$ for all i, moreover the exceptional divisors can have at most normal crossings. Considering this, the construction of the ideal of the lemma just describes the points of $\mathrm{Sing}(E^a)$. □

Let $C = (a_1,\dots,a_m)$ represent $\langle (g_1,c_1),\dots,(g_m,c_m)\rangle$ as in definition 2.39. We define $\mathrm{Sing}(C) = \mathrm{Sing}(\langle (g_1,c_1),\dots,(g_m,c_m)\rangle)$.

Lemma 2.49. *With the above notation we have*

$$\mathrm{Sing}(C) = \bigcap_{i=1}^{m} \mathbf{Z}\Big(\bigcap_{\substack{\{p_{i_1},\dots,p_{i_l}\}\subseteq \mathrm{IND} \\ \sum_{j=1}^{l} a_i(p_{i_j})\geq 1}} \langle p_{i_1},\dots,p_{i_l}\rangle\Big).$$

Proof. The singular locus of C is the intersection of the singular loci of the principal ideals generated by pairs (g_i,c_i), coming from the a_i functions as in definition 2.39. Now by lemma 2.48, the singular locus of $\langle (g_i,c_i)\rangle$ is

$$\mathbf{Z}\Big(\bigcap_{\substack{\{p_{i_1},\dots,p_{i_l}\}\subseteq \mathrm{IND} \\ \sum_{j=1}^{l} a_i(p_{i_j})\geq 1}} \langle p_{i_1},\dots,p_{i_l}\rangle\Big).$$

□

Weighted objects

Consider a weighted object as in definition 2.37, and use the notation of definition 2.41 assuming $c > 0$. Let the generalized ideal of the weighted object as in definition 2.41 be I_w with rational component l. We define the singular locus of the weighted object to be $\mathrm{Sing}(I_w)$.

The algorithm changes the rational components of J and the function values of C in the maxorder computation (see subsection 2.6.2). However, we need the original rational values to compute the singular locus, which we can retrieve by taking $J_{1/c}$, and C_c (the inverse operation). Therefore we can assume without the loss of generality that $c = 1$.

Lemma 2.50. *With the notation described above we have*

$$\begin{aligned}\mathrm{Sing}(I) = {} & \mathrm{Sing}(\langle (M,d)\rangle) \cup (\mathrm{Sing}(J) \cap \mathrm{Sing}(C)) \cup \\ & \bigcup_{r=1}^{l-1} \Big(\mathrm{Sing}(J_{\frac{r}{l}}) \cap \mathrm{Sing}(C_{\frac{l}{r}}) \cap \mathbf{Z}\big(\bigcap_{\substack{\{p_{i_1},\dots,p_{i_l}\}\subseteq E \\ \sum_{j=1}^{l} a(p_{i_j})+\frac{r}{l}\geq 1}} \langle p_{i_1},\dots,p_{i_l}\rangle\big)\Big).\end{aligned}$$

Proof. Consider that $\mathrm{Sing}(I_w)$ consists of those points $a \in U$ where $\mathrm{ord}_a(f_i^{l/b_i}) + \mathrm{ord}_a(M^{l/d}) \geq l$ and $\mathrm{ord}_a(g_j^{l/c_j}) + \mathrm{ord}_a(M^{l/d}) \geq l$ for all i and j. The extreme cases are when the monomial part or the proper part is already at least of order l:

1. $\mathrm{ord}_a(M^{l/d}) \geq l$,

2. $\mathrm{ord}_a(f_i^{l/b_i}) \geq l$ and $\mathrm{ord}_a(g_j^{l/c_j}) \geq l$ for all i, j.

Points of type 1 are exactly the ones in $\mathrm{Sing}(\langle\langle(M, d)\rangle\rangle)$, while of points of type 2 are the ones in $\mathrm{Sing}(J) \cap \mathrm{Sing}(C)$.

In the intermediate cases, for $r = 1, \ldots, l-1$, we require $\mathrm{ord}_a(f_i^{l/b_i}) \geq r$, $\mathrm{ord}_a(g_j^{l/c_j}) \geq r$ and $\mathrm{ord}_a(M^{l/d}) \geq l - r$. These points are exactly the ones in

$$\begin{aligned}&\mathrm{Sing}(\langle\langle(f_1^{l/b_1}, r), \ldots, (f_k^{l/b_k}, r), (g_1^{l/c_1}, r), \ldots, (g_m^{l/c_m}, r)\rangle\rangle) \cap \\ &\qquad \mathrm{Sing}(\langle\langle(M^{l/d}, l - r)\rangle\rangle) = \\ &\qquad\qquad \mathrm{Sing}(J_{r/l}) \cap \mathrm{Sing}(C_{l/r}) \cap \mathrm{Sing}(\langle\langle(M, d(1 - r/l))\rangle\rangle).\end{aligned}$$

We take the union of all these points to get the singular locus of I_w. ☐

When $c = 0$ and the weighted object is not resolved, it is monomial. In this case $\mathrm{Sing}(I_w) = \mathrm{Sing}(E^a)$ which can be constructed with the method of lemma 2.48.

Tagged objects

Consider a tagged object $\mathcal{T}$ as in definition 2.42, with I_t its generalized ideal as in definition 2.43. The singular locus of $\mathcal{T}$ is $\mathrm{Sing}(I_t)$.

Lemma 2.51. *With the above notation we have* $\mathrm{Sing}(I_t) = \mathrm{Sing}(J) \cap \mathrm{Sing}(C)$.

Proof. The lemma is just a special case of lemma 2.50, namely $c = 1$ and the monomial factor is 1. ☐

Balanced objects

Consider a balanced object $\mathcal{B}$ as in definition 2.44, with its generalized ideal I_b as in definition 2.45. The singular locus of $\mathcal{B}$ is $\mathrm{Sing}(I_b)$.

Lemma 2.52. *With the above notation we have* $\mathrm{Sing}(I_b) = \mathrm{Sing}(J) \cap \mathrm{Sing}(C) \cap \bigcap_{p_i \in H} \mathbf{Z}(p_i)$.

Proof. It is clear by lemma 2.51 and the fact that the rational component is 1 for all the generators of I_b coming from $p_i \in H$. ☐

2.6.2 Maxorder

We use the notation of definition 2.41, denote the generalized ideal of the weighted object by I_w. The *maxorder* of the weighted object $\mathcal{W}$ is the maximal order of the proper part (see definition 2.39) within $\mathrm{Sing}(I_w)$.

Lemma 2.53. *With the notation as in the above paragraph, the maxorder of* $\mathcal{W}$ *is* i/l*, where* i *is the maximal natural number so that* $\mathrm{Sing}(J_{i/l}) \cap \mathrm{Sing}(C_{l/i}) \cap \mathrm{Sing}(I_w)$ *is not empty, or 0 if no such* i *exists.*

Proof. If all the rational components of the generators of J as well as the function values in C are 0, let i be 0. (In this case the proper part is the exceptional generalized ideal $\langle(1,0)\rangle$.)

Otherwise consider the representation of J and C as in definition 2.39 and make the generators have the same rational component as in definition 2.41, namely $\langle(f_1^{l/b_1},l),\ldots,(f_k^{l/b_k},l),(g_1^{l/c_1},l),\ldots,(g_m^{l/c_m},l)\rangle$ with $l>0$.

In this case by remark 2.23(ii) we have

$$\begin{aligned}\mathrm{ord}_{\mathrm{Sing}(I_w)}(\langle(f_1^{l/b_1},l),\ldots,(f_k^{l/b_k},l),(g_1^{l/c_1},l),\ldots,(g_m^{l/c_m},l)\rangle) =\\ \mathrm{ord}_{\mathrm{Sing}(I_w)}(\langle f_1^{l/b_1},\ldots,f_k^{l/b_k},g_1^{l/c_1},\ldots,g_m^{l/c_m}\rangle)/l.\end{aligned}$$

Let i be the numerator of the right hand side of the above equation. □

Remark 2.54. If $i=0$ the object is either resolved or monomial; in these cases we do not need to consider the proper part of $\mathcal{W}$ to proceed. Otherwise, the updated generalized ideal is $J_{i/l}$, the C list gets updated to $C_{l/i}$, and c is updated by setting it to $c\cdot i/l$.

2.6.3 Equiorder

Let $\mathcal{W}=(W,J,c,C,E,a)$ be a weighted object, the input of the *EquiOrder* algorithm. We assume that the rational numbers in J, C, as well as c are updated by the maxorder computation. Notice that we must have $c>0$, otherwise $\mathcal{W}$ is either resolved or monomial (see section 1.4.1). The *EquiOrder* algorithm for the new representation of objects performs the following computations:

– When $c\geq 1$, the tagged object $\mathcal{T}=(W,J,c=1,C,E,E_-=E)$.

– If $0<c<1$, we define a new function a' on IND as:

$$a'(p)=\begin{cases}a(p)/(1-c) & \text{if } p\in E\\ 0 & \text{otherwise.}\end{cases}$$

The new tagged object is $\mathcal{T}=(W,J,c=1,\mathrm{append}(C,a'),E,E_-=E)$.

Using the notation of definition 2.46, let $\mathcal{W}'$ be a weighted object in the old representation that defines the same resolution problem as $\mathcal{W}$. Let $\mathcal{T}'=$ *EquiOrder*$(\mathcal{W}')$ using the old representation, while $\mathcal{T}=$ *EquiOrder*$(\mathcal{W})$ using generalized ideals in the new representation.

Lemma 2.55. *With the above notation, we have $\mathcal{T}'$ and $\mathcal{T}$ defining the same resolution problem.*

Proof. If $c\geq 1$, by definition 2.46 the lemma is trivial.

Let $\mathcal{T}'$ have (J_t',b_t'), and let I_t be the generalized ideal of $\mathcal{T}$ as in definition 2.43. Assume now that we have $0<c<1$ (equivalently $0<\bar b'<b'$), so that $c=\bar b'/b'$ and $b_t'=\bar b'(b'-\bar b')$. Moreover let $\langle(M,b')\rangle$ be the monomial part of $\mathcal{W}$ which yields $M=\prod_{p\in E}p^{a'(\mathbf{Z}(p))}$. Then we have

$$\begin{aligned}\mathrm{Tr}(J_t',\bar b'(b'-\bar b'))\approx \mathrm{Tr}(\overline{J}'^{b'-\bar b'},\bar b'(b'-\bar b'))+\mathrm{Tr}(M^{\bar b},\bar b'(b'-\bar b'))\approx\\ \approx \mathrm{Tr}(\overline{J}',\bar b')+\mathrm{Tr}(M,b'-\bar b')\approx(\overline{I}_w)_c+\langle(M,b'(1-c))\rangle\approx I_t\end{aligned}$$

using lemma 2.25, lemma 2.29 and the construction of definition 2.43. □

2.6.4 Support

The computation of the support of a tagged object $\mathcal{T} = (W, J, C, E, E_-)$ is done as presented in the algorithm *Support2*, with the difference that the generalized ideal is not modified. In the else branch, when all the exceptional divisors from E_- have common singularities, the *Support2* algorithm performs the following computations for the new representation of objects:

- When $E = E_-$, the balanced object is $\mathcal{B} = (W, J, C, E - E_-, H = E_-)$.
- When $E \neq E_-$, assume $E_- = (p_1, \ldots, p_r)$, $r \geq 1$ and for $i = 2, \ldots, r$ we define new functions on IND:

$$a_i(p) = \begin{cases} 1 & \text{if } p = p_i \\ 0 & \text{otherwise.} \end{cases}$$

  The new balanced object is $\mathcal{B} = (W, J, \mathrm{append}(C, (a_2, \ldots, a_r)), E - E_-, H = (p_1))$.

Let I_t be the generalized ideal of $\mathcal{T}$ and by using the notation of definition 2.46 let $\mathcal{T}'$ be a tagged object in the old representation that defines the same resolution problem as $\mathcal{T}$. Let $\mathcal{B}' = \mathit{Support2}(\mathcal{T}')$ and let $\mathcal{B} = \mathit{Support2}(\mathcal{T})$ over the same chart.

The previous assumption contains a proposition, namely that the chart structures generated by the two function calls are the same. This is easy to see because we have $\mathrm{Sing}(J', b') = \mathrm{Sing}(I_t)$ by remark 2.23(ii), and $E'_- = \mathbf{Z}\{E_-\}$. Because of this we may assume that the singular loci are generated by the same polynomials, so that the subdivision of the input chart is performed in exactly the same steps in both cases.

Lemma 2.56. *With the above notation we have $\mathcal{B}'$ and $\mathcal{B}$ defining the same resolution problem.*

Proof. Without the loss of generality we can assume that $E_- = (p_1, \ldots, p_r)$ has common singularities, when we also have $E'_- = \mathbf{Z}\{E_-\}$.

Let then J'_b be the ideal of $\mathcal{B}'$ and I_b be the ideal of $\mathcal{B}$ as in definition 2.45. Either $E = E_-$ and $E' = E'_-$ or $E \neq E_-$ and $E' \neq E'_-$ we have

$$\mathrm{Tr}(J'_b, b') \approx \mathrm{Tr}(J'_t, b') + \langle (p_1^{b'}, b'), \ldots, (p_r^{b'}, b') \rangle \approx I_t + \langle (p_1, 1), \ldots, (p_r, 1) \rangle \approx I_b.$$

$\square$

2.6.5 Coefficient ideal

Let $\mathcal{B} = (W, J, C, E, H)$ be a balanced object, and let V be a list of regular parameters defining hypersurfaces of maximal contact (either coming from H or computed explicitly in *ResolveBalanced2*), and let $Z = \mathbf{Z}\{V\}$. The coefficient object is

$$\mathcal{W} = (W \cap \mathbf{Z}(V), \mathrm{Coeff}(J, Z), c = 1, \mathrm{Coeff}(C, Z), E, 0_E),$$

where 0_E is the zero function on E, and $\mathrm{Coeff}(C, Z)$ is defined below.

For a function $a \in C$ we define $\mathrm{Coeff}(a, Z)$ to be the zero function (on IND) if $\sum_{p\in V} a(p) \geq 1$, else

$$\mathrm{Coeff}(a,Z)(p) = \begin{cases} 0 & \text{if } p \in V, \\ a(p)/(1-\sum_{p\in V} a(p)) & \text{otherwise.} \end{cases}$$

We define $\mathrm{Coeff}(C, Z) = (\mathrm{Coeff}(a, Z) \mid a \in C,\ \mathrm{Coeff}(a, Z) \neq 0_{\mathrm{IND}})$.

Using the notation of definition 2.46, let $\mathcal{B}'$ be a balanced object in the old representation that defines the same resolution problem as $\mathcal{B}$. Let $\mathcal{W}$ and $\mathcal{W}'$ be the coefficient objects of $\mathcal{B}$ and $\mathcal{B}'$ respectively computed with the list Z of hypersurfaces of maximal contact.

Lemma 2.57. *With the above notation we have $\mathcal{W}'$ and $\mathcal{W}$ defining the same resolution problem.*

Proof. Let us assume $V = (p_1)$, $Z = \mathbf{Z}(p_1)$. As in definition 2.45, let

$$I_b = \langle (f_1^{l/b_1}, l), \dots, (f_k^{l/b_k}, l), (g_1^{l/c_1}, l), \dots, (g_m^{l/c_m}, l), (p_1, 1)\rangle,$$

where the generator $(p_1, 1)$ might not be present if $H = ()$ and V has been computed directly. However, $\mathrm{Coeff}(\{(p_1, 1)\}, Z) = 0$, therefore this generator does not play any role in the new resolution problem.

First we have to show that the coefficient ideal computation on C is in one to one correspondence with the same operation on the corresponding generators of I_b. By theorem 2.35 we have $\mathrm{Coeff}(\{(g_i^{l/c_i}, l)\}, Z) \approx \mathrm{Coeff}(\{(g_i, c_i)\}, Z)$, and what we need to show is $\mathrm{Coeff}(\{(g_i, c_i)\}, Z) \approx \{(h_i, d_i)\}$, for $i = 1, \dots, m$, where (h_i, d_i) comes from $\mathrm{Coeff}(a_i, Z)$ for $a_i \in C$.

Let $g_i = p_1^{e_1} \dots p_r^{e_r}$; if $e_1 \geq c_i$, p_1 is a factor in every polynomial in the generators of $\mathrm{Coeff}(\{(g_i, c_i)\}, Z)$; therefore all of them are zero in $k[Z]$.

If $e_1 < c_i$, p_1 is a factor in $\partial^j_{p_1} g_i$ for $j = 0, \dots, e_1 - 1$, and $\partial^j_{p_1} g_i = 0$ for $j = e_1 + 1, \dots, \lceil c_i \rceil - 1$. The only nonzero generator in $k[Z]$ is $(h_i, d_i) = (\partial^{e_1}_{p_1} g_i, c_i - e_1) = (e_1! p_2^{e_2} \dots p_r^{e_r}, c_i - e_1)$, which is the corresponding generator for the function $\mathrm{Coeff}(a_i, Z) = (0, e_2/(c_i - e_1), \dots, e_r/(c_i - e_1), 0, \dots)$.

Let I_w be the generalized ideal of $\mathcal{W}$ as in definition 2.41, and let I'_w be the ideal of $\mathcal{W}'$. By the assumption we have $\mathrm{Tr}(J', b') \approx I_b$, so by remark 2.36, theorem 2.35 and the previous argument we have

$$\mathrm{Tr}(I'_w, b'!) \approx \mathrm{Coeff}(I_b, V) \approx \mathrm{Coeff}(J, V) + \langle (h_1, d_1), \dots, (h_m, d_m)\rangle.$$

Because the other components of $\mathcal{W}'$ and $\mathcal{W}$ correspond to each other directly, they represent the same resolution problem.

If $V = (p_1, \dots, p_r)$, $Z = \mathbf{Z}(p_1, \dots, p_r)$, apply the recursive definition of $\mathrm{Coeff}(I_b, Z)$ and $\mathrm{Coeff}(J', b', Z)$. Since all intermediate weighted objects will be simple, one can get the next balanced objects in a straightforward way (see section 2.2), without a change in the ideal. Therefore the lemma can be applied for the intermediate coefficient object computations, yielding weighted objects $\mathcal{W}'_r$, $\mathcal{W}_r$ representing the same resolution problem. ☐

2.6.6 Transform along blowing up

Let the chart U, described as in section 1.2, be blown up along $Z = \mathbf{Z}(p_1, \dots, p_r)$, and let the chart with exceptional divisor p_i^* be U_i. We write $\mathrm{Transform}(\cdot, p_i^*)$

to indicate over which chart we consider the transformation of the object. We describe the transform operation only for weighted and tagged objects, since the balanced objects are resolved iff their coefficient objects are resolved; therefore they are not transformed, rather recomputed from their ancestors every time it is necessary.

The transform of the generalized ideal is performed by the Transform function (see definition 2.18), its compatibility with the Tr function is shown by theorem 2.24.

Let $\mathcal{W}$ be a weighted object as in definition 2.37. The transformed of the function a is

$$\mathrm{Transform}(a, p_i^*)(p^*) = \begin{cases} c - 1 + \sum_{p_j \in E,\, 1 \leq j \leq r} a(p_j) & \text{if } p^* = p_i^*, \\ a(p) & \text{otherwise.} \end{cases}$$

Let $\mathcal{W}'$ be a weighted object in the old representation that defines the same resolution problem as $\mathcal{W}$. Using the notation as in definition 2.46, considering that $c = \bar{b}'/b'$ and that $a(p) = a'(\mathbf{Z}(p))/b'$ for $p \in E$, we can easily see that the above computation is analogous to the one given in subsection 1.4.1. In particular $\mathrm{Transform}(a, p_i^*)(p_i^*) = \big(\bar{b}' - b' + \sum_{p_j \in E,\, 1 \leq j \leq r} a'(\mathbf{Z}(p_j))\big)/b'$.

The transform of a function $a \in C$ is

$$\mathrm{Transform}(a, p_i^*)(p^*) = \begin{cases} \sum_{j=1}^{r} a(p_j) \ - 1 & \text{if } p^* = p_i^*, \\ a(p) & \text{otherwise.} \end{cases}$$

Then we define $\mathrm{Transform}(C, p_i^*) = (\mathrm{Transform}(a, p_i^*) \mid a \in C)$.

Let $\langle (g_1, c_1), \ldots, (g_m, c_m) \rangle$ be the generalized ideal represented by C and let $\langle (h_1, d_1), \ldots, (h_m, d_m) \rangle$ be the generalized ideal represented by $\mathrm{Transform}(C, p_i^*)$, both as in definition 2.39.

Lemma 2.58. *With the above notation we have*

$$\mathrm{Transform}(\langle (g_1, c_1), \ldots, (g_m, c_m) \rangle, p_i^*) = \langle (h_1, d_1), \ldots, (h_m, d_m) \rangle.$$

Proof. Since the transform computation is generator-wise, it is enough to show that $(g_k^* : {p_i^*}^{c_k}, c_k)$ is represented by $\mathrm{Transform}(a_k, p_i)$, where $a_k \in C$ defines (g_k, c_k), (recall from definition 2.39 that c_k is actually a natural number). Without the loss of generality we can assume $g_k = p_1^{e_1} \ldots p_r^{e_r}$ and $i = 1$. Then we have $g_k^* = {p_1^*}^{\sum_{j \in I,\, j \leq r} e_j} {p_2^*}^{e_2} \ldots {p_r^*}^{e_r}$; therefore

$$(g_k^* : {p_1^*}^{c_k}, c_k) = ({p_1^*}^{\sum_{j \in I,\, j \leq r} e_j - c_k} {p_2^*}^{e_2} \ldots {p_r^*}^{e_r}, c_k),$$

which is just (h_k, d_k), the generator given by $\mathrm{Transform}(a_k, p_i^*)$. $\square$

Let $\mathcal{W} = (W, J, c, C, E, a)$ be a weighted object and let I_w be the generalized ideal as in definition 2.41. Using the notation of definition 2.46, let $\mathcal{W}'$ be a weighted object in the old representation that defines the same resolution problem as $\mathcal{W}$. Keep in mind that the maxorder computation might have updated the rational components of J and the values of the functions in C; therefore with the order updates in definition 2.39 we just set back the original values. Let $\mathcal{W}_T$ and $\mathcal{W}'_T$ be the transforms of $\mathcal{W}$ and $\mathcal{W}'$ respectively, along the indicated blowing up over the chart U_i with exceptional divisor p_i^*.

Lemma 2.59. *With the above notation we have that $\mathcal{W}'_T$ and $\mathcal{W}_T$ represent the same resolution problem.*

Proof. We have that W_T is the same for both objects and that the changes in E'_T are in direct correspondence with the changes in E_T; $l'_T = l'$, $c_T = c$. If $c = 0$, the object is monomial, and only a and a' get transformed by a simple computation. The straightforward argument which shows that $a'_T = (a_T(p)b' \mid p \in E_T)$ can be found at the beginning of this subsection. Assume now $c > 0$.

Recall that we have $\mathrm{Tr}(\mathrm{Transform}(J', b', p_i^*), b') \approx \mathrm{Transform}(I_w, p_i^*)$ according to theorem 2.24 and remark 2.28(vii); therefore what we have to show is that the transform of J, C and a corresponds to the transform of I_w. Let I_{wT} be the generalized ideal constructed as in definition 2.41 from the transformed data: $\mathrm{Transform}(J, p_i^*)$, $\mathrm{Transform}(C, p_i^*)$, $\mathrm{Transform}(a, p_i^*)$.

Since Transform is a generator-wise operation, it is enough to show is that the generators of $\mathrm{Transform}(I_w, p_i^*)$ are in one to one correspondence with the generators of I_{wT}. This ideal is generated by $(f_j^{*l/B_j} M^{*l/d} : p_i^{*l}, l)$ and $(g_e^{*l/C_e} M^{*l/d} : p_i^{*l}, l)$, where $B_j = b_j/c$, $C_e = c_e/c$, and without the loss of generality, by lemma 2.13, as in definition 2.41, we may assume that b_j, c_e are natural numbers, and l can be chosen so that lc, $l(1-c)$, l/B_j, l/C_e and l/d are also natural numbers for $j = 1, \ldots, k$, $e = 1, \ldots, m$.

We have $\mathrm{Transform}(J, p_i^*)_{1/c}$ generated by $\{(f_j^* : p_i^{*b_j}, B_j)\} \approx \{(f_j^{*l/B_j} : p_i^{*cl}, l)\}$, $j = 1, \ldots, k$. By the construction of $\mathrm{Transform}(a, p_i^*)$ we get

$$\langle (M^* : p_i^{d(1-c)}, d) \rangle \approx \langle (M^{*l/d} : p_i^{*l(1-c)}, l) \rangle$$

for the monomial part, where whenever $c > 1$ the division actually turns into multiplication. Then, along the lines of definition 2.41, for these generators we get $((f_j^{*l/B_j} : p_i^{*cl})(M^{*l/d} : p_i^{*l(1-c)}), l) = (f_j^{*l/B_j} M^{*l/d} : p_i^{*l}, l)$ the corresponding generator of $\mathrm{Transform}(I_w, p_i^*)$.

The same argument applies to the generators of $\mathrm{Transform}(I_w, p_i^*)$ of the form $(g_e^{*l/C_e} M^{*l/d} : p_i^{*l}, l)$, and by lemma 2.58 we get that $\mathrm{Transform}(C, p_i)$ performs the operation on the functions in C which just corresponds to the transform of $\{(g_e, c_e)\}$, $e = 1, \ldots, m$.

Finally we got that $I_{wT} = \mathrm{Transform}(I_w, p_i^*)$; therefore $\mathcal{W}'_T$ and $\mathcal{W}_T$ describe the same resolution problem over U_i. □

The point in computing the transform of the weighted object by transforming J, C and a separately is to avoid costly Gröbner basis computations on the generators of I_w. Though the transform of a generalized ideal is computed by polynomial division on the generators (see definition 2.18), this division is computed in the coordinate ring of the ambient space of the object $k[W]$, so we cannot avoid using the method of Gröbner bases. Furthermore, this way we transform proper to proper and monomial to monomial part, saving additional computations in (the first line of) algorithm *EquiOrder*.

Lemma 2.60. *The claim of lemma 2.59 in the setup for tagged objects holds.*

Proof. This case is much easier, because of the lack of the monomial part. The lemma holds by theorem 2.20 and lemma 2.58, using a simplified argument of lemma 2.59. □

In case of a tagged object p_i^* is dropped from the transformed E_- when it contains it.

Remark 2.61. Balanced objects do not get transformed.

2.6.7 Detection of good points

In certain situations one can resolve weighted objects much easier than the way Villamayor's stratifying function prescribes. Such a situation is when the singular locus of the weighted object consists only of so called "exceptional and good" points. Using the original notation of a weighted object (W, J, b, E, a) (see section 1.4), a point $x \in W$ is *exceptional and good* for $J = \langle p_{i_1}^{a(p_{i_1})} \cdots p_{i_k}^{a(p_{i_k})} \rangle \overline{J}$ with b, if $\operatorname{ord}_x(\langle p_{i_1}^{a(p_{i_1}) \bmod b} \cdots p_{i_k}^{a(p_{i_k}) \bmod b} \rangle \overline{J}) < b$. For more details we refer to section 3 in [EV98].

If we have this condition fulfilled by all the points of the singular locus, we can make the weighted object resolved by blowing up along exceptional divisors. Since blowing ups along hypersurfaces are isomorphisms, we can compute these operations by simply updating the function values of a by subtracting b from the a-value of the exceptional divisor of the blowing up center.

In the new representation, for a weighted object (W, J, C, c, E, a), where we have maxorder computation done (see section 2.6.2), we can detect easily whether the singular locus of a weighted object consists only of exceptional and good points: We need only to check whether there is $p \in E$ such that $a(p) \geq 1$ and whether $c + \sum_{p \in E}\{a(p)\} < 1$, where $\{\}$ denotes the fraction function.

We also have to remark that if the manifold W of the weighted object is of smaller dimension than the underlying chart, the basic objects, whose manifold contain W are also blown up along centers of codimension higher than one. This means that the blowing up operation for them is not an isomorphism. Nevertheless, it is still much more efficient to resolve the "good" weighted object this way than using the stratifying function.

2.7 Equivalence of algorithms

We can formulate the resolution algorithm using the new representation of weighted, tagged and balanced objects, presented in section 2.5, and applying the new computational techniques presented in section 2.6 and in section 2.4. Let this new algorithm be called *Villamayor_resolution3*. We overload function names like *ResolveWeighted*, so that their new versions accept basic objects in the new representation, performing the optimized operations on them. We show that the algorithms *Villamayor_resolution2*, presented in section 2.3, and *Villamayor_resolution3* compute equivalent desingularizations.

Theorem 2.62. *Let the $(\mathbb{A}^n, \mathbf{Z}(f))$ be a hypersurface, where $f \in k[x_1, \ldots, x_n]$. Let the weighted objects in the old (see section 1.4) and in the new (see section 2.5) representations be respectively $\mathcal{W}' = (W = \mathbb{A}^n, J' = \langle f \rangle, b' = 1, E' = (), a' = 0_E)$, and $\mathcal{W} = (W = \mathbb{A}^n, J = \langle (f, 1) \rangle, c = 1, C = (), E = (), a = 0_{E'})$.*

The resolution of the singularities of the hypersurface $(\mathbb{A}^n, \mathbf{Z}(f))$, by the function calls ResolveWeighted$(\mathcal{W}')$ *and* ResolveWeighted$(\mathcal{W})$ *over the natural chart for $\mathbb{A}^n$ presented in section 1.2, compute the same sequence of blowing up and cover operations.*

Proof. We use induction on the newly generated and the transformed objects of the algorithm. Primes always denote objects or data entries in the old representation.

Because $\mathrm{Tr}(\langle f\rangle, 1) = \langle (f, 1)\rangle$, $C = (), E = ()$, by definition 2.41 the generalized ideal I of $\mathcal{W}$ is J, thus the singular loci for $\mathcal{W}'$ and $\mathcal{W}$ coincide by remark 2.23(iii). Since $E' = ()$ and $C = (), E = ()$, the translation of the proper part $\overline{J}'$ (see section 1.4.1) is the same as $\overline{J}$ (see definition 2.39), thus $\mathcal{W}'$ and $\mathcal{W}$ represent the same resolution problem. By theorem 2.47, $\mathcal{W}'$ is resolved iff $\mathcal{W}$ is resolved.

In the in induction step we show that *Villamayor_resolution2* and *Villamayor_resolution3* generate objects that represent the same resolution problem from ones with this property. We also know by theorem 2.47 that at each stage of the algorithms two corresponding objects representing the same resolution problem are resolved (or not) at the same time. We denote the underlying chart of the objects by U.

Assume that $\mathcal{W}'$ and $\mathcal{W}$ represent the same resolution problems. If they are monomial, the procedures of the two algorithms are the same, thus by the induction step along blowing ups (see below) the transformed objects represent the same resolution problems. Otherwise, by lemma 2.55 the new tagged objects $\mathcal{T}'$ and $\mathcal{T}$ represent the same resolution problems.

Assume that $\mathcal{T}'$ and $\mathcal{T}$ represent the same resolution problems. Because of this, their singular loci must be the same, and without the loss of generality we may assume that they are defined by the same polynomials in $k[U]$. We must have $E'_- = \mathbf{Z}\{E_-\}$; therefore the *Support2* algorithm (see section 2.1) computes the same subchart covering of U. Without the loss of generality we may assume that this covering consists of only U, i.e. no subdivision was necessary. Then by lemma 2.56 the new balanced objects $\mathcal{B}'$ and $\mathcal{B}$ represent the same resolution problems.

Assume that $\mathcal{B}'$ and $\mathcal{B}$ represent the same resolution problems. Then in *ResolveBalanced2* we must have $H' = \mathbf{Z}\{H\}$; again the singular loci must be the same, so without the loss of generality we may assume that they are generated by the same polynomials in $k[U]$, and that the hypersurface(s) of maximal contact $(V_1, \ldots, V_k)$ are generated by the same $k[U]$ elements. Then by remark 2.36 the coefficient ideals are zero at the same time; so if blowing up is to be performed, the transforms represent the same resolution problems by the induction step along blowing ups (see below). Otherwise, by lemma 2.57 the new weighted objects $\mathcal{W}'$ and $\mathcal{W}$ represent the same resolution problems.

Assume that $\mathcal{W}'$ and $\mathcal{W}$ represent the same resolution problems, and that the algorithms blow up U along the same permissible centers (see section 1.4), generated by the same elements of IND. The transform $\mathcal{W}'_T$ of $\mathcal{W}'$ and the transform $\mathcal{W}_T$ of $\mathcal{W}$ represent the same resolution problems by lemma 2.59. In the above setup the same holds for tagged objects by lemma 2.60. Balanced objects do not get transformed because they are resolved iff their coefficient objects are resolved. $\square$

Finally, let us present some test data run on both the original and the new algorithm. Table 2.1 shows the benefit of using generalized ideals for representing algebraic sets in the resolution algorithm. Question marks denote computations that failed to complete because the computer algebra system refused to work with the polynomials popped up in intermediate steps of the algorithm. The results show that it is inevitable to use generalized ideals in the algorithm to resolve singularities of higher order.

Equation	t_1	t_2
$x_1^4 - x_1^2 + x_2^2 x_3^2$	6672	6641.1
$x_1^2 x_2^2 x_3^2 - x_1^4 + x_2^4$	63.0	21.1
$(x_1^2 + x_2^2 - 1)^2 x_3^2 - x_3^4$	1760.8	25.2
$x_1 x_2 x_3$	1611.9	51.4
$x_1^2 + x_2^3 x_3^3$	1398.2	187.6
$x_1^6 + x_2^6 - x_1 x_2 x_3$	1579.9	268.7
$(x_1^2 + x_2^2 + x_3^2 - 1)^2 (x_1^2 + (x_2 - 1)^2 - 1)^2$	?	173.2
$x_1^2 x_2^2 x_3^2 - x_1^4 + x_2^4 - x_3^4$	?	250.5
$x_1^4 - x_1^2 x_2 (x_3 - 1)$	?	1162.6
$x_1^6 - x_2^4 + x_1 x_2 x_3^3$	?	59322.0
$(x_1^2 + x_2^2 + x_3^2 - 1)^2 x_4^2 - x_4^4$	?	43.1

Table 2.1: Notation: $t_1(t_2)$ the used processor time in seconds with ordinary (generalized) ideals.

2.8 Elimination of chart redundancies

During the resolution process it can often be the case that when the algorithm refines the covering of the nonsingular ambient variety W, some newly generated charts are subsets of unions of already existing ones. This, of course, introduces extra overhead, since singularities lying in multiple charts must be resolved in all of them.

What we can do is trying to avoid such redundant charts to be generated. In this section we present a simple but powerful method that achieves this goal, and even more. We use the definitions and notation introduced in section 1.2.

We extend the chart data structure with a new entry:

– A list FOCUS $= (f_1, \dots, f_k) \subset k[U]$, which is initially empty.

We also extend the previous list of postulates with the following one:

5 The zero sets of the FOCUS entries of the chart covering form a decomposition of W, where some of the components may be empty. More precisely, assume $W = \bigcup_{i=1}^r U_i$; we require for all global points $\overline{a} \in W$ that there is a unique U_i such that the corresponding local representative $a \in \mathbf{Z}(\mathrm{FOCUS}_{U_i})$.

This postulate is obviously fulfilled by the initial setup.

Since only the cover and the blowing up operations can change the structure of the chart covering, we need to give the transformation of FOCUS only in these operations. The original algorithms with the correctness proofs can be found in section 1.3; therefore we prove compatibility only with the new postulate. In the algorithms we use the notation as in the definition of the chart data.

The *cover* operation takes as input a chart and a finite list $F \subset k[U]$, with the property $1 \in \langle F \rangle$. It returns a list of charts with the properties:

1. Each new chart is isomorphic to an affine open subset of the input chart.
2. The new charts cover the input chart.
3. The postulates are fulfilled by the new charts.

The operation is performed by the algorithm *Cover*.

Algorithm *Cover*((VAR, DEP, IND, PDER, MAP, IMAP, FOCUS), F);
    $L := ()$; $M := ()$;
    for each $f \in F$ do
        VAR$'$:= append x_{m+1} to VAR;
        DEP$'$:= append $x_{m+1}f - 1$ to DEP;
        $R := (-x_{m+1}^2 \partial_k f \mid k = 1, \ldots, n)$;
        PDER$'$:= append R as the $(m+1)$-st row to PDER;
        IMAP$'$:= attach IMAP$(1/f)$ to IMAP for x_{m+1};
        FOCUS$'$:= append M to FOCUS;
        L := attach (VAR$'$, DEP$'$, IND, PDER$'$, MAP, IMAP$'$, FOCUS$'$) to L;
        M := append f to M;
    return L;

Theorem 2.63. *Assume that $W = \bigcup_{i=1}^r U_i$ and the chart covering fulfills postulate 5. Let $U = U_j$ be a chart of this covering, let $f_1, \ldots, f_s \in k[U]$ with $\mathbf{Z}(f_1, \ldots, f_s) = \emptyset$, and let $(U'_1, \ldots, U'_s) = Cover(U, (f_1, \ldots, f_s))$. Then $W = \left(\bigcup_{\substack{i=1,\ldots,r \\ i \neq j}} U_i\right) \cup \left(\bigcup_{k=1}^s U'_k\right)$ also fulfills postulate 5.*

Proof. Let $i_k : U'_k \to U$ be the inclusion maps. We have that $\mathbf{Z}(\mathrm{FOCUS}_{U_i})$ form a decomposition of W; moreover, $i_k(\mathbf{Z}(\mathrm{FOCUS}_{U'_k})) \subseteq \mathbf{Z}(\mathrm{FOCUS}_U)$ for all k. Therefore there cannot be any $\overline{a} \in W$ and $i \neq j$ for which the local representatives $a_{U_i} \in \mathbf{Z}(\mathrm{FOCUS}_{U_i})$ and $a_{U'_k} \in \mathbf{Z}(\mathrm{FOCUS}_{U'_k})$ for some k.

So what we have to show is that $U = \bigcup_{k=1}^s U'_k$ fulfills the postulate (formulated now for $\mathbf{Z}(\mathrm{FOCUS}_U)$). But this is immediate because $\mathbf{Z}(f_1, \ldots, f_s) = \emptyset$ and

$$\begin{aligned}
i_1(\mathbf{Z}(\mathrm{FOCUS}_{U'_1})) &= \mathbf{Z}(\mathrm{FOCUS}_U) \setminus \mathbf{Z}(f_1), \\
i_2(\mathbf{Z}(\mathrm{FOCUS}_{U'_2})) &= \mathbf{Z}(\mathrm{FOCUS}_U, f_1) \setminus \mathbf{Z}(f_2), \\
&\vdots \\
i_s(\mathbf{Z}(\mathrm{FOCUS}_{U'_s})) &= \mathbf{Z}(\mathrm{FOCUS}_U, f_1, \ldots, f_{s-1}) \setminus \mathbf{Z}(f_s).
\end{aligned}$$

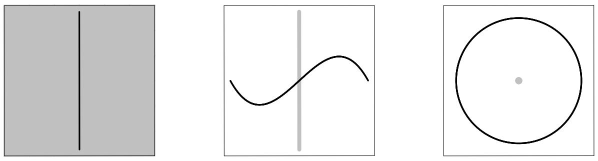
□

Example 2.64. Let us perform cover operation on $\mathbb{A}^2$, given by (VAR, DEP, IND, PDER, MAP, IMAP, FOCUS) $= ((x_1, x_2), (), (x_1, x_2), \mathrm{id}_{2\times 2}, (x_1, x_2), (x_1, x_2), ())$, with its open subsets defined by $x_1 \neq 0, x_1 - x_1^3 - x_2 \neq 0, x_1^2 + x_2^2 - 1 \neq 0$. On the pictures of figure 2.1 the zero sets of the FOCUS component of the charts are grayed, while the missing parts are black. Figure 2.1(a) shows the fist new chart: $((x_1, x_2, x_3), (x_1x_3 - 1), (x_1, x_2), \mathrm{PDER}_1, (x_1, x_2), (x_1, x_2, 1/x_1), ())$, figure 2.1(b) shows the second: $((x_1, x_2, x_3), ((x_1 - x_1^3 - x_2)x_3 - 1), (x_1, x_2),$

$\mathrm{PDER}_2, (x_1, x_2), (x_1, x_2, 1/(x_1 - x_1^3 - x_2)), (x_1))$, and figure 2.1(c) describes the third: $((x_1, x_2, x_3), ((x_1^2+x_2^2-1)x_3-1), (x_1, x_2), \mathrm{PDER}_3, (x_1, x_2), (x_1, x_2, 1/(x_1^2+x_2^2-1)), (x_1, x_1 - x_1^3 - x_2))$.

$$\mathrm{PDER}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x_3^2 & 0 \end{pmatrix}, \qquad \mathrm{PDER}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 3x_1^2x_3^2 - x_3^2 & x_3^2 \end{pmatrix},$$

$$\mathrm{PDER}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2x_1x_3^2 & -2x_2x_3^2 \end{pmatrix}.$$



(a) (b) (c)

Figure 2.1: An atlas of $\mathbb{A}^2$; grey: "relevant" subsets, black: excluded subsets

The *blowing up* operation takes as input a chart and a list of different elements of IND. For the sake of better readability we assume that the blowing up center is given by $(p_1, \ldots, p_s)$. The operation returns a list of charts with the following properties:

1. Each new chart is isomorphic to an affine open subset of the blownup.
2. The new charts cover the blownup.
3. The postulates are fulfilled by the new charts.

The operation is performed by the algorithm *BlowUp*.

Theorem 2.65. *Assume that $W = \bigcup_{i=1}^r U_i$ and the chart covering fulfills the postulate 5. Let $U = U_j$ be a chart of this covering, let $\pi : W' \to W$ be a blowing up along the center in U that is disjoint from U_i for $i \neq j$, given by $p_1, \ldots, p_s \in \mathrm{IND}_U$, and let $(U_1', \ldots, U_s') = BlowUp(U, (p_1, \ldots, p_s))$. Then $W' = \left(\bigcup_{\substack{i=1,\ldots,r \\ i \neq j}} U_i\right) \cup \left(\bigcup_{k=1}^s U_k'\right)$ also fulfills postulate 5.*

Proof. We assume that the blowing up center lies only in U in order to simplify the structure of the blownup. In the general case, the intersection of the center and U_i has to be blown up for each i simultaneously to get the global transformation.

Again, as in theorem 2.63, since $\mathbf{Z}(\mathrm{FOCUS}_{U_i})$ form a decomposition of W and $\pi(\mathbf{Z}(\mathrm{FOCUS}_{U_k'})) \subseteq \mathbf{Z}(\mathrm{FOCUS}_U)$, the new zero sets cannot interfere with the ones in U_i, $i \neq j$. So what we have to show is that $\pi^{-1}(U) = \bigcup_{k=1}^s U_k'$

```
Algorithm BlowUp((VAR, DEP, IND, PDER, MAP, IMAP, FOCUS),
                                                  (p_1, ..., p_r));
    L := ();
    for i from 1 to r do
        VAR' := append (x_{m+j} | j = 1, ..., r, j ≠ i) to VAR;
        DEP' := append (x_{m+j} p_i − p_j | j = 1, ..., r, j ≠ i) to DEP;
        IND' := replace p_j by x_{m+j} in IND for j = 1, ..., r, j ≠ i;
        for j from 1 to m do
            PDER'[j, i] := p_{ji} + Σ_{k=1,k≠i}^{r} x_{m+k} p_{jk};
            for k from 1 to r except i do
                PDER'[j, k] := p_i p_{jk};
            for k from r + 1 to n do
                PDER'[j, k] := p_{jk};
        for j from 1 to r except i do
            R := a list of zeros of length n with a 1 at the jth position;
            PDER' := append R as a new row to PDER';
        IMAP' := attach IMAP(p_j/p_i) to IMAP for x_{m+j}, j = 1, ..., r, j ≠ i;
        if i > 1 then
            FOCUS' := append (x_{m+j} | j = 1, ..., i − 1) to FOCUS;
        else
            FOCUS' := FOCUS;
        L := append (VAR', DEP', IND', PDER', MAP, IMAP',
                                                  FOCUS') to L;
    return L;
```

fulfills the postulate (formulated now for $\pi^{-1}(\mathbf{Z}(\mathrm{FOCUS}_U))$). The answer is analogous to the one in theorem 2.63 since the new charts form the principal open covering of the preimage of U:

$$\begin{aligned}
i_1(\mathbf{Z}(\mathrm{FOCUS}_{U_1'})) &= \mathbf{Z}(\mathrm{FOCUS}_U^*) \setminus \mathbf{Z}(p_1^*),\\
i_2(\mathbf{Z}(\mathrm{FOCUS}_{U_2'})) &= \mathbf{Z}(\mathrm{FOCUS}_U^*, p_1^*) \setminus \mathbf{Z}(p_2^*),\\
&\vdots\\
i_s(\mathbf{Z}(\mathrm{FOCUS}_{U_s'})) &= \mathbf{Z}(\mathrm{FOCUS}_U^*, p_1^*, \ldots, p_{s-1}^*) \setminus \mathbf{Z}(p_s^*),
\end{aligned}$$

where $i_k : U_k' \to \pi^{-1}(U)$ are the inclusion maps, and in the chart U_k, $k = 2, \ldots, s$, we have the pullback of $p_l^* = x_{m+l}$ for $l = 1, \ldots, k-1$.

In the general case, when the blowing up center lies in several charts, analogous arguments can be applied for every local blowing up. Moreover, because the FOCUS entries define a decomposition of W, their preimages define a decomposition of W'. Because of this, the decomposition of the preimages of the FOCUS-subsets of the charts extends to a decomposition of W'. □

Now we have the tool in our hands to deal with redundant charts. It is easy to see that the construction we presented provides in every chart the subset that is "relevant" in the global structure, in the sense that it is not covered by relevant subsets of other charts. This gives rise to the following uses of the FOCUS entry:

1. Whenever a newly generated chart U'_k has $\mathbf{Z}(\mathrm{FOCUS}_{U'_k}) = \emptyset$, we may immediately drop it, because it carries no new information.

2. Whenever the resolution algorithm computes the next blowing up center in a chart U, generated by $p_1, \ldots, p_s \in \mathrm{IND}$, and we have $\mathbf{Z}(p_1, \ldots, p_s) \cap \mathbf{Z}(\mathrm{FOCUS}) = \emptyset$, we may replace the costly blowing up operation with the computationally cheaper cover.

The justification of this latter speedup idea is that, since the blowing up center is disjoint from the relevant subset of the chart, according to postulate 5 there must be other charts whose relevant parts contain it. So there must be a charts where the singularity gets treated.

In the case of point 2, in order to retain the patching of the charts of the atlas without applying blowing up on U and also because the output charts must not contain unresolved singularities, we still have to get rid of the singular points. We achieve this by a seemingly unusual cover operation. Assume the situation as in point 2 and consider the following function call $(U'_1, \ldots, U'_{s+1}) := \mathit{Cover}(U, (p_1, \ldots, p_s, 1))$. All the output charts U'_k for $k = 1, \ldots, s$ are obviously disjoint from the blowing up center, thus from the singularity we were going to resolve. The only questionable chart is the last one, which is isomorphic to U. Nevertheless, we have $\mathrm{FOCUS}_{U'_{s+1}} = \mathrm{FOCUS} + (p_1, \ldots, p_s)$, whose zero set is empty by the assumption; therefore the last chart is to be dropped according to point 1.

The results of the tests on the implementation of the algorithm showed that the number of charts in the atlas of the computed manifold dropped on average by 46% when we applied the above speedup criteria. The used processor time dropped proportionally by 36.7% on average, however, the benefit increased with the complexity of the resolution. Table 2.2 shows that with very simple examples we might find no charts that we can eliminate, thus the computations with FOCUS actually increase the time of the resolution. On the other hand, with nontrivial examples the speedup was over a factor of four.

Equation	n_1	t_1	n_2	t_2
$x_1x_2 - x_3x_4$	8	3.4	8	3.8
$x_1^2x_2 + x_2^2x_3 + x_3^2x_1$	24	12.7	13	11.6
$x_1^2x_2^2x_3^2 - x_1^4 + x_2^4$	24	50.0	9	21.1
$(x_1^2 + x_2^2 - 1)^2x_3^2 - x_3^4$	33	50.8	16	25.2
$x_1^2 - x_2x_3^2$	371	199.7	203	112.7
$x_1^6 + x_2^6 - x_1x_2x_3$	344	1116.9	136	268.7
$(x_1^2 + x_2^2 + x_3^2 - 1)^2(x_1^2 + (x_2 - 1)^2 - 1)^2$	18	225.5	9	173.2
$x_1^2x_2^2x_3^2 - x_1^4 + x_2^4 - x_3^4$	?	>70000.0	14	250.5
$x_1^4 - x_1^2x_2(x_3 - 1)$	?	>100000.0	219	1162.6
$x_1^4 - x_1^2 + x_2^2x_3^2$	?	>100000.0	89	6641.1
$(x_1^2 + x_2^2 + x_3^2 - 1)^2x_4^2 - x_4^4$	48	83.6	23	43.1

Table 2.2: Notation: $n_1(n_2)$ the number of charts in the atlas without (with) the FOCUS used; $t_1(t_2)$ the used processor time in seconds respectively.

2.9 An extended example

We take the hypersurface defined by $x_1^6 - x_2^4 + x_1x_2x_3^3$ in $\mathbb{A}^3$, and follow the original and the optimized algorithms in the resolution process as in section 1.6. The full data of the resolution, whose computation took 57792 seconds on a PIII 450MHz machine, is far too large to be printed on paper. However, the interested reader can take a look at it under the URL `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/gb/e2/tree.html` (unfortunately the huge HTML table appears to be a problem for web browsers to display, thus we would like to ask the reader to use Netscape, with which the errors are minimal).

Here we present one branch of the resolution moving from one chart to another along the operations performed by the algorithms. To make the correspondence of the charts and the basic objects clear we use a double indexing. Upper indices count charts, while lower indices count the basic objects over the same chart. We also use the concise description for the progress of the algorithms introduced in section 1.6. We denote basic objects in the original algorithm with the $\circ$ symbol and in the optimized algorithm with the $\bullet$ symbol in the itemizations.

Because the redundant chart elimination strategy, presented in section 2.8, can also be used with the original algorithm, we utilize it in both cases. We chose a short branch, shown in figure 2.2, for presentation intentionally.

The initial setup consists of the chart U^1: $\mathrm{VAR}^1 = (x_1, x_2, x_3)$, $\mathrm{DEP}^1 = ()$, $\mathrm{IND}^1 = (x_1, x_2, x_3)$, $\mathrm{PDER}^1 = \mathrm{id}_{3\times 3}$, $\mathrm{MAP}^1 = (x_1 \mapsto x_1, x_2 \mapsto x_2, x_3 \mapsto x_3)$, $\mathrm{IMAP}^1 = (x_1 \mapsto x_1, x_2 \mapsto x_2, x_3 \mapsto x_3)$, $\mathrm{FOCUS}^1 = ()$, and the weighted object in the original algorithm:

- $\circ$ $\mathcal{W}_1^1$: $W_1^1 = U^1$, $J_1^1 = \langle x_1^6 - x_2^4 + x_1x_2x_3^3\rangle$, $b_1^1 = 1$, , $E_1^1 = ()$, $a_1^1 = 0_{E_1^1}$,

and in the optimized algorithm

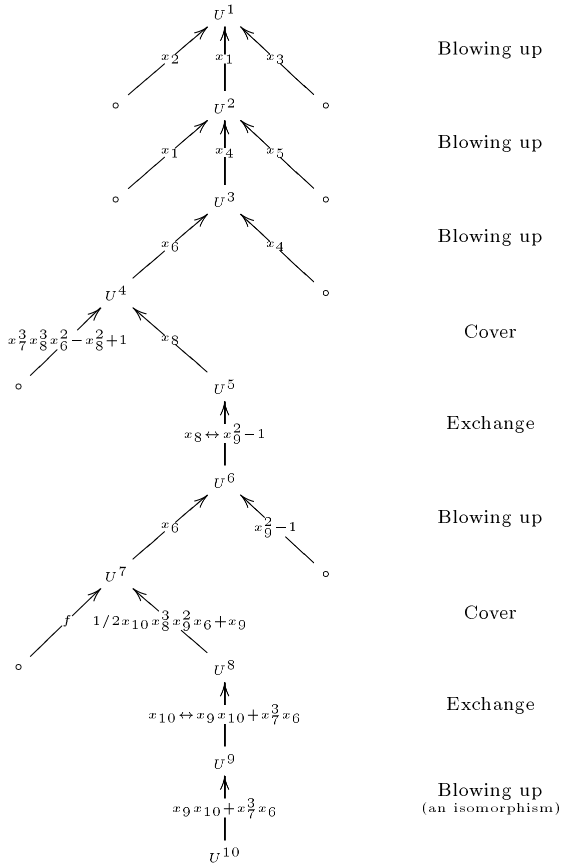
- $\bullet$ $\mathcal{W}_1^1$: $W_1^1 = U^1$, $J_1^1 = \langle (x_1^6 - x_2^4 + x_1x_2x_3^3, 1)\rangle$, $c_1^1 = 1$, $C_1^1 = ()$, $E_1^1 = ()$, $a_1^1 = 0_{E_1^1}$.

The objects are not resolved and not monomial, so the algorithms set up sequences of tagged-, balanced- and weighted objects through the corresponding subroutines. We present them in one stroke:

- $\circ$ $\mathcal{T}_2^1$: $W_2^1 = U^1$, $J_2^1 = \langle x_1^6 - x_2^4 + x_1x_2x_3^3\rangle$, $b_2^1 = 4$, $E_2^1 = ()$, $E_{2-}^1 = ()$,
- $\circ$ $\mathcal{B}_3^1$: $W_3^1 = U^1$, $J_3^1 = \langle x_1^6 - x_2^4 + x_1x_2x_3^3\rangle$, $b_3^1 = 4$, $E_3^1 = ()$, the hypersurface of maximal contact is $V_3^1 = \mathbf{Z}(x_2)$,
- $\circ$ $\mathcal{W}_4^1$: $W_4^1 = V_3{}^1$, $J_4^1 = \langle x_1^36, x_1^8x_3^24\rangle$, $b_4^1 = 24$, $E_4^1 = ()$, $a_4^1 = {}_{E_4^1}$,
- $\circ$ $\mathcal{T}_5^1$: $W_5^1 = V_3{}^1$, $J_5^1 = \langle x_1^36, x_1^8x_3^24\rangle$, $b_5^1 = 32$, $E_5^1 = ()$, $E_{5-}^1 = ()$,
- $\circ$ $\mathcal{B}_6^1$: $W_6^1 = V_3{}^1$, $J_6^1 = \langle x_1^36, x_1^8x_3^24\rangle$, $b_6^1 = 32$, $E_6^1 = ()$, the hypersurface of maximal contact is $V_6^1 = \mathbf{Z}(x_1)$.

At this point the algorithm would have to compute $C(J_6^1, b_6^1, V_6^1)$ which is $\langle (x_3^24)^{32!/(32-8)}\rangle \subset k[U^1 \cap V_3^1 \cap V_6^1]$, but the computer algebra system (understandably) refused to work with such polynomials. Therefore this is the end of the presentation of the computation of the original algorithm.

The optimized version updates $\mathcal{W}_1^1$ and computes the following sequence of objects:



$$f = -1/2x_7^3x_8^3x_6 - 1/2x_7^3x_8^3x_6x_9^2 - 2x_{10}$$

Figure 2.2: The presented path of the resolution

- $\mathcal{W}_1^1$: $W_1^1 = U^1$, $J_1^1 = \langle (x_1^6 - x_2^4 + x_1x_2x_3^3, 4)\rangle$, $c_1^1 = 4$, $C_1^1 = ()$, $E_1^1 = ()$, $a_1^1 = 0_{E_1^1}$
- $\mathcal{T}_2^1$: $W_2^1 = U^1$, $J_2^1 = \langle (x_1^6 - x_2^4 + x_1x_2x_3^3, 4)\rangle$, $c_2^1 = 1$, $C_2^1 = ()$, $E_2^1 = ()$, $E_{2-}^1 = ()$,
- $\mathcal{B}_3^1$: $W_3^1 = U^1$, $J_3^1 = \langle (x_1^6 - x_2^4 + x_1x_2x_3^3, 4)\rangle$, $C_3^1 = ()$, $E_3^1 = ()$, $H_3^1 = ()$, the hypersurface of maximal contact is $V_3^1 = \mathbf{Z}(x_2)$,
- $\mathcal{W}_4^1$: $W_4^1 = V_3^1$, $J_4^1 = \langle (x_1^6, 16/3), (x_1x_3^3, 4)\rangle$, $c_4^1 = 4/3$, $C_4^1 = ()$, $E_4^1 = ()$, $a_4^1 = 0_{E_4^1}$
- $\mathcal{T}_5^1$: $W_5^1 = {V_3}^1$, $J_5^1 = \langle (x_1^6, 16/3), (x_1x_3^3, 4)\rangle$, $c_5^1 = 1$, $C_5^1 = ()$, $E_5^1 = ()$, $E_{5-}^1 = ()$,
- $\mathcal{B}_6^1$: $W_6^1 = {V_3}^1$, $J_6^1 = \langle (x_1^6, 16/3), (x_1x_3^3, 4)\rangle$, $C_6^1 = ()$, $E_6^1 = ()$, $H_6^1 = ()$, the hypersurface of maximal contact is $V_6^1 = \mathbf{Z}(x_1)$,
- $\mathcal{W}_7^1$: $W_7^1 = V_6^1$, $J_7^1 = \langle (x_3^3, 3)\rangle$, $c_7^1 = 1$, $C_7^1 = ()$, $E_7^1 = ()$, $a_7^1 = 0_{E_4^1}$

- $\mathcal{T}_8^1$: $W_8^1 = {V_6}^1$, $J_8^1 = \langle (x_3^3, 3) \rangle$, $c_8^1 = 1$, $C_8^1 = ()$, $E_8^1 = ()$, $E_{8-}^1 = ()$,

- $\mathcal{B}_9^1$: $W_9^1 = {V_6}^1$, $J_9^1 = \langle (x_3^3, 3) \rangle$, $C_9^1 = ()$, $E_9^1 = ()$, $H_9^1 = ()$, the hypersurface of maximal contact is $V_9^1 = \mathbf{Z}(x_3)$, which leads to a blowing up.

Please note that the weighted objects got the generalized ideals updated, where the c components contain the update.

The blowing up center is $\mathbf{Z}(x_2, x_1, x_3)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_1, U^2: $\mathrm{VAR}^2 = (x_1, x_4, x_5)$, $\mathrm{DEP}^2 = ()$, $\mathrm{IND}^2 = (x_1, x_4, x_5)$, $\mathrm{PDER}^2 = \mathrm{id}_{3\times 3}$, $\mathrm{MAP}^2 = (x_1 \mapsto x_1, x_2 \mapsto x_4 x_1, x_3 \mapsto x_5 x_1)$, $\mathrm{IMAP}^2 = (x_1 \mapsto x_1, x_4 \mapsto x_2/x_1, x_5 \mapsto x_3/x_1)$, $\mathrm{FOCUS}^2 = (x_4)$. Notice that the generators x_2, x_3 were eliminated by the algebraic dependencies $x_4 x_1 - x_2, x_5 x_1 - x_3$ using the technique described in section 1.2.

The transform of $\mathcal{W}_1^1$ is $\mathcal{W}_1^2$: $W_1^2 = U^2$, $J_1^2 = \langle (x_4 x_5^3 x_1 - x_4^4 + x_1^2, 4) \rangle$, $c_1^2 = 4$, $C_1^2 = ()$, $E_1^2 = (\mathbf{Z}(x_1))$, $a_1^2 = (\mathbf{Z}(x_1) \mapsto 3)$. The other objects got resolved therefore they are dropped and the recursion is returned to the resolution of $\mathcal{W}_1^2$, which is neither resolved nor monomial, so the algorithm updates it, and creates new objects to improve the resolution of $\mathcal{W}_1^2$. The sequence of objects up to the next blowing up is:

- $\mathcal{W}_1^2$: $W_1^2 = U^2$, $J_1^2 = \langle (x_4 x_5^3 x_1 - x_4^4 + x_1^2, 2) \rangle$, $c_1^2 = 2$, $C_1^2 = ()$, $E_1^2 = (x_1)$, $a_1^2 = (x_1 \mapsto 3)$,

- $\mathcal{T}_2^2$: $W_2^2 = U^2$, $J_2^2 = \langle (x_4 x_5^3 x_1 - x_4^4 + x_1^2, 2) \rangle$, $c_2^2 = 1$, $C_2^2 = ()$, $E_2^2 = (x_1)$, $E_{2-}^2 = (x_1)$,

- $\mathcal{B}_3^2$: $W_3^2 = U^2$, $J_3^2 = \langle (x_4 x_5^3 x_1 - x_4^4 + x_1^2, 2) \rangle$, $C_3^2 = ()$, $E_3^2 = ()$, $H_3^2 = (x_1)$, the hypersurface of maximal contact is $V_3^1 = \mathbf{Z}(x_1)$,

- $\mathcal{W}_4^2$: $W_4^2 = V_3^2$, $J_4^2 = \langle (-x_4^4, 4), (x_4 x_5^3, 2) \rangle$, $c_4^2 = 2$, $C_4^2 = ()$, $E_4^2 = ()$, $a_4^2 = 0_{E_4^2}$,

- $\mathcal{T}_5^2$: $W_5^2 = {V_3}^2$, $J_5^2 = \langle (-x_4^4, 4), (x_4 x_5^3, 2) \rangle$, $c_5^2 = 1$, $C_5^2 = ()$, $E_5^2 = ()$, $E_{5-}^2 = ()$,

- $\mathcal{B}_6^2$: $W_6^2 = {V_3}^2$, $J_6^2 = \langle (-x_4^4, 4), (x_4 x_5^3, 2) \rangle$, $C_6^2 = ()$, $E_6^2 = ()$, $H_6^2 = ()$, the hypersurface of maximal contact is $V_3^1 = \mathbf{Z}(x_4)$,

- $\mathcal{W}_7^2$: $W_7^2 = V_6^2$, $J_7^2 = \langle (x_5^3, 3) \rangle$, $c_4^2 = 3$, $C_7^2 = ()$, $E_7^2 = ()$, $a_7^2 = 0_{E_7^2}$,

- $\mathcal{T}_8^2$: $W_8^2 = {V_6}^2$, $J_8^2 = \langle (x_5^3, 3) \rangle$, $c_8^2 = 1$, $C_8^2 = ()$, $E_8^2 = ()$, $E_{8-}^2 = ()$,

- $\mathcal{B}_9^2$: $W_9^2 = {V_6}^2$, $J_9^2 = \langle (x_5^3, 3) \rangle$, $C_9^2 = ()$, $E_9^2 = ()$, $H_9^2 = ()$, the hypersurface of maximal contact is $V_3^1 = \mathbf{Z}(x_5)$, which leads to a blowing up.

Please note the slight change of types, namely that the E entries, according to the definitions given in section 2.5, contain now the generators of the exceptional divisors, not the hypersurfaces themselves.

The blowing up center is $\mathbf{Z}(x_1, x_4, x_5)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_4, U^3: $\mathrm{VAR}^3 = (x_4, x_6, x_7)$, $\mathrm{DEP}^3 = ()$, $\mathrm{IND}^3 = (x_6, x_4, x_7)$,

$$\mathrm{PDER}^3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathrm{MAP}^3 = (x_1 \mapsto x_6 x_4, x_2 \mapsto x_4^2 x_6, x_3 \mapsto x_7 x_4^2 x_6)$, $\mathrm{IMAP}^3 = (x_4 \mapsto x_2/x_1, x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2)$, $\mathrm{FOCUS}^3 = (x_4, x_6)$.

The non-resolved basic objects of U^2 are transformed in U^3 as follows:

- $\mathcal{W}_1^3$: $W_1^3 = U^3$, $J_1^3 = \langle (x_6 x_7^3 x_4^3 - x_4^2 + x_6^2, 2) \rangle$, $c_1^3 = 2$, $C_1^3 = ()$, $E_1^3 = (x_6, x_4)$, $a_1^3 = (x_6 \mapsto 3, x_4 \mapsto 4)$,
- $\mathcal{T}_2^3$: $W_2^3 = U^3$, $J_2^3 = \langle (x_6 x_7^3 x_4^3 - x_4^2 + x_6^2, 2) \rangle$, $c_2^3 = 1$, $C_2^3 = ()$, $E_2^3 = (x_6, x_4)$, $E_{2-}^3 = (x_6)$,
- $\mathcal{B}_3^3$: because balanced objects are resolved iff their coefficient objects are, we do not need to compute the transform for them; the preimage of V_3^2 is $V_3^3 = \mathbf{Z}(x_6)$,
- $\mathcal{W}_4^3$: $W_4^3 = {V_3}^3$, $J_4^3 = \langle (1, 0) \rangle$, $c_4^3 = 0$, $C_4^3 = ()$, $E_4^3 = (x_4)$, $a_4^3 = (x_4 \mapsto 1)$.

The object $\mathcal{W}_4^3$ is monomial thus the *ResolveMonomial* subroutine is called, where the blowing up center can easily be determined from a_4^3, leading to the next transformation.

The blowing up center is $\mathbf{Z}(x_6, x_4)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_6, U^4: $\mathrm{VAR}^4 = (x_6, x_7, x_8)$, $\mathrm{DEP}^4 = ()$, $\mathrm{IND}^4 = (x_6, x_8, x_7)$,

$$\mathrm{PDER}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$\mathrm{MAP}^4 = (x_1 \mapsto x_6^2 x_8, x_2 \mapsto x_8^2 x_6^3, x_3 \mapsto x_7 x_8^2 x_6^3)$, $\mathrm{IMAP}^4 = (x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2, x_8 \mapsto x_2^2/x_1^3)$, $\mathrm{FOCUS}^4 = (x_6)$. Notice that $x_4 \in \mathrm{FOCUS}^3$ got replaced with $x_8 x_6$ by the elimination equation $x_8 x_6 - x_4$, becoming a redundant generator in FOCUS^4.

The transform of the non-resolved object in U^4 is $\mathcal{W}_1^4$: $W_1^4 = U^4$, $J_1^4 = \langle (x_7^3 x_8^3 x_6^2 - x_8^2 + 1, 2) \rangle$, $c_1^4 = 2$, $C_1^4 = ()$, $E_1^4 = (x_8, x_6)$, $a_1^4 = (x_8 \mapsto 4, x_6 \mapsto 8)$. This object is not monomial and not resolved, thus it is updated and the a tagged object is generated for its resolution:

- $\mathcal{W}_1^4$: $W_1^4 = U^4$, $J_1^4 = \langle (x_7^3 x_8^3 x_6^2 - x_8^2 + 1, 1) \rangle$, $c_1^4 = 1$, $C_1^4 = ()$, $E_1^4 = (x_8, x_6)$, $a_1^4 = (x_8 \mapsto 4, x_6 \mapsto 8)$,
- $\mathcal{T}_2^4$: $W_2^4 = U^4$, $J_2^4 = \langle (x_7^3 x_8^3 x_6^2 - x_8^2 + 1, 1) \rangle$, $c_2^4 = 1$, $C_2^4 = ()$, $E_2^4 = (x_8, x_6)$, $E_{2-}^4 = (x_8, x_6)$.

This time the *Support2* algorithm has to apply a cover operation in order to have all the exceptional divisors indicated by E_{2-}^4 having common singularities or to have the object resolved in the subcharts (for more details see section 2.1).

In the implementation the *Cover* algorithm drops redundant generators from its second argument to reduce the number of output charts as much as possible.

Because $1 \in \langle x_7^3 x_8^3 x_6^2 - x_8^2 + 1, x_8 \rangle_{k[U^4]}$, we actually do not need the other exceptional divisor from E^4_{2-}. We follow the course of the resolution on the branch where x_8 becomes invertible, U^5: $\mathrm{VAR}^5 = (x_6, x_7, x_8, x_9)$, $\mathrm{DEP}^5 = (x_9x_8 - 1)$, $\mathrm{IND}^5 = (x_6, x_8, x_7)$,

$$\mathrm{PDER}^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -x_9^2 & 0 \end{pmatrix},$$

$\mathrm{MAP}^5 = (x_1 \mapsto x_6^2 x_8, x_2 \mapsto x_8^2 x_6^3, x_3 \mapsto x_7 x_8^2 x_6^3)$, $\mathrm{IMAP}^5 = (x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2, x_8 \mapsto x_2^2/x_1^3, x_9 \mapsto x_1^3/x_2^2)$, $\mathrm{FOCUS}^5 = (x_6)$.

The algorithm updates the $\mathcal{W}_1^5$, $\mathcal{T}_2^5$ objects so that exceptional divisors which are outside U^5 got dropped from the E and E_- entries, and continues the resolution by generating new objects:

- $\mathcal{W}_1^5$: $W_1^5 = U^5$, $J_1^5 = \langle (x_7^3 x_8^3 x_6^2 - x_8^2 + 1, 1) \rangle$, $c_1^5 = 1$, $C_1^5 = ()$, $E_1^5 = (x_6)$, $a_1^5 = (x_6 \mapsto 8)$,
- $\mathcal{T}_2^5$: $W_2^5 = U^5$, $J_2^5 = \langle (x_7^3 x_8^3 x_6^2 - x_8^2 + 1, 1) \rangle$, $c_2^5 = 1$, $C_2^5 = ()$, $E_2^5 = (x_6)$, $E_{2-}^5 = (x_6)$,
- $\mathcal{B}_3^5$: $W_3^5 = U^5$, $J_3^5 = \langle (x_7^3 x_8^3 x_6^2 - x_8^2 + 1, 1) \rangle$, $C_3^5 = ()$, $E_3^5 = ()$, $H_3^5 = (x_6)$,the hypersurface of maximal contact is $V_3^5 = \mathbf{Z}(x_6)$,
- $\mathcal{W}_4^5$: $W_4^5 = {V_3}^5$, $J_4^5 = \langle (1 - x_8^2, 1) \rangle$, $c_4^5 = 1$, $C_4^5 = ()$, $E_4^5 = (x_6)$, $a_4^5 = 0_{E_4^5}$,
- $\mathcal{T}_5^5$: $W_5^5 = {V_3}^5$, $J_5^5 = \langle (1 - x_8^2, 1) \rangle$, $c_5^5 = 1$, $C_5^5 = ()$, $E_5^5 = ()$, $E_{5-}^5 = ()$,
- $\mathcal{B}_6^5$: $W_6^5 = {V_3}^5$, $J_6^5 = \langle (1 - x_8^2, 1) \rangle$, $C_6^5 = ()$, $E_6^5 = ()$, $H_6^5 = ()$, the hypersurface of maximal contact is $V_6^5 = \mathbf{Z}(1 - x_8^2)$.

At this point a generator for V_6^5 has to be included into IND^5, and the exchange of x_8 and $x_9^2 - 1$ is possible ($1 - x_8^2$ and $x_9^2 - 1$ represent the same element of $k[U^5]$).

The output chart of the *Exchange* algorithm is not really new because only the extended data, IND and PDER, is changed, but for the clearer presentation we assign a new index to it: U^6: $\mathrm{VAR}^6 = (x_6, x_7, x_8, x_9)$, $\mathrm{DEP}^6 = (x_9x_8 - 1)$, $\mathrm{IND}^6 = (x_6, x_9^2 - 1, x_7)$,

$$\mathrm{PDER}^6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1/2x_8^3 & 0 \\ 0 & 1/2x_8^3 x_9^2 & 0 \end{pmatrix},$$

$\mathrm{MAP}^6 = (x_1 \mapsto x_6^2 x_8, x_2 \mapsto x_8^2 x_6^3, x_3 \mapsto x_7 x_8^2 x_6^3)$, $\mathrm{IMAP}^6 = (x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2, x_8 \mapsto x_2^2/x_1^3, x_9 \mapsto x_1^3/x_2^2)$, $\mathrm{FOCUS}^6 = (x_6)$.

Of course, the basic objects has also not changed: $\mathcal{W}_1^6 = \mathcal{W}_1^5$, $\mathcal{T}_2^6 = \mathcal{T}_2^5$, $\mathcal{B}_3^6 = \mathcal{B}_3^5$, $\mathcal{W}_4^6 = \mathcal{W}_4^5$, $\mathcal{T}_5^6 = \mathcal{T}_5^5$, $\mathcal{B}_6^6 = \mathcal{B}_6^5$.

Because the coefficient ideal $\mathrm{Coeff}(J_6^6, V_6^6)$ is zero, where V_6^6 is just the preimage of V_6^5 along the open immersion of U^6 into U^5, a blowing up operation is performed. The center is $\mathbf{Z}(x_6, x_9^2 - 1)$, and we follow the course of the resolution along the chart with exceptional divisor given by x_6, U^7: $\mathrm{VAR}^7 = (x_6, x_7, x_8, x_9, x_{10})$, $\mathrm{DEP}^7 = (x_9x_8 - 1, x_{10}x_6 + 1 - x_9^2)$, $\mathrm{IND}^7 = (x_6, x_{10}, x_7)$,

$$\mathrm{PDER}^7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1/2x_{10}x_8^3 & -1/2x_8^3x_6 & 0 \\ 1/2x_{10}x_8^3x_9^2 & 1/2x_8^3x_9^2x_6 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$\mathrm{MAP}^7 = (x_1 \mapsto x_6^2x_8, x_2 \mapsto x_8^2x_6^3, x_3 \mapsto x_7x_8^2x_6^3)$, $\mathrm{IMAP}^7 = (x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2, x_8 \mapsto x_2^2/x_1^3, x_9 \mapsto x_1^3/x_2^2, x_{10} \mapsto (x_1^6 - x_2^4)/(x_2^3x_1^2))$, $\mathrm{FOCUS}^7 = (x_6)$.

The transforms of the non-resolved objects in U^7 are:

- $\mathcal{W}_1^7$: $W_1^7 = U^7$, $J_1^7 = \langle (x_7^3x_8^3x_6 + x_8^2x_{10}, 1)\rangle$, $c_1^7 = 1$, $C_1^7 = ()$, $E_1^7 = (x_6)$, $a_1^5 = (x_6 \mapsto 8)$,
- $\mathcal{T}_2^7$: $W_2^7 = U^7$, $J_2^7 = \langle (x_7^3x_8^3x_6 + x_8^2x_{10}, 1)\rangle$, $c_2^7 = 1$, $C_2^7 = ()$, $E_2^7 = (x_6)$, $E_{2-}^5 = ()$.

The tagged object $\mathcal{T}_2^7$ is not resolved; therefore a balanced object is created:

- $\mathcal{B}_3^7$: $W_3^7 = U^7$, $J_3^7 = \langle (x_7^3x_8^3x_6 + x_8^2x_{10}, 1)\rangle$, $C_3^7 = ()$, $E_3^7 = (x_6)$, $H_3^7 = ()$,the hypersurface of maximal contact is $V_3^7 = \mathbf{Z}(x_7^3x_8^3x_6 + x_8^2x_{10})$.

Again, a generator for V_3^7 has to be included into the system of regular parameters to make blowing up possible. This operation has to be prepared by a cover over U^7. We are in case 2 of the distinction presented in section 1.5, and the reduced set of polynomials for the cover operation is $\{-1/2x_7^3x_8^3x_6 - 1/2x_7^3x_8^3x_6x_9^2 - 2x_{10}, 1/2x_{10}x_8^3x_9^2x_6 + x_9\}$. We follow the course of the resolution over the chart where the second polynomial gets invertible, U^8: $\mathrm{VAR}^8 = (x_6, x_7, x_8, x_9, x_{10}, x_{11})$, $\mathrm{DEP}^8 = (x_9x_8 - 1, x_{10}x_6 + 1 - x_9^2, x_{11}(1/2x_{10}x_8^3x_9^2x_6 + x_9) - 1)$, $\mathrm{IND}^8 = (x_6, x_{10}, x_7)$,

$$\mathrm{PDER}^8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1/2x_{10}x_8^3 & -1/2x_8^3x_6 & 0 \\ 1/2x_{10}x_8^3x_9^2 & 1/2x_8^3x_9^2x_6 & 0 \\ 0 & 1 & 0 \\ \begin{pmatrix} -(1/2x_{10}x_8^3x_9^2 - \\ 3/4x_{10}^2x_8^5x_9^2x_6 + \\ 1/2(x_{10}x_8^3x_9x_6 + 1) \\ x_{10}x_8^3x_9^2)x_{11}^2 \end{pmatrix} & \begin{pmatrix} (-3/4x_{10}x_8^5x_9^2x_6^2 + \\ 1/2(x_{10}x_8^3x_9x_6 + 1) \\ x_8^3x_9^2x_6 + 1/2x_8^3x_9^2x_6)x_{11}^2 \end{pmatrix} & 0 \end{pmatrix},$$

$\mathrm{MAP}^8 = (x_1 \mapsto x_6^2x_8, x_2 \mapsto x_8^2x_6^3, x_3 \mapsto x_7x_8^2x_6^3)$, $\mathrm{IMAP}^8 = (x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2, x_8 \mapsto x_2^2/x_1^3, x_9 \mapsto x_1^3/x_2^2, x_{10} \mapsto (x_1^6 - x_2^4)/(x_2^3x_1^2), x_{11} \mapsto 2x_2^2x_1^3/(3x_1^6 - x_2^4))$, $\mathrm{FOCUS}^8 = (x_{10}, x_6)$.

The basic objects are not changed: $\mathcal{W}_1^8 = \mathcal{W}_1^7$, $\mathcal{T}_2^8 = \mathcal{T}_2^7$, $\mathcal{B}_3^8 = \mathcal{B}_3^7$.

Now over U^8 the exchange of x_{10} and $x_9x_{10} + x_7^3x_6$ is possible ($x_7^3x_8^3x_6 + x_8^2x_{10}$ and $x_9x_{10} + x_7^3x_6$ represent the same element over $k[U^8]$). The output of the exchange operation is U^9: $\mathrm{VAR}^9 = (x_6, x_7, x_8, x_9, x_{10}, x_{11})$, $\mathrm{DEP}^9 = (x_9x_8 -$

$1, x_{10}x_6 + 1 - x_9^2, x_{11}(1/2x_{10}x_8^3x_9^2x_6 + x_9) - 1)$, $\mathrm{IND}^9 = (x_6, x_9x_{10} + x_7^3x_6, x_7)$, $\mathrm{PDER}^9 =$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \begin{pmatrix} -1/2x_{10}x_8^3+ \\ 1/2x_{11}x_8^3x_6(x_7^3+ \\ 1/2x_{10}^2x_8^3x_9^2) \end{pmatrix} & -1/2x_{11}x_8^3x_6 & 3/2x_{11}x_8^3x_6^2x_7^2 \\ \begin{pmatrix} 1/2x_{10}x_8^3x_9^2- \\ 1/2x_{11}x_8^3x_9^2x_6(x_7^3 \\ +1/2x_{10}^2x_8^3x_9^2) \end{pmatrix} & 1/2x_{11}x_8^3x_9^2x_6 & -3/2x_{11}x_8^3x_9^2x_6^2x_7^2 \\ \begin{pmatrix} -x_{11}(x_7^3+ \\ 1/2x_{10}^2x_8^3x_9^2) \end{pmatrix} & x_{11} & -3x_{11}x_6x_7^2 \\ \begin{pmatrix} -(1/2x_{10}x_8^3x_9^2- \\ 3/4x_{10}^2x_8^5x_9^2x_6+ \\ 1/2(x_{10}x_8^3x_9x_6+1) \\ x_{10}x_8^3x_9^2)x_{11}^2 + x_{11}^3 \\ (-3/4x_{10}x_8^5x_9^2x_6^2+ \\ 1/2(x_{10}x_8^3x_9x_6+1) \\ x_8^3x_9^2x_6 + 1/2x_8^3x_9^2x_6) \\ (x_7^3 + 1/2x_{10}^2x_8^3x_9^2) \end{pmatrix} & \begin{pmatrix} -x_{11}^3(-3/4 \\ x_{10}x_8^5x_9^2x_6^2+ \\ 1/2(x_{10}x_8^3x_9x_6 \\ +1)x_8^3x_9^2x_6+ \\ 1/2x_8^3x_9^2x_6) \end{pmatrix} & \begin{pmatrix} 4x_{11}^3(-3/4x_{10} \\ x_8^5x_9^2x_6^2+ \\ 1/2(x_{10}x_8^3x_9x_6 \\ +1)x_8^3x_9^2x_6+ \\ 1/2x_8^3x_9^2x_6)x_6x_7^2 \end{pmatrix} \end{pmatrix},$$

$\mathrm{MAP}^9 = (x_1 \mapsto x_6^2x_8, x_2 \mapsto x_8^2x_6^3, x_3 \mapsto x_7x_8^2x_6^3)$, $\mathrm{IMAP}^9 = (x_6 \mapsto x_1^2/x_2, x_7 \mapsto x_3/x_2, x_8 \mapsto x_2^2/x_1^3, x_9 \mapsto x_1^3/x_2^2, x_{10} \mapsto (x_1^6 - x_2^4)/(x_2^3x_1^2), x_{11} \mapsto 2x_2^2x_1^3/(3x_1^6 - x_2^4))$, $\mathrm{FOCUS}^9 = (x_{10}, x_6)$.

The basic objects are unchanged: $\mathcal{W}_1^9 = \mathcal{W}_1^8$, $\mathcal{T}_2^9 = \mathcal{T}_2^8$, $\mathcal{B}_3^9 = \mathcal{B}_3^8$, and since $\mathrm{Coeff}(J_3^9, V_3^9)$ is zero, where V_3^9 is the preimage of V_3^7 along the open immersion of $U^8 (= U^9)$ into U^7, a blowing up follows.

The center is the regular hypersurface $\mathrm{Z}(x_9x_{10} + x_7^3x_6) \subset U^9$; therefore the blowing up is an isomorphism: $U^{10} = U^9$. The updated transform of the weighted object $\mathcal{W}_1^9$ in U^{10} is

- $\mathcal{W}_1^{10}$: $W_1^{10} = U^{10}$, $J_1^{10} = \langle (x_8^3, 0) \rangle$, $c_1^{10} = 0$, $C_1^{10} = ()$, $E_1^{10} = (x_6, x_9x_{10} + x_7^3x_6)$, $a_1^{10} = (x_6 \mapsto 8, x_9x_{10} + x_7^3x_6 \mapsto 0)$.

This object is monomial and it takes eight more blowing ups to complete its resolution, but since these blowing ups are also isomorphisms (their only effect is to reduce the values of a_1^{10} one by one), we omit to present them.

Eventually we arrive to a chart, isomorphic to U^{10}, which is the member of the atlas of the nonsingular algebraic set of the embedded resolution of $(\mathbb{A}^3, \mathbf{Z}(x_1^6 - x_2^4 + x_1x_2x_3^3))$.

Chapter 3

Unifying sequences of blowing ups

It is known for quite some time that blowing up operations are the elementary building blocks of a broad variety of birational transformations. It is proved that for any birational projective morphism $f : Z \to X$, there exists a sheaf of ideals I on X such that Z is isomorphic to the blownup W in the blowing up $\pi : W \to X$ along I (see Theorem 7.17 on page 166 in [Har77]). The cited proof is non-constructive.

The constructive solution of a specialization of the above problem, when f is given as the composition of consecutive blowing ups, was posed to the author as an open problem by H. Hauser. In this chapter we present an algorithm to compute unifications of given sequences blowing ups.

In section 3.1 we develop the elementary step of unifying two consecutive blowing ups. Then in section 3.2 we present a simple recursive algorithm that unifies a given sequence of blowing ups which is represented by a tree of charts. Finally, in section 3.3 we present an extended example for the method.

3.1 Unification of blowing ups

Let $R = k[W]$ be the coordinate ring of the algebraic set W, where k is a field of zero characteristic. Let $I_1 := \langle f_1, \ldots, f_n \rangle \subseteq R$ be a nonzero ideal. We describe the blowing up of R along I_1 in terms of ring homomorphisms.

Definition 3.1. Using the above notation, let $R_1' = R[y_1, \ldots, y_n]$, and let $\varphi_1 : R_1' \to R[t]$ be a ring homomorphism defined as $a \mapsto a$ for $a \in R$, $y_i \mapsto tf_i$, for $i = 1, \ldots, n$. Let $I_1' := \ker(\varphi_1)$; the blownup ring is $R_1 = R_1'/I_1'$. The ring homomorphism of the blowing up is the natural ring inclusion $R \to R_1$, $a \mapsto a$.

We use the convention that we identify elements in factor structures by their representatives, e.g. we write simply y_i instead of $[y_i]$.

Remark 3.2. The correspondence between definition 3.1 and definition 1.3 is straightforward. Notice that R_1 is just the coordinate ring of W' in the blowing up $\pi_1 : W' \to W$, in the sense of definition 1.3, along the center $\mathbf{Z}(I_1) \subset W$.

The ring R_1 is graded with $\deg_1(a) = 0$ for $a \in R$ and $\deg_1(y_i) = 1$, for $i = 1, \ldots, n$. The well definedness of the grading comes from the fact that I_1 is a homogeneous ideal in the y_is, thus no y_i can be eliminated by a $y_i - a$ polynomial.

Lemma 3.3. *Let $I = \langle f_1, \ldots, f_n \rangle = \langle g_1, \ldots, g_m \rangle$; $\varphi_1 : R[y_1, \ldots, y_n] \to R[t]$, $a \mapsto a$ for $a \in R$, $y_i \mapsto tf_i$, $R_1 = R[y_1, \ldots, y_n]/\ker(\varphi_1)$; $\varphi_2 : R[z_1, \ldots, z_m] \to R[t]$, $a \mapsto a$ for $a \in R$, $z_i \mapsto tg_i$, $R_2 = R[z_1, \ldots, z_m]/\ker(\varphi_2)$. Then we have $R_1 \simeq R_2$.*

Proof. Let $g_i = \sum_{j=1}^n h_{ij} f_j$, for $i = 1, \ldots, m$ and $f_j = \sum_{i=1}^m q_{ji} g_i$ for $j = 1, \ldots, n$. Let $\psi_1 : R_1 \to R_2$ take $a \mapsto a$ for $a \in R$ and $y_j \mapsto \sum_{i=1}^m q_{ji} g_i$; $\psi_2 : R_2 \to R_1$ take $a \mapsto a$ for $a \in R$ and $z_i \mapsto \sum_{j=1}^n h_{ij} f_j$.

We have $\psi_2(\psi_1(y_j)) = \sum_{i=1}^m q_{ji} \sum_{k=1}^n h_{ik} y_k$, and $\varphi_1(\psi_2(\psi_1(y_j))) = tf_j$; therefore $\psi_2(\psi_1(y_j)) = y_j$ modulo $\ker(\varphi_1)$.

We can apply the same argument for $\psi_1(\psi_2(z_i))$, thus ψ_1 is a ring isomorphism from R_1 to R_2 with inverse ψ_2. □

The blownup can also be described by representing the graded ring with its dehomogenizations, giving ring homomorphisms between superrings of the dehomogenized rings. The method corresponds directly to the representation of the blownup algebraic set via an atlas of charts.

Definition 3.4. Using the notation of definition 3.1, the ith dehomogenization of R_1 is $R_1^{(i)} = R_1'/(I_1' + \langle y_i - 1 \rangle)$, for $i = 1, \ldots, n$.

The ring $R_1^{(i)}$ is not $\deg_1$-graded anymore.

Lemma 3.5. *Using the notation of definition 3.4, we have $R_1^{(i)} \simeq R[\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}]$, where we omit $f_i/f_i = 1 \in k$ from the generators.*

Proof. We always have for $i \neq j$, $f_i y_j - f_j y_i \in I_1'$ as the simplest homogeneous relations in $\ker(\varphi_1)$. This means that in $I_1' + \langle y_i - 1 \rangle$, where the new generator provides an elimination equation for y_i, we have $f_i y_j - f_j$, for $j \neq i$.

By assuming $y_i = 1, f_i \neq 0$, the algebraic dependencies between the f_j/f_is and the y_js modulo $\ker(\varphi_1)$ are the same, so the isomorphism between the rings follows naturally: $a \leftrightarrow a$ for $a \in R$, $y_j \leftrightarrow f_j/f_i$ for $j \neq i$. □

In this representation the blownup is given by $R_1^{(i)} \simeq R[\frac{f_2}{f_i}, \ldots, \frac{f_n}{f_i}]$, for $i = 1, \ldots, n$. The ring homomorphisms (that correspond to the chart change maps of atlases) can be defined for superrings of the dehomogenized rings. For $R_1^{(i)}$ and $R_1^{(j)}$ we define $\varphi_{ij} : R_1^{(i)}[(f_j/f_i)^{-1}] \to R_1^{(j)}[(f_i/f_j)^{-1}]$, taking $a \mapsto a$, for $a \in R$, $f_l/f_i \mapsto f_l/f_j(f_i/f_j)^{-1}$, for $l \neq i, j$, $f_j/f_i \mapsto (f_i/f_j)^{-1}$, $(f_j/f_i)^{-1} \mapsto f_i/f_j$.

Definition 3.6. Let $I = \langle h_1, \ldots, h_{s_0} \rangle \subset R_1$ be a homogeneous ideal, where h_j is a form of degree $\deg_1(h_j)$. We define the 0th *inverse saturation* $I_{(0)} = I$, and for $i \in \mathbb{N}$, assuming that $I_{(i)}$ is generated finitely by polynomials $h_{j(i)}$ which are forms of degree $\deg_1(h_{j(i)}) \geq i$, $j = 1, \ldots, s_i$, we set

$$I_{(i+1)} = \langle \{h_{j(i)} \,|\, 1 \leq j \leq s_i,\ \deg_1(h_{j(i)}) > i\} \cup \bigcup_{k=1}^{n} \{y_k h_{j(i)} \,|\, 1 \leq j \leq s_i,\ \deg_1(h_{j(i)}) = i\} \rangle.$$

We learned this well known method from [Sch00]. For an ideal $I \subseteq R_1$ we denote $I^{(i)} = I \cap R_1^{(i)}$ and $I_{(j)}^{(i)} = (I_{(j)})^{(i)}$.

Lemma 3.7. *Using the notation of definition 3.6, we have for any $j \in \mathbb{N}$, $I^{(i)} = I_{(j)}^{(i)}$, for $i = 1, \ldots, n$.*

Proof. We proceed by induction on j. For $j = 0$ the claim is trivial, so assume we have the lemma for $j > 0$. Now observe that $I_{(j+1)}^{(i)}$ contains all the generators of $I_{(j)}^{(i)}$, because dehomogenization with respect to y_i cancels the multiplication with it. On the other hand the rest of the generators of $I_{(j+1)}^{(i)}$ are of the form $y_k h$, where h is a generator of $I_{(j)}^{(i)}$. Therefore, by the induction hypothesis $I_{(j+1)}^{(i)} = I_{(j)}^{(i)} = I^{(i)}$. □

Let the nonzero homogeneous ideal I, as in definition 3.6, describe the second blowing up center. Let $R_2' = R_1[z_1, \ldots, z_{s_0}]$, and $\varphi_2 : R_2' \to R_1[t_1, t_2]$ a ring homomorphism defined as $a \mapsto a$ for $a \in R_1$, $z_j \mapsto h_j t_1^{\deg_1(h_j)} t_2$, for $j = 1, \ldots, s_0$. Let $I_2' = \ker(\varphi_2)$; the blownup ring is $R_2 = R_2'/I_2'$.

The ring R_2 is doubly graded with $\deg_1(a) = 0$ for $a \in R$, $\deg_1(y_i) = 1$, $\deg_1(z_j) = \deg_1(h_j)$; $\deg_2(a) = 0$ for $a \in R_1$ and $\deg_2(z_j) = 1$.

Let $\varphi : R_1 \to R$ be a ring homomorphism defined as $a \mapsto a$ for any $a \in R$ and $y_i \mapsto f_i$ for $i = 1, \ldots, n$. The dehomogenization of the jth generator of I with respect to the ith homogeneous variable results $h_j^{(i)} = h_{j*}/f_i^{\deg_1(h_j)}$, where $h_{j*} = \varphi(h_j)$.

Lemma 3.8. *Using the notation introduced so far, we have that the blownup whose coordinate ring is R_2 is being represented via the patching local blownups of $R_1^{(i)}$ along $I^{(i)}$.*

Proof. The dehomogenization of the blownup ring of $R_1^{(i)}$ along $I^{(i)}$ with respect to $h_j^{(i)}$ is

$$R_2^{(i,j)} \simeq R_1^{(i)}[\frac{h_1^{(i)}}{h_j^{(i)}}, \ldots, \frac{h_{s_0}^{(i)}}{h_j^{(i)}}] = R[\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}, \frac{h_{1*}}{h_{j*}} \frac{f_i^{\deg_1(h_j)}}{f_i^{\deg_1(h_1)}}, \ldots, \frac{h_{s_0*}}{h_{j*}} \frac{f_i^{\deg_1(h_j)}}{f_i^{\deg_1(h_{s_0})}}].$$

The dehomogenization of R_2 with respect to z_j is

$$R_2^{(j)} \simeq R_1[\frac{h_1}{h_j}, \ldots, \frac{h_{s_0}}{h_j}],$$

which is graded by $\deg_1$. Its dehomogenization with respect to y_i yields

$$R[\frac{h_{1*}}{h_{j*}} \frac{f_i^{\deg_1(h_j)}}{f_i^{\deg_1(h_1)}}, \ldots, \frac{h_{s_0*}}{h_{j*}} \frac{f_i^{\deg_1(h_j)}}{f_i^{\deg_1(h_{s_0})}}, \frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}] \simeq R_2^{(i,j)}.$$

□

Lemma 3.9. *Using the notation of definition 3.6, the blownup rings of R_1 belonging to the centers I and $I_{(j)}$ are are coordinate rings of isomorphic algebraic sets for any $j \in \mathbb{N}$.*

Proof. According to lemma 3.8, the blownup of R_1 along I and $I_{(j)}$ can be represented by the blownups of $R_1^{(i)}$ along $I^{(i)}$ and $I_{(j)}^{(i)}$ for $i = 1, \ldots, n$. But by lemma 3.7 these blowing ups are exactly the same. □

Let $d = \max_{j=1,\ldots,s_0}(\deg_1(h_j))$ for I, the ideal of the second blowing up center. Then we set $I_2 = \langle g_1, \ldots, g_m \rangle = I_{(d)}$, the dth inverse saturation of I, which has $\deg_1(g_j) = d$, thus $g_j^{(i)} = g_{j*}/f_i^d$, for $j = 1, \ldots, m$.

We describe the blowing up of R_1 along I_2, which is essentially the same as the blowing up of R_1 along I, by lemma 3.9. Let $R_2' = R_1[z_1, \ldots, z_m]$, and $\varphi_2 : R_2' \to R_1[t]$ a ring homomorphism defined as $a \mapsto a$ for $a \in R_1$, and $z_i \mapsto tg_i$, for $i = 1, \ldots, m$. Let $I_2' = \ker(\varphi_2)$, and $R_2 = R_2'/I_2'$.

The ring R_2 is doubly graded with $\deg_1(a) = 0$ for $a \in R$, $\deg_1(y_i) = 1$, $\deg_1(z_i) = d$; $\deg_2(a) = 0$ for $a \in R_1$ and $\deg_2(z_i) = 1$.

If we expand the second blowing up according to lemma 3.8, we get that the jth dehomogenization of the blownup of $R_1^{(i)}$ is

$$R_2^{(i,j)} \simeq R_1^{(i)}[\frac{g_{1*}}{g_{j*}}, \ldots, \frac{g_{m*}}{g_{j*}}] = R[\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}, \frac{g_{1*}}{g_{j*}}, \ldots, \frac{g_{m*}}{g_{j*}}],$$

where g_{j*}/g_{j*} and f_i/f_i are omitted from the generators.

Consider now the blowing up of R along the ideal

$$I_3 = \langle f_1, \ldots, f_n \rangle \langle g_{1*}, \ldots, g_{m*} \rangle = \langle f_1 g_{1*}, \ldots, f_1 g_{m*}, \ldots, f_n g_{1*}, \ldots, f_n g_{m*} \rangle.$$

We describe the blowing up of R along I_3 using definition 3.1. Let $R_3' = R[v_{1,1}, \ldots, v_{n,m}]$, and $\varphi_3 : R_3' \to R[t]$ a ring homomorphism defined as $a \mapsto a$ for $a \in R$, and $v_{i,j} \mapsto tf_i g_{j*}$, for $i = 1, \ldots, n,\ j = 1, \ldots, m$. Let $I_3' = \ker(\varphi_3)$, and $R_3 = R_3'/I_3'$. The ring R_3 is graded with $\deg_3(a) = 0$ for $a \in R$ and $\deg_3(v_{i,j}) = 1$.

Theorem 3.10. *Using the notation introduced so far, we have that R_2 and R_3 are coordinate rings of isomorphic algebraic sets.*

Proof. The blowing up of R along I_3 in the dehomogenized representation is

$$\begin{aligned}
R_3^{(1,1)} &= R[\frac{f_1 g_{2*}}{f_1 g_{1*}}, \ldots, \frac{f_1 g_{m*}}{f_1 g_{1*}}, \ldots, \frac{f_n g_{1*}}{f_1 g_{1*}}, \ldots, \frac{f_n g_{m*}}{f_1 g_{1*}}], \\
&\vdots \\
R_3^{(1,m)} &= R[\frac{f_1 g_{1*}}{f_1 g_{m*}}, \ldots, \frac{f_1 g_{m-1*}}{f_1 g_{m*}}, \ldots, \frac{f_n g_{1*}}{f_1 g_{m*}}, \ldots, \frac{f_n g_{m*}}{f_1 g_{m*}}], \\
&\vdots \\
R_3^{(n,1)} &= R[\frac{f_1 g_{1*}}{f_n g_{1*}}, \ldots, \frac{f_1 g_{m*}}{f_n g_{1*}}, \ldots, \frac{f_n g_{2*}}{f_n g_{1*}}, \ldots, \frac{f_n g_{m*}}{f_n g_{1*}}], \\
&\vdots \\
R_3^{(n,m)} &= R[\frac{f_1 g_{1*}}{f_n g_{m*}}, \ldots, \frac{f_1 g_{m*}}{f_n g_{m*}}, \ldots, \frac{f_n g_{1*}}{f_n g_{m*}}, \ldots, \frac{f_{n-1} g_{m-1*}}{f_n g_{m*}}].
\end{aligned}$$

We omit $f_i g_{j*}/f_i g_{j*} = 1 \in k$ from the generators of $R_3^{(i,j)}$. We observe that the generators of $R_3^{(i,j)}$, for any $i = 1, \ldots, n,\ j = 1, \ldots, m$ contain

$$\frac{f_l}{f_i} = \frac{f_l g_{j*}}{f_i g_{j*}} \quad l = 1, \ldots, n,\ l \neq i,$$
$$\frac{g_{o*}}{g_{j*}} = \frac{f_i g_{o*}}{f_i g_{j*}} \quad o = 1, \ldots, m,\ o \neq j,$$

and that we can eliminate all the other generators by the equations

$$\frac{f_l g_{o*}}{f_i g_{j*}} = \left(\frac{f_l}{f_i}\right)\left(\frac{g_{o*}}{g_{j*}}\right) \quad l \neq i \text{ and } o \neq j.$$

We got $R_3^{(i,j)} = R_2^{(i,j)}$, so because these rings are the same subrings of the total quotient ring of the blownup (i.e. the ring homomorphisms for the corresponding localizations of $R_3^{(i,j)}$ and $R_3^{(k,l)}$ are the same as for $R_2^{(i,j)}$ and $R_2^{(k,l)}$), the blownup of R along I_3 is the coordinate ring of an algebraic set that is isomorphic to the one with coordinate ring R_2, the blownup of R_1 along I_2, where R_1 is the blownup of R along I_1. $\square$

The *ComputeUnifiedCenter* algorithm summarizes the results of this section. Its input is: R, the ring of the ideal of the first blowing up center I_1; R_1, the blownup ring with φ_1 as in definition 3.1; I, the homogeneous ideal of the second blowing up center in R_1 and $d \in \mathbb{N}$, the degree up to which the inverse saturation should go (when unifying two blowing ups d can be set to the maximal degree in which I is generated). The output is an ideal in R, with the property that the blowing up along it is isomorphic to the composition of the two blowing ups corresponding to the input centers. We remark that the deg function in the algorithm assigns $\deg(a) = 0$ for $a \in R$ and $\deg(y_i) = 1$.

Algorithm *ComputeUnifiedCenter*$(R, I_1 = \langle f_1, \ldots, f_n\rangle,$
    $R_1 = R[y_1, \ldots, y_n]/\ker(\varphi_1), I = \langle h_1, \ldots, h_s\rangle,\ d)$
  $I_2 = I_{(d)}$;
  $I_{2*} =$ substitute $\{y_1 \mapsto f_1, \ldots, y_n \mapsto f_n\}$ in I_2;
  return $I_1 \cdot I_{2*}$;

3.2 Computing global centers

To be able to compute the ideal of a given sequence of blowing ups (represented as a tree of charts in which the charts of each intermediate level form atlases for intermediate blownups), we have to compute global blowing up centers from local data. For this we assume that the local ideals define sheafs over the blownups.

Let $\pi : W \to U$ be a blowing up as in definition 1.3, along the center $\mathbf{Z}(J)$, with $J \subset k[U]$, $W = \bigcup_{i=1}^{n} U_i$, and let $\mathbf{Z}(I_i)$ be the centers of the next blowing up, with $I_i \subseteq k[U_i]$. We need to compute $I \subseteq k[W]$ with the property that $I^{(i)} = I_i$ in $k[U_i]$ for all $i = 1, \ldots, n$.

Definition 3.11. Let $R = k[x_1, \dots, x_m]$,

$$f = \sum_{i=0}^{d} \sum_{d_1+\cdots+d_m=i} c_{d_1,\dots,d_m} x_1^{d_1} \dots x_m^{d_m}$$

in R, with $d, d_j \in \mathbb{N}$. The *homogenization* of f with respect to a new indeterminate y is

$$f^{[y]} = \sum_{i=0}^{d} \sum_{d_1+\cdots+d_m=i} c_{d_1,\dots,d_m} x_1^{d_1} \dots x_m^{d_m} y^{d-i}$$

in $R[y]$. For an ideal $I \subseteq R$ we set $I^{[y]} = \langle f^{[y]} \mid f \in I \rangle_{R[y]}$.

We have $k[U_i] \simeq k[W]^{(i)}$; we denote the homogenization of I_i with respect to the ith new generator (conventionally y_i) of $k[W]$ with $I_i^{[i]}$.

Lemma 3.12. *Using the notation introduced so far, if there exists $I' \subseteq k[W]$ such that ${I'}^{(i)} = I_i$ then we have for $I = \bigcap_{i=1}^n I_i^{[i]} \subseteq k[W]$, $I^{(i)} = I_i$.*

Proof. Let $k[W] = R[y_1, \dots, y_n]$ graded in the y_is. Let $f \in k[W]$, $d = \deg(f)$, $d_i = \deg(f^{(i)})$ for any $1 \leq i \leq n$, then we have $f = (f^{(i)})^{[i]} y_i^{d-d_i}$.

Assume $f^{(i)} \in I_i$, then we have $f \in I$ because the above equality generates f in $I_i^{[i]}$ for any i; therefore $f^{(i)} \in I^{(i)}$. On the other hand, let $f^{(i)} \in I^{(i)}$, which means $f \in I$, but then $f \in I_j^{[j]}$ for all $j = 1, \dots, n$, resulting $f^{(i)} \in I_i$. ☐

Remark 3.13. In the context of lemma 3.12, according to lemma 3.8 the blowing up along I is isomorphic to the blowing up along I'.

In certain situations, like in the resolution algorithm, we may wish to change the representation of the global algebraic set by sub-chart covering (see section 1.3). Let $U = \bigcup_{i=1}^n U_i$, computed in a *Cover* operation along $f_1, \dots, f_n \in k[U]$, and let the algorithm compute $\mathbf{Z}(I_i)$, the centers of the next blowing ups over the charts. We need to compute $I \subseteq k[U]$ such that $\langle \varphi_i(I) \rangle = I_i$, where $\varphi_i : k[U] \to k[U_i]$ is the natural ring inclusion.

We have $k[U_i] \simeq k[U][y]/\langle f_i y - 1 \rangle$, where y is a new indeterminate and $f_i \in k[U]$ defines the open subset of U with which U_i is isomorphic. We can easily compute $\varphi_i^{-1}(I_i)$ by eliminating y from I_i using the method of Gröbner basis (see for instance [BW93]).

Lemma 3.14. *Using the above notation, if there exists $I \subseteq k[U]$ such that $\langle \varphi_i(I) \rangle = I_i$ for all $i = 1, \dots, n$, we have $I = \bigcap_{i=1}^n \varphi_i^{-1}(I_i)$.*

Proof. We remark that because y depends on the generators of $k[U]$ in $k[U_i]$, $\langle \varphi_i(\varphi_i^{-1}(I_i)) \rangle = \langle I_i \cap k[U] \rangle_{k[U][y]/\langle f_i y - 1 \rangle} = I_i$.

If $f \in I$, $\varphi_i(f) \in I_i$, and because φ_i is just a ring inclusion having $\varphi_i(f) \in I_i \cap k[U]$, $f = \varphi_i^{-1}(\varphi_i(f)) \in \varphi_i^{-1}(I_i)$ for all i. On the other hand, if $f \in \bigcap_{i=1}^n \varphi_i^{-1}(I_i)$, $\varphi_i(f) \in I_i = \langle \varphi_i(I) \rangle$, thus $f = \varphi_i^{-1}(\varphi_i(f)) \in I$. ☐

The results are summarized in the *Unify* algorithm, which takes as input the tree of charts generated by consecutive blowing ups of patching local centers at each intermediate stage. The algorithm forks to compute the unified centers for each subtree rooted in a child of the root of the input tree. From that data it

Algorithm *Unify*(T, l)
    $U = \mathrm{root}(T)$;
    if $\mathrm{children}(U) = \emptyset$ then
        return $\langle 1 \rangle$;
    if the operation on U is *BlowUp*, $\pi : W \to U$
        for each T_i maximal subtree in T such that $\mathrm{root}(T_i) \in \mathrm{children}(U)$ do
            $I_i =$ fork *Unify*$(T_i, l+1)$;
        $I = \bigcap_i (I_i)^{[i]}$;
        $d = \max(\deg_W(h_i) \mid \langle h_1, \ldots, h_s \rangle = I)$;
        $\mathrm{export}(d, l)$;
        $d_l = \max(\mathrm{synchronize}(l))$;
        $I_1 =$ the ideal of the blowing up center in U;
        $I_2 =$ *ComputeUnifiedCenter*$(k[U], I_1, k[W], I, d_l)$;
        return $I_1 \cdot I_2$;
    else (the operation on U is *Cover* with $\varphi_i : k[U] \to k[\mathrm{child}_i(U)]$)
        for each T_i maximal subtree in T such that $\mathrm{root}(T_i) \in \mathrm{children}(U)$ do
            $I_i =$ fork *Unify*(T_i, l);
        return $\bigcap_i \varphi_i^{-1}(I_i)$;

computes the ideal of the blowing up center on the $l+1$st local level (where intermediate subchart covering does not count as new level).

Then it exports the maximal degree of polynomials in a finite generating set of this ideal for the other threads computing the lth level of the tree. The degree computation is with respect to the generators of $k[W]$ that are introduced by the blowing up.

In the next line these threads are synchronized so that the degree for inverse saturation can be determined all over the $l+1$st level (whose local data form an atlas for the $l+1$st blownup). The "synchronize" function returns the list of exported degrees over the given level. Because the original ideals define a sheaf over the $l+1$st blownup, their inverse saturation up to the same $\deg_W$-degree will also form a sheaf.

Finally, the coherently computed local ideal of the unified blowing up of the $l+1$st and the lth level is returned.

In the process of the construction of the input tree of the algorithm one has to take special care of the naming of the generators of the k-algebras. It is desirable that each new level introduces its own new set of variables, with respect to which the inverse saturation takes place. The naming of the generators introduced by local refinements of the charts should also not interfere with the names introduced by blowing ups.

Remark 3.15. The *Villamayor_resolution* algorithm resolves the singularities of the input hypersurface by blowing up patching local nonsingular centers. Thus the tree of charts that records the whole sequence of *BlowUp* and *Cover* operations, with additional "global timestamps" and modifications in the variable naming, can be used as input for the *Unify* algorithm.

3.3 An extended example

The example presents first the blowing up of the union of two coordinate axes in three-space using the radical vanishing ideal of the center. This blowing up results a singular blownup, where singularity is the at the origin of one of the affine charts. In the second step we describe the blowing up of this singular point, getting now a regular blownup. Finally, we compute the unified blowing up that stands for the composition of the first two transformations.

Let $R = k[x_1, x_2, x_3]$. Blow up $I_1 = \langle x_1x_2, x_3\rangle \subset R$, i.e. the radical ideal for the center consisting of the union of two coordinate axes (x_1, x_2) in $\mathbb{A}^3$.

We get the blownup ring $R_1 = k[x_1, x_2, x_3, y_0, y_1]/\langle x_1x_2z_1 - x_3z_0\rangle$, with charts

- $R_{1,1} = k[x_1, x_2, x_3, z_1]/\langle x_1x_2z_1 - x_3\rangle \simeq k[x_1, x_2, x_3, \frac{x_3}{x_1x_2}] \simeq k[x_1, x_2, \frac{x_3}{x_1x_2}]$,
- $R_{1,2} = k[x_1, x_2, x_3, z_0]/\langle x_1x_2 - x_3z_0\rangle \simeq k[x_1, x_2, x_3, \frac{x_1x_2}{x_3}]$.

The ring homomorphism $\varphi_1 : R_1 \to R$ takes $x_i \mapsto x_i$, $z_0 \mapsto x_1x_2$, $z_1 \mapsto x_3$. The grading is $\deg_{R_1}(x_i) = 0$, $\deg_{R_1}(z_j) = 1$.

The second chart is singular, and the a Gröbner basis of the generators of the points that are at least of order two is $\{x_1, x_2, x_3, z_0\}/\langle x_1x_2 - x_3z_0\rangle$. The singular point is the origin of U_2, which lies in the unique variety of the chart, that is, the center is disjoint from U_1.

Let us blow up the ideal $\langle x_1, x_2, x_3, z_0\rangle \subset R_{1,2}$. We get

$$\begin{aligned} R_2 = k[x_1, x_2, x_3, z_0, y_0, y_1, y_2, y_3]/\langle & z_0y_2 - x_3y_3, y_0y_1 - y_3y_2, z_0y_1 - x_2y_3, \\ & y_1x_3 - y_2x_2, y_1x_1 - x_3y_3, z_0y_0 - x_1y_3, x_3y_0 - y_2x_1, \\ & y_0x_2 - x_3y_3, x_1x_2 - x_3z_0\rangle, \end{aligned}$$

with charts

- $R_{2,1} = k[x_1, x_2, x_3, z_0, y_1, y_2, y_3]/\langle z_0y_2 - x_3y_3, y_1 - y_3y_2, z_0y_1 - x_2y_3, y_1x_3 - y_2x_2, y_1x_1 - x_3y_3, z_0 - x_1y_3, x_3 - y_2x_1, x_2 - x_3y_3, x_1x_2 - x_3z_0\rangle \simeq$
$$k[x_1, x_2, x_3, \frac{x_1x_2}{x_3}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_2}{x_3}] \simeq k[x_1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_2}{x_3}],$$
- $R_{2,2} = k[x_1, x_2, x_3, z_0, y_0, y_2, y_3]/\langle z_0y_2 - x_3y_3, y_0 - y_3y_2, z_0 - x_2y_3, x_3 - y_2x_2, x_1 - x_3y_3, z_0y_0 - x_1y_3, x_3y_0 - y_2x_1, y_0x_2 - x_3y_3, x_1x_2 - x_3z_0\rangle \simeq$
$$k[x_1, x_2, x_3, \frac{x_1x_2}{x_3}, \frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{x_1}{x_3}] \simeq k[x_2, \frac{x_1}{x_2}, \frac{x_3}{x_2}, \frac{x_1}{x_3}],$$
- $R_{2,3} = k[x_1, x_2, x_3, z_0, y_0, y_1, y_3]/\langle z_0 - x_3y_3, y_0y_1 - y_3, z_0y_1 - x_2y_3, y_1x_3 - x_2, y_1x_1 - x_3y_3, z_0y_0 - x_1y_3, x_3y_0 - x_1, y_0x_2 - x_3y_3, x_1x_2 - x_3z_0\rangle \simeq$
$$k[x_1, x_2, x_3, \frac{x_1x_2}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_1x_2}{x_3^2}] \simeq k[x_3, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_1x_2}{x_3^2}],$$
- $R_{2,4} = k[x_1, x_2, x_3, z_0, y_0, y_1, y_2]/\langle z_0y_2 - x_3, y_0y_1 - y_2, z_0y_1 - x_2, y_1x_3 - y_2x_2, y_1x_1 - x_3, z_0y_0 - x_1, x_3y_0 - y_2x_1, y_0x_2 - x_3, x_1x_2 - x_3z_0\rangle \simeq$
$$k[x_1, x_2, x_3, \frac{x_1x_2}{x_3}, \frac{x_3}{x_2}, \frac{x_3}{x_1}, \frac{x_3^2}{x_1x_2}] \simeq k[x_1, x_2, \frac{x_1x_2}{x_3}, \frac{x_3}{x_2}, \frac{x_3}{x_1}].$$

The ring homomorphism $\varphi_2 : R_2 \to R_1$ takes $x_i \mapsto x_i$, $z_0 \mapsto z_0$, $y_j \mapsto x_{j+1}$ for $j = 0, 1, 2$ and $y_3 \mapsto z_0$. Now the global algebraic set, represented by $R_{1,1}, R_{2,1}, R_{2,2}, R_{2,3}, R_{2,4}$, is nonsingular. The global blownup can also be computed by blowing up R_1 along the homogeneous ideal $I_2 = \langle x_1, x_2, x_3, z_0\rangle \subset R_1$, getting

$$\begin{aligned} R_3 = k[x_1, x_2, x_3, z_0, z_1, y_0, y_1, y_2, y_3]/\langle & z_0y_2 - x_3y_3, z_0y_1 - x_2y_3, \\ & y_1x_3 - y_2x_2, z_0y_0 - x_1y_3, x_3y_0 - y_2x_1, y_0x_2 - y_1x_1, \\ & z_1y_1y_0 - y_3y_2, x_1z_1y_1 - x_3y_3, x_1x_2z_1 - x_3z_0\rangle. \end{aligned}$$

The ring homomorphism is $\varphi_3 : R_3 \to R_1$ takes $x_i \mapsto x_i$, $z_j \mapsto z_j$, $y_j \mapsto x_{j+1}$ for $j = 0, 1, 2$ and $y_3 \mapsto z_0$. The gradings are $\deg_{1,R_3}(x_i) = 0$, $\deg_{1,R_3}(z_j) = 1$, $\deg_{1,R_3}(y_k) = \deg_{R_1}(\varphi_3(y_j))$; $\deg_{2,R_3}(x_i) = 0$, $\deg_{2,R_3}(z_j) = 0$, $\deg_{2,R_3}(y_k) = 1$.

Let us determine the ideal J in R whose blowing up is isomorphic with the above construction. We have the blowing up center in the global algebraic set of R_1 generated by the homogeneous ideal I_2. The inverse saturation of I_2 requires only one step, since it is generated in maximal homogeneous degree one (according to $\deg_{R_1}$). We take $I_2 \cap S_0$ generated by $\{x_1, x_2, x_3\} \subset R_1$ and the generators of of degree one with respect to $\deg_{R_1}$: z_0, z_1. Their product is

$$I_2' = \langle x_1z_0, x_2z_0, x_3z_0, x_1z_1, x_2z_1, x_3z_1\rangle,$$

thus the saturated ideal (after simplification) is

$$I_{2S} = \langle x_1z_1, x_2z_1, x_3z_1, z_0\rangle.$$

Finally, we compute $J = I_1 \cdot \varphi_1(I_{2S})$

$$\begin{aligned} J = & \langle x_1x_2, x_3\rangle \cdot \langle x_1x_3, x_2x_3, x_3^2, x_1x_2\rangle = \\ & \langle x_1^2x_2x_3, x_1x_2^2x_3, x_1x_2x_3^2, x_1^2x_2^2, x_1x_3^2, x_2x_3^2, x_3^3, x_1x_2x_3\rangle = \\ & \langle x_1^2x_2^2, x_1x_2x_3, x_1x_3^2, x_2x_3^2, x_3^3\rangle. \end{aligned}$$

The blowing up of R along J is

$$\begin{aligned} R_4 = k[x_1, x_2, x_3, y_0, y_1, y_2, y_3, y_4]/\langle & y_3y_2 - y_4y_1, x_3y_3 - x_2y_4, \\ & x_3y_2 - x_1y_4, y_2x_2 - x_1y_3, y_1x_3 - x_1y_3, x_1y_3^2 - x_2y_4y_1, x_1y_3y_1 - y_4y_0, \\ & x_2y_1^2 - y_3y_0, x_1y_1^2 - y_0y_2, x_1x_2y_1 - x_3y_0, x_1^2x_2y_3 - x_3^2y_0, x_1^2x_2^2y_4 - x_3^3y_0\rangle, \end{aligned}$$

with charts

- $R_{4,1} = k[x_1, x_2, x_3, y_1, y_2, y_3, y_4]/\langle y_3y_2 - y_4y_1, x_3y_3 - x_2y_4, x_3y_2 - x_1y_4, y_2x_2 - x_1y_3, y_1x_3 - x_1y_3, x_1y_3^2 - x_2y_4y_1, x_1y_3y_1 - y_4, x_2y_1^2 - y_3, x_1y_1^2 - y_2, x_1x_2y_1 - x_3, x_1^2x_2y_3 - x_3^2, x_1^2x_2^2y_4 - x_3^3\rangle \simeq$

$$k[x_1, x_2, x_3, \frac{x_3}{x_1x_2}, \frac{x_3^2}{x_1x_2^2}, \frac{x_3^2}{x_1^2x_2}, \frac{x_3^3}{x_1^2x_2^2}] \simeq k[x_1, x_2, \frac{x_3}{x_1x_2}] \simeq R_{1,1},$$

- $R_{4,2} = k[x_1, x_2, x_3, y_0, y_2, y_3, y_4]/\langle y_3y_2 - y_4, x_3y_3 - x_2y_4, x_3y_2 - x_1y_4, y_2x_2 - x_1y_3, x_3 - x_1y_3, x_1y_3^2 - x_2y_4, x_1y_3 - y_4y_0, x_2 - y_3y_0, x_1 - y_0y_2, x_1x_2 - x_3y_0, x_1^2x_2y_3 - x_3^2y_0, x_1^2x_2^2y_4 - x_3^3y_0\rangle \simeq$

$$k[x_1, x_2, x_3, \frac{x_1x_2}{x_3}, \frac{x_3}{x_2}, \frac{x_3}{x_1}, \frac{x_3^2}{x_1x_2}] \simeq k[x_1, x_2, \frac{x_1x_2}{x_3}, \frac{x_3}{x_2}, \frac{x_3}{x_1}] \simeq R_{2,4},$$

- $R_{4,3} = k[x_1, x_2, x_3, y_0, y_1, y_3, y_4]/\langle y_3 - y_4 y_1, x_3 y_3 - x_2 y_4, x_3 - x_1 y_4, x_2 - x_1 y_3, y_1 x_3 - x_1 y_3, x_1 y_3^2 - x_2 y_4 y_1, x_1 y_3 y_1 - y_4 y_0, x_2 y_1^2 - y_3 y_0, x_1 y_1^2 - y_0, x_1 x_2 y_1 - x_3 y_0, x_1^2 x_2 y_3 - x_3^2 y_0, x_1^2 x_2^2 y_4 - x_3^3 y_0\rangle \simeq$

$$k[x_1, x_2, x_3, \frac{x_1 x_2^2}{x_3^2}, \frac{x_2}{x_3}, \frac{x_2}{x_1}, \frac{x_3}{x_1}] \simeq k[x_1, \frac{x_2}{x_3}, \frac{x_2}{x_1}, \frac{x_3}{x_1}] \simeq R_{2,1},$$

- $R_{4,4} = k[x_1, x_2, x_3, y_0, y_1, y_2, y_4]/\langle y_2 - y_4 y_1, x_3 - x_2 y_4, x_3 y_2 - x_1 y_4, y_2 x_2 - x_1, y_1 x_3 - x_1, x_1 - x_2 y_4 y_1, x_1 y_1 - y_4 y_0, x_2 y_1^2 - y_0, x_1 y_1^2 - y_0 y_2, x_1 x_2 y_1 - x_3 y_0, x_1^2 x_2 - x_3^2 y_0, x_1^2 x_2^2 y_4 - x_3^3 y_0\rangle \simeq$

$$k[x_1, x_2, x_3, \frac{x_1^2 x_2}{x_3^2}, \frac{x_1}{x_3}, \frac{x_1}{x_2}, \frac{x_3}{x_2}] \simeq k[x_2, \frac{x_1}{x_3}, \frac{x_1}{x_2}, \frac{x_3}{x_2}] \simeq R_{2,2},$$

- $R_{4,5} = k[x_1, x_2, x_3, y_0, y_1, y_2, y_3]/\langle y_3 y_2 - y_1, x_3 y_3 - x_2, x_3 y_2 - x_1, y_2 x_2 - x_1 y_3, y_1 x_3 - x_1 y_3, x_1 y_3^2 - x_2 y_1, x_1 y_3 y_1 - y_0, x_2 y_1^2 - y_3 y_0, x_1 y_1^2 - y_0 y_2, x_1 x_2 y_1 - x_3 y_0, x_1^2 x_2 y_3 - x_3^2 y_0, x_1^2 x_2^2 - x_3^3 y_0\rangle \simeq$

$$k[x_1, x_2, x_3, \frac{x_1^2 x_2^2}{x_3^3}, \frac{x_1 x_2}{x_3^2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}] \simeq k[x_3, \frac{x_1 x_2}{x_3^2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}] \simeq R_{2,3}.$$

The ring homomorphism is $\varphi_4 : R_4 \to R$ takes $x_i \mapsto x_i$, $y_0 \mapsto x_1^2 x_2^2$, $y_1 \mapsto x_1 x_2 x_3$, $y_2 \mapsto x_1 x_3^2$, $y_3 \mapsto x_2 x_3^2$, $y_4 \mapsto x_3^3$. The grading is $\deg_{R_4}(x_i) = 0$, $\deg_{R_4}(y_j) = 1$.

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