The Finite Element Approximation to the Soap Film Problems on Convex Domains

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Abstract

In this paper, we will discuss the nonlinear finite element discrete schemes to one kind of minimal surface equations which are reduced from soap film probems. The convergence property of the numerical solutions will be proved and this method will be applied to generating the soap film graphically later.

Key Words: Minimal surface, Finite element method, Variational form, Convexity, Brouwers fixed point theorem.

1 Introduction

The study of minimal surfaces is a branch of differential geometry, because the methods of differential calculus are applied to geometrical problems. One of the oldest questions is: "What is the surface of smallest area spanned by a given contour?" The question is nontrivial despite the fact that every physical soap film appears to know the answer.

From the point of view of local geometry, a minimal surface can also be defined as *the surface which has the zero average mean curvature at each inner point*, e.g., a saddle shape.

Previously unknown and certainly unexpected minimal surfaces were found by David Hoffman and his collaborators at GANG, the Center for Geometry, Analysis, Numerics, and Graphics at the University of Massachusetts in 1985 (c.f. [8],[9] and [10] etc.). They first used their MESH computer graphics system

to find these surfaces, and then later proved their existence with fully rigorous mathematics. This truly excited the minimal surface community and piqued their interest in computer graphics.

Many problems in optimal geometry require specialized software systems because there is often no explicit parametrization of a desired minimal surface. Although in the past years, lots of softwares have been produced for generating minimal surfaces graphically, (e.g, the "Surface Evolver"(c.f. [3]) worked out by the geometric center of the University of Minnesota which applies the evolution techniques) there is very few numerical theory or error analysis for the approximation procedures (c.f. [2], etc.). One finite element approximation to the minimal surfaces has been proposed in [12] but its discrete form is still not practically solvable. In this paper we will concentrate on discussing the solution of one kind of the typical minimal surface problems, namely soap film problems by the nonlinear finite element approach in a reasonable way, and then give the proof of its convergence properties of the discrete solution.

In the next section we will first propose the soap film equations and give their variational forms based on both the continuous Sobolev space and the finite element subspace. Some critical error analysis for the standard linear finite element approximation will be given in Section 3, and we will focus on the proving of the convergence property of the discrete solution of the soap film problem mainly by Brouwer's fixed point theorem in Section 4.

2 Preliminaries

Due to the fact that the minimal surface has the property of having zero average mean curvature at each inner surface point (c.f. [22]), we can always define the soap film problems in terms of nonlinear partial differential equations, which satisfying proper given boundary conditions, (e.g., one closed curve in three-dimensional space) on certain domain Ω. (Sometimes we need a rotation in the 3-dimensional space in advance, if possible, to make the border shape one to one correspondence to $\partial\Omega$.)

Especially, if the soap film (a minimal surface) can be represented explicitly by

$$z = u(x, y), \quad (x, y) \in \Omega$$

satisfying the boundary restriction

$$u = f(x, y), \quad (x, y) \in \partial\Omega,$$

then u can be defined by the following system of differential equations:

$$\begin{cases} -div((1+u_y^2)u_x,(1+u_x^2)u_y)+6u_{xy}u_xu_y=0 \ \text{ in } \ \Omega, \\ \quad u=f(x,y), \ \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a strictly convex domain. Here the (strictly) convexity of the domain Ω is especially required for purpose of the finite element analysis. And furthermore, that is sufficient to assume that the solution to the previous equation is unique if $f(x,y)$, which satisfies the bounded slope condition (c.f. [14]), is the restriction to $\partial\Omega$ of a function in the Sobolev space $W_q^2(\Omega)$, for some $q>2$.

Set the nonlinear operator

$$A(u,v)=\int_\Omega[(1+u_y^2)u_xv_x+(1+u_x^2)u_yv_y+6u_{xy}u_xu_yv],$$

where $u_x,u_{xy}...$ denote $\partial_x u,\partial_x(\partial_y u)...$, etc. Then the equation (2.1) has the following variational form:

Looking for $w\in H_0^1(\Omega)\cap H^2(\Omega)\cap C_1(\Omega)$, such that

$$A(w+\bar{f},v)=0, \ \text{ in } \Omega,\forall v\in H_0^1(\Omega)\cap H^2(\Omega)\cap C_1(\Omega) \tag{2.2}$$

where $\bar{f}$ is the "regular" extension of f from $\partial\Omega$ to Ω, i.e., $\bar{f}\in W_q^2(\Omega), q>2$. (Obviously, $w+\bar{f}=u$ according to the previous uniqueness assumption.)

We now consider the discrete scheme for (2.2). Let $\Omega_h\subset\Omega$ be divided by mesh T_h with size h. $S_h(\Omega_h)$ is the finite element subspace of $H_0^1(\Omega_h)\cap H^2(\Omega_h)\cap C_1(\Omega_h)$ defined as

$$S_h(\Omega_h)=\{v\in H_0^1(\Omega_h)\cap H^2(\Omega_h)\cap C_1(\Omega_h): v|_e \text{ are bounded-degree polynomials}$$

$$\text{generated by some } C_1 \ \text{ interpolation}, \forall e\in T_h\}$$

and I_h is the corresponding interpolation operator on $S_h(\Omega_h)$. It is also assumed that the mesh partion is regular and all the grid nodes on $\partial\Omega_h$ also located on $\partial\Omega$.

Then the finite element solution $w_h\in S_h(\Omega)$ is defined by (c.f. [6])

$$A(w_h+\bar{f},v)=0, \ \text{ in } \Omega_h \qquad \forall v\in S_h(\Omega_h). \tag{2.3}$$

and thus $w_h+\bar{f}=u_h$ will be taken as the numerical solution to the soap films (minimal surfaces).

For the continuous solution $w^h=w|_{\Omega_h}\in H_0^1(\Omega_h)\cap H^2(\Omega_h)\cap C_1(\Omega_h)$ also satisfying

$$A(w^h+\bar{f},v)=0, \ \text{ in } \Omega \qquad \forall v\in H_0^1(\Omega_h)\cap H^2(\Omega_h)\cap C_1(\Omega_h)$$

, it is natual to consider the oringinal problem on a certain convex polygon domain and we can still take w^h as w, Ω_h as Ω, etc.,for convenience. Then we will prove that w_h convergence to the real soap film w when the mesh size h goes to zero in the following discussions.

3 Some Error Resolutions for the Linear Finite Element Method

Before we discuss the convergence property of the finite element solution w_h, let's introduce some important fact of the finite element approximation results to the linear problems.

Let L be a linear operator defined on $S_h(\Omega)$ by

$$L(u) = -div(a_{11}u_x + a_{21}u_y, a_{12}u_x + a_{22}u_y) + b_1u_x + b_2u_y + cu,$$

where $a_{ij}, b_i,\ i,j = 1,2,\ c \geq 0$ are continuous functions satisfying

$$a_{11}v_x^2 + a_{22}v_y^2 + (a_{12} + a_{21})v_xv_y \geq c_0(\nabla v)^2 \qquad \forall v \in S_h(\Omega)$$

for some $c_0 > 0$.

It can be infered that the bilinear operator

$$(L(u_h), v) = (f, v) \quad \forall v \in S_h(\Omega),$$

where $(\cdot,\cdot)$ denote the inner product on $L_2(\Omega)$, is an isomorphism. (c.f. [18])

We also define the corresponding variational form as:

$$A'(u,v) = \int_\Omega [a_{11}u_xv_x + a_{22}u_yv_y + (a_{12} + a_{21})u_xv_y + (b_1u_x + b_2u_y + cu)v], \quad (3.1)$$

on continuous space and take P_hu as the discrete solution to the u in (3.1), which satisfying

$$A'(u - P_hu, v) = 0 \quad \forall v \in S_h(\Omega),$$

(usually we named such operator P_h as the Galerkin's projection operator), then we have (c.f. [4],[7],[19] and [26])

Theorem 3.1 *For $h << 1$, P_hu admits the following estimate*

$$\|u - P_hu\|_{1,\infty} \leq Ch\|u\|_{2,\infty}, \quad (3.2)$$

for $C > 0$.

Moreover, we can always define the discrete Green functions as, given $z \in \Omega$, the Green functions $g^z_{h,x}, g^z_{h,y} \in S_h(\Omega)$ satisfying

$$A'(v, g^z_{h,x}) = v_x(z), \quad A'(v, g^z_{h,y}) = v_y(z) \quad \forall v \in S_h(\Omega). \tag{3.3}$$

It has been proved that (c.f. [4],[23] and [26])

$$\|g^z_{h,x}\|_{1,1} \leq C|\log h|, \tag{3.4}$$

$$\|g^z_{h,y}\|_{1,1} \leq C|\log h|, \tag{3.5}$$

for constant $C > 0$.

Last, the inverse inequality will also be used in the next section(c.f. [6]).

Theorem 3.2 *For $h << 1$, and $v \in S_h(\Omega)$, we have*

$$\|v\|_{2,\infty} \leq Ch^{-1}\|v\|_{1,\infty}, \tag{3.6}$$

for $C > 0$.

4 The Finite Element Approximation to the Soap Film Problems

The following lemma will show the sufficient and necessary condition of the finite element solution of (2.3).

Lemma 4.1 *For any w, w_h and $v \in S_h(\Omega)$,*

$$A(w_h + \bar{f}, v) = A(w + \bar{f}, v) + A'(w + \bar{f}; w_h - w, v) + R(w + \bar{f}, w_h + \bar{f}, v), \tag{4.1}$$

where $A'(w + \bar{f}; ., .)$ is the bilinear operator defined from (3.1) when we take the coefficients $a_{ij}, b_i, c,\ i, j = 1, 2$, respectively according to the first argument as

$$a_{11} = 1 + u_y^2,\ a_{22} = 1 + u_x^2,$$

$$a_{12} = a_{21} = -u_y u_x,$$

$$b_1 = 3u_y u_{xy} - 3u_{yy} u_x,$$

$$b_2 = 3u_x u_{xy} - 3u_{xx} u_y,$$

and

$$c = 0.$$

Then $w_h \in S_h(\Omega)$ solves (2.3) if and only if

$$A'(w+\bar{f}; w-w_h, v) = R(w+\bar{f}, w_h+\bar{f}, v), \quad \forall v \in S_h(\Omega). \tag{4.2}$$

Further more, if $\|w_h\|_{2,\infty} \leq K$, then the remainder R satisfying

$$R(w+\bar{f}, w_h+\bar{f}, v) \leq C(K)\|w-w_h\|_{1,\infty}^2 \|v\|_{1,1}. \tag{4.3}$$

Proof. Set $G(t) = A(w+\bar{f}+t(w_h-w), v)$. then it follows from the elementary identity

$$G(1) = G(0) + G_t(0) + \int_0^1 G_{tt}(t)(1-t)dt.$$

It is easy to compute by local intergration that

$$G_t(0) = A'(w+\bar{f}; w_h - w, v)$$

where the coefficients of A' satisfying the given conditions from the lemma. And, by taking

$$R(w+\bar{f}, w_h+\bar{f}, v) = \int_0^1 G_{tt}(t)(1-t)dt.$$

A straightforward calculation shows that

$$\begin{aligned} & |R(w+\bar{f}, w_h+\bar{f}, v)| \\ \leq & \max|G_{tt}(t)| \\ \leq & C(K)\|w-w_h\|_{1,\infty}\|v\|_{1,1}. \end{aligned}$$

Finally, if w_h solves (2.3), then $G(1) = 0$ and complete the proof.

Let P_h be the Galerkin projection with respect to the bilinear from $A'(w+\bar{f}; .,.)$, then we have the following theorem:

Theorem 4.1 *For problems of the form (2.2), there exist a constant $C > 0$, such that for mesh T_h with its size $h << 1$, the corresponding finite element equation (2.3) has a solution w_h satisfying*

$$\|w_h - P_h w\|_{1,\infty} \leq Ch,$$

and thus we have the following error estimation

$$\|w_h - w\|_{1,\infty} \leq Ch$$

Proof of theorem 2.1.

Define a nonlinear operator $\Phi : S_h(\Omega) \to S_h(\Omega)$ by

$$A'(w + \bar{f}, \Phi(v) - w, \phi) = R(w + \bar{f}, v + \bar{f}, \phi) \quad \forall \phi \in S_h(\Omega)$$

It is easy to see by the analysis of operator A' in section 3 that, Φ is well defined and a contiunous operator, according to the result of Theorem 3.1 we can obtain that

$$\|w - P_h w\|_{1,\infty} \leq Ch.$$

Then Define a set $B = \{v \in S_h(\Omega) : \|v - P_h w\|_{1,\infty} \leq Ch\}$, and by the inverse estimation from (3.6), there exist $C_1, C_2, C_3, C_4 > 0$, such that

$$\begin{aligned}
&\|v - w\|_{2,\infty} \\
&\leq \|v - I_h w\|_{2,\infty} + \|I_h w - w\|_{2,\infty} \\
&\leq C_1 h^{-1} \|v - I_h w\|_{1,\infty} + C_2 \\
&\leq C_1 h^{-1} (\|v - P_h w\|_{1,\infty} + \|I_h w - P_h w\|_{1,\infty}) + C_2 \\
&\leq C_3,
\end{aligned}$$

then we get $\|v\|_{2,\infty}$ is uniformly bounded.

Now we can prove that $\Phi(B) \subset B$ which means Φ is a contracting operator from B to B. In fact, when we substitute ϕ by each kind of discrete Green Function $\phi = g^z_{h,x}$ or $\phi = g^z_{h,y}$ into

$$A'(w + \bar{f}; \Phi(v) - P_h w, \phi) = R(w + \bar{f}, v + \bar{f}, \phi),$$

according to Lemma 4.1, (3.4), (3.5) and Poincare inequality (c.f. [6]), for all $v \in B$, we get

$$\begin{aligned}
&\|\Phi(v) - P_h w\|_{1,\infty} \\
&\leq C_0 |\log h| \|w - v\|^2_{1,\infty} \\
&\leq 2C_0 |\log h| (\|P_h w - v\|^2_{1,\infty} + \|w - P_h w\|^2_{1,\infty}) \\
&\leq 2C_0 |\log h| (C_1^2 h^2 + C_1^2 h^2) \\
&\leq Ch
\end{aligned}$$

Where $C_0, C_1, C > 0$.

By Brouwer's fixed point theorem, there exist a solution $w_h \in B$, such that $\Phi(w_h) = w_h$. By the sufficent and necessary condition given in Lemma 4.1, w_h solves (2.3) and satisfies

$$\|w_h - P_h w\|_{1,\infty} \leq Ch.$$

Thus completes the proof.

For uniqueness, let's consider that u_h of should has the minimal area over domain Ω, which is, according to the knowledge base of differential geometry (c.f. [22])

$$Area(u_h) = \int_\Omega (1 + |\nabla u_h|^2)^{1/2}$$

should has the mininal value. By the fact that the given mapping $Area(\cdot)$ is strictly convex (c.f. [12],[14]) and $S_h(\Omega)$ is finite dimensional, it then follows easily that there exists a unique minimizing function u_h.

5 Graphical Examples

To illustrate the features of the approximation approach, we will show some model problems defined on convex domain Ω.

Example 1.

For $\Omega = [0,1] \times [0,1]$, we use the piecewise bilinear finite elements with mesh size $h = 16$. The border curves are the restriction of smooth function $\bar{f} = 0.25 - (x - 0.5)^2$. (see Figure.1) Then the numerical result after five Newton iteration steps to the discrete system is generated in Figure.2

Example 2.

This example shows the convergence property by given the initial border function the restriction of $xy(1-x)(1-y)$ on the same domain (see Figure.3), comparing to the exact continous solution — a trival rectangular plane.

Figure.4 shows the maxium error will not be greater than $2 * 10^{-4}$ between the numerical solution and the exact one.

Both two examples are generated by Maple.V work sheets.

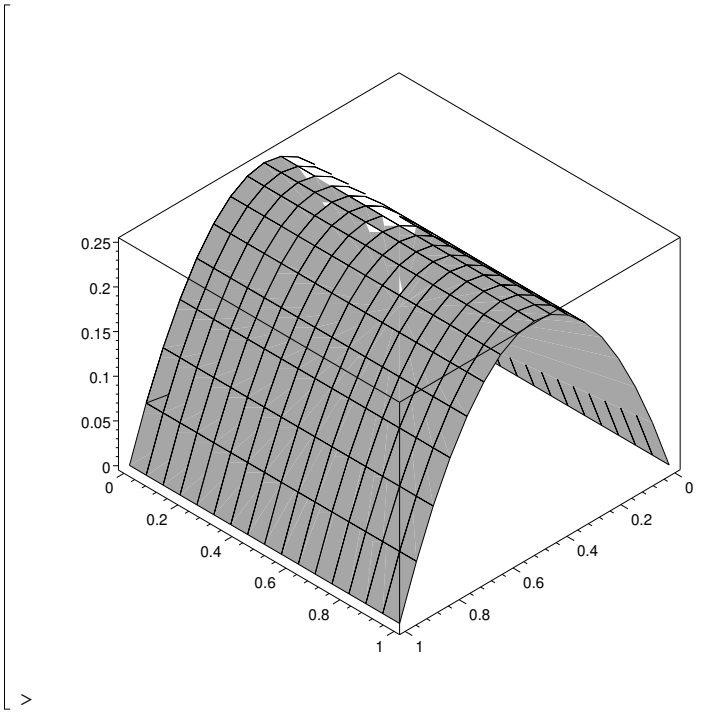

Figure 1: The initial surface with the fixed border

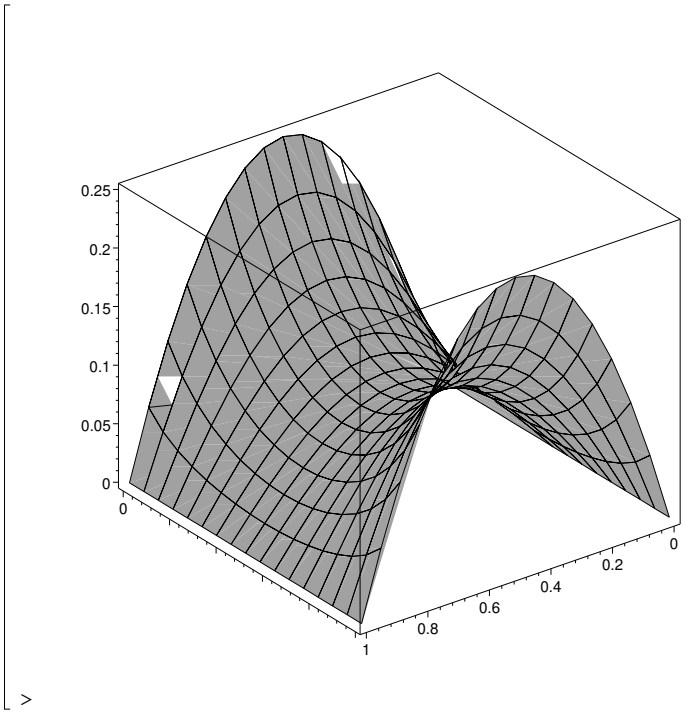

Figure 2: The final form of the soap film

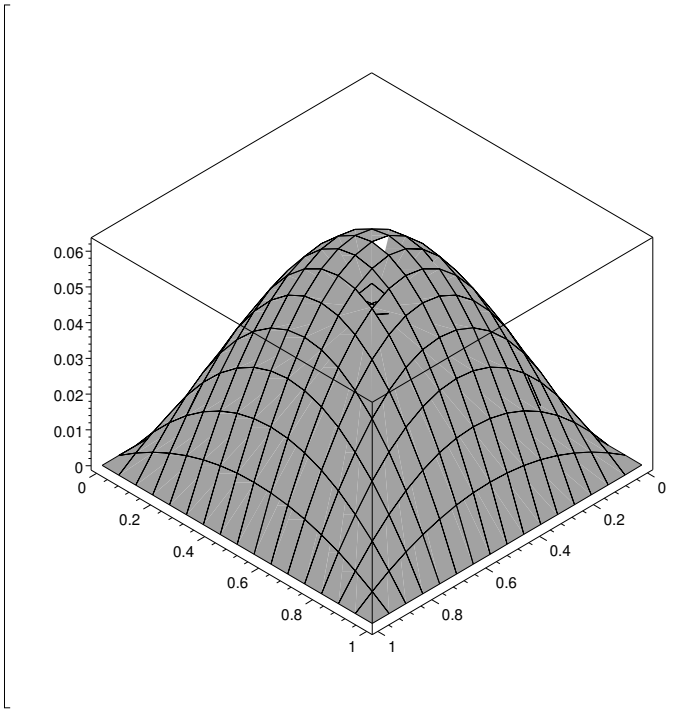

Figure 3: The initial surface with the fixed border

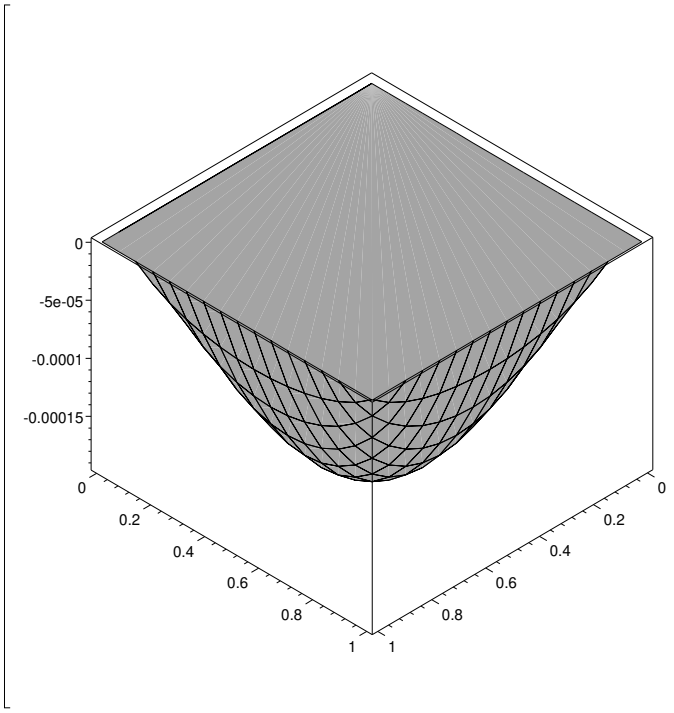

Figure 4: The final form of the soap film

6 Conclusion

In this paper, we discussed the nonlinear finite element method for solving the soap film problems on convex domains, and give the error analysis of the approximation approachs. Since we also proved the uniqueness property of the discrete solution, and the discrete system is practically a systems of polynomails which can be solved either by Newton iteration method or by symbolic eliminate method, then the soap film can be reasonably approximated by such finite element technique and the discrete solution will surely convergence to the real soap film.

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