Normal Parametrizations of Algebraic Plane Curves over Fields of Characteristic Zero*

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Abstract

In this paper we present a theoretical and algorithmic analysis on the normality of rational parametrizations of algebraic plane curves over arbitrary fields of characteristic zero. If the field is algebraically closed we give an algorithm to decide whether a parametrization is proper and, if not, a normal parametrization is computed. If the field is not algebraically closed the problem is more complecated, degenerated situation may appear. We classify the degenerations in strong and weak degenerations, and an algorithm to decide this phenomenon is derived. Furthermore, we prove that if the parametrization is strongly degenerated then the curve can not be normally parametrized, but weak degenerations can be resolved, and an algorithm to reparametrized the input weakly degenerated parametrization into a normal one is given. In addition, we show how these results can be applied and improved to the case of real rational curves.

1 Introduction

Rational curves, and in particular their rational parametric representation, play an important role in many applications in computer aided geometric design (see e.g. [7], [8], [9]). This situation motivates a reciprocal relationship of interes between the fields of applications and development of constructive methods in algebraic geometry. On one hand, geometric problems derived from the applied frame generate interesting questions for algebraic geometers, and on the other hand symbolic algorithmic solutions of problems in algebraic geometry, mainly for curves and surfaces, provide answers that can be used in applications. An example of this affirmation is the construction of

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injective (i.e. proper), at almost every place, rational parametrations (see e.g. [12] or [14]).

A natural question related to the injectivity of parametrizations is the computation of surjective parametrizations, i.e. normal parametrizations, of rational varieties. Any rational parametrization induces a natural rational mapping from the ground field onto the variety that is dominant, that is the image of the mapping is dense in the variety. Thus, in general, the mapping might not be surjetive (i.e. normal), and hence some points of the algebraic set are missed. This phenomenon may generate unexpected complications in applications; for instance in the problem of plotting of geometric objects in the screen of a computer. Therefore, the question of deciding whether a rational parametrization is normal and if not computing a normal parametrization, if possible, arises.

This problem was approach in [4], where a method, based on Ritt-Wu's decomposition algorithm, to decide whether a parametrization of an algebraic variety, over an algebraically closed field of characteristic zero, of arbitrary dimension is normal is given. In addition, normal parametrizations for conics and some quadrics were computed. In [3] normal parametrizations for the remainding quadrics are presented. Also, in [3], the authors provide a method to construct normal parametrizations over the field of real numbers for parametrizations where no real point on the variety corresponds only to complex parameter values. Moreover, they leave the study of the normality of parametrizations, where there exist real points corresponding only to complex parameter values, as an open problem.

In this paper, we deal with the problems described above for the case of rational plane curves over an arbitrary field of characteristic zero.

We start analyzing the question when the ground field is algebraically closed (see Section 2.). From a theoretical point of view, the question is clear: if a parametrization is not normal it can be extended to a projective parametrization, that is surjective, by taking care of the point on the curve that is the image of the infinity of the ground field (see [15] Chapter I, Section 5, Theorem 2). However, we are interested here in approaching the problem algorithmically. In Section 2, we provide a complete characterization for deciding when a parametrization is normal. This result provides an algorithm to decide when a parametrization is normal, that in combination with a theorem due to Prof. Tomás Recio yields to an algorithm to normally parametrized any rational curve over any algebraically closed field of characteristic zero. The geometric idea of this result is to take a projective parametrization that sends the infinity of the field into a point of the curve at infinity. The algorithm for the normality test basically involves gcd computation and therefore is much simpler, from the complexity point of view, than the algorithm presented in [4]. In [5] a similar test algorithm, approched from the theory of rational function fields, is given.

In Section 3, we study the normality for the case of fields that are not algebraically closed. The first remark is that in this case it may happen that points on the variety may only be generated by means of parameter values out of the groung field (see the above comment on the open problem stated in [3]). This yields to the notion of degeneration. We prove that, for the case of plane curves, the set of all points in this degenerated situation are finite, and we provide an algorithm to compute them. Furthermore, we distinguish between two types of degenerations, called strong and weak degeneration. Strong degenerations occur when there is a point different to the critical point of the parametrization (see Definition 3), i.e. different to the image of the infinity by the parametrization, that is only generated by parameter values out of the ground field. Weak degenerations happen when the critical point is the only point in this situation.

These two types of degenerations play a fundamental role in this study since we prove that, on one hand, if strong degeneration appears then the curve can not be normally parametrized, even non–properly, over the ground field, and on the other hand, if weak degeneration appears then the degeneration can be avoided, and the curve can be normally parametrized over the ground field. The geometric intuitive explanation for the strong degeneration behavior, at least for the real case, is that these degenerated points different to the critical point are the singularities of the curve which tangent cone is completely imaginary, and this property does not depend on the parametrization.

Therefore, we give a complete answer to the problem stated in [3] for the case of plane curves and for arbitrary fields. In addition, if the input is not degenerated we prove that the normality over the field and over its algebraic clousure is equivalent.

The last section (Section 3) is devoted to apply an improve these results to the case of the field of real numbers.

In the sequel, we use the following notations. $\mathbb{K}$ is an algebraically closed field of characteristic zero, $\mathbb{L} \subset \mathbb{K}$ is a subfield of $\mathbb{K}$, and $\mathbb{R}$, $\mathbb{C}$ are the field of real numbers and the field of complex numbers, respectively. For a parametrization $\mathcal{P}(t)$ of an affine rational curve $\mathcal{C}$ over $\mathbb{K}$ we write its components as

$$\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)} \right).$$

Also, we will assume in the paper that all rational parametrizations are given in reduced form, that is $\gcd(p_i, q_i) = 1$ for $i = 1, 2$. Furthermore, for any given rational parametrization $\mathcal{P}(t)$ we consider the polynomials

$$H_1^{\mathcal{P}}(x,t) = xq_1(t) - p_1(t), \;\; H_2^{\mathcal{P}}(y,t) = yq_2(t) - p_2(t).$$

Moreover, we assume w.l.o.g. that the given curves are not lines. Note that for lines the study of normal parametrizations is trivial.

2 Normal Over Algebraically Closed Fields.

In this section we study the normality problem for rational curves over algebraically closed fields.

A rational parametrization $\mathcal{P}(t)$ of an affine curve $\mathcal{C}$ over $\mathbb{K}$ induces a dominant rational map $\varphi_{\mathcal{P}}$ from the field $\mathbb{K}$ on the curve. That is,

$$\begin{array}{rccc} \varphi_{\mathcal{P}}: & \mathbb{K} & \longrightarrow & \mathcal{C} \\ & t & \longmapsto & \mathcal{P}(t) \end{array}$$

where $\varphi_{\mathcal{P}}(\mathbb{K})$ is a dense subset of $\mathcal{C}$ in the Zariski topology. However, in general $\varphi_{\mathcal{P}}(\mathbb{K}) \neq \mathcal{C}$, and therefore $\varphi_{\mathcal{P}}$ might not be surjective. Rational parametrizations such that $\varphi_{\mathcal{P}}(\mathbb{K}) = \mathcal{C}$ are called normal. More precisely, one has the following definition.

Definition 1. A parametrization $\mathcal{P}(t)$ is *normal* if for all $P \in \mathcal{C}$ there exists $t_0 \in \mathbb{K}$ such that $\mathcal{P}(t_0) = P$. If there exists a normal parametrization of $\mathcal{C}$ we say that $\mathcal{C}$ can be *normally parametrized*.

In [4] a sufficient condition for a rational parametrization to be normal is given.

Theorem 1. Let $\mathcal{P}(t)$ be a rational parametrization such that $\deg_t(p_i(t)) > \deg_t(q_i(t))$ for some $i \in \{1, 2\}$, then $\mathcal{P}(t)$ is normal.

Proof. See Theorem 2.8. in [4]. □

From this result one inmediatly gets the following corollary

Corollary 1. Any polynomial parametrization, i.e. a parametrization with polynomial components, is normal.

An additional natural question is to decide when the rational map $\varphi_{\mathcal{P}}$ is injective. This leads to the definition of proper parametrization.

Definition 2. A parametrization $\mathcal{P}(t)$ is *proper* if the rational map $\varphi_{\mathcal{P}}$ induced by $\mathcal{P}(t)$ is birational.

Therefore, if $\mathcal{P}(t)$ is proper, then the map $\varphi_{\mathcal{P}}$ is injective for almost all values in $\mathbb{K}$. There exists algorithmic criteria to decide whether a parametrization is proper (see e.g. [12] or [14]). However, the question of deciding when $\varphi_{\mathcal{P}}$ is injective for all elements in $\mathbb{K}$ needs a further analysis. As a first approach, one sees that a necessary condition is that the parametrization is polynomial, since otherwise $\varphi_{\mathcal{P}}$ is not defined for the roots of the denominators. Nevertheless, this condition is not sufficient. For instance, consider the polynomial parametrization $\mathcal{P}(t) = (t^2 + t, t^3 + t^2 + 1)$. It is proper, and therefore injective for almost all values in $\mathbb{K}$. However, $\mathcal{P}(0) = \mathcal{P}(-1) = (0, 1)$.

Applying the previous corollary and Section 3 in [14] on gets directly the following result.

Corollary 2. Let $\mathcal{P}(t)$ be a rational parametrization of an affine plane curve $\mathcal{C}$, let $\varphi_{\mathcal{P}}$ be the rational map induced by $\mathcal{P}(t)$, and let

$$G_i(s,t) = q_i(s)p_i(t) - q_i(t)p_i(t), \quad i = 1, 2.$$

If $\mathcal{P}(t)$ is polynomial and

$$\mathrm{Res}_t\left(\frac{G_1}{gcd(G_1, G_2)}, \frac{G_2}{\gcd(G_1, G_2)}\right) \in \mathbb{K},$$

then $\varphi_{\mathcal{P}}$ is bijective as a map from $\mathbb{K}$ to $\mathcal{C}$.

Remark. In the example given before the corollary the resultant is $s^2 + s$. □

Considering again the question of the surjectivity, we show in the next example that Theorem 1 does not characterize normal parametrizations of curves.

Example 1. We consider the parametrization

$$\mathcal{P}(t) = \left(\frac{t^2-1}{t^3}, \frac{t-1}{t^2}\right)$$

of the plane cubic defined by the polynomial

$$f(x, y) = y^3 + 2y^2 - 3xy + x^2.$$

Clearly, $\mathcal{P}(t)$ does not satisfy the condition in Theorem 1. Nevertheless, let us see that $\mathcal{P}(t)$ is normal. First of all, we observe that $\mathcal{P}(t)$ is a proper parametrization. We compute the inverse of the parametrization (see e.g. [11]), that can be expressed as:

$$\mathcal{P}^{-1}(t) = \frac{2y - x}{y^2}.$$

Thus, one deduces that for every $(a, b) \in \mathcal{C} \setminus \{(0, 0)\}$

$$(a, b) = \mathcal{P}\left(\frac{2b - a}{b^2}\right).$$

Furthermore, $\mathcal{P}(1) = (0, 0)$. Therefore, $\mathcal{P}(t)$ is normal. □

For the case of algebraically closed fields, if $\mathcal{P}(t)$ is not normal it can be extended to a projective parametrization, that is surjective, by taking care of the point on $\mathcal{C}$ that is the image of the infinity of $\mathbb{K}$ (see [15] Chapter I, Section 5, Theorem 2). We approach here the problem of doing that algorithmically.

In [4], Theorem 2.6., the authors describe a method, based on Ritt-Wu's decomposition algorithm to decide whether a parametrization of an algebraic variety of arbitrary dimension is normal. In Theorem 2 we prove that, for the case of plane curves, this can be done by computing a gcd of two univariate polynomials. For this purpose, we first state the following lemma. This algorithm can also be found in [5] Remark 1.6. The approach in [5] is derived using rational functions field theory.

Lemma 1. Let $\mathcal{P}(t)$ be a rational parametrization of an affine plane curve $\mathcal{C}$ with defining polynomial $f(x, y)$, let $\varphi_{\mathcal{P}}$ be the rational map induced by $\mathcal{P}(t)$, and let $\ell_1(x), \ell_2(y)$ be the leading coefficient w.r.t. t of the polynomials $H_1^{\mathcal{P}}(x,t), H_2^{\mathcal{P}}(y,t)$ respectively. Then, it holds that

$$\varphi_{\mathcal{P}}(\mathbb{K}) = \{(a,b) \in \mathcal{C} \mid gcd(H_1^{\mathcal{P}}(a,t), H_2^{\mathcal{P}}(b,t)) \neq 1\}.$$

Furthermore,

$$\mathcal{C} \setminus \varphi_{\mathcal{P}}(\mathbb{K}) \subset \{(a,b) \in \mathbb{K}^2 \,|\, f(a,b) = \ell_1(a) = \ell_2(b) = 0\}.$$

Proof. First, we prove that

$$\varphi_{\mathcal{P}}(\mathbb{K}) = \{(a,b) \in \mathcal{C} \mid gcd(H_1^{\mathcal{P}}(a,t), H_2^{\mathcal{P}}(b,t)) \neq 1\}.$$

Clearly, if $(a,b) \in \varphi_{\mathcal{P}}(\mathbb{K})$ then $(a,b) \in \mathcal{C}$, and there exists $t_0 \in \mathbb{K}$ such that $\mathcal{P}(t_0) = (a,b)$. Thus, t_0 is a common root of $H_1^{\mathcal{P}}(a,t)$ and $H_2^{\mathcal{P}}(b,t))$. Conversely, if $(a,b) \in \mathcal{C}$ and $t_0 \in \mathbb{K}$ is a common root of $H_1^{\mathcal{P}}(a,t)$ and $H_2^{\mathcal{P}}(b,t))$, then $q_1(t_0)q_2(t_0) \neq 0$, since otherwise $H_1^{\mathcal{P}}(a,t_0) = p_1(t_0) = 0$ or $H_2^{\mathcal{P}}(b,t_0) = p_2(t_0) = 0$ which is impossible because $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$. Therefore, $\mathcal{P}(t_0) = (a,b)$, and hence $(a,b) \in \varphi_{\mathcal{P}}(\mathbb{K})$.

Now, for every $(a,b) \in \mathcal{C}$ such that $\ell_1(a)\ell_2(b) \neq 0$ we consider the evaluation homomorphism $\psi_{(a,b)} : \mathbb{K}[x,y][t] \to \mathbb{K}[t]$ defined as $\psi_{(a,b)}(M(x,y,t)) = M(a,b,t)$, and let us assume w.l.o.g. that $\deg_t(H_1^{\mathcal{P}}(x,t)) \geq \deg_t(H_2^{\mathcal{P}}(x,t))$. Then, taking into account the behavior of the resultant under a homomorphism (see e.g. Lemma 7.3.1. in [10]), one has that

$$\psi_{(a,b)}(\mathrm{Res}_t(H_1^{\mathcal{P}}(x,t), H_2^{\mathcal{P}}(y,t))) =$$
$$= \ell_1(a)^{\deg_t(H_1^{\mathcal{P}}(x,t)) - \deg_t(H_2^{\mathcal{P}}(x,t))}\mathrm{Res}_t(H_1^{\mathcal{P}}(a,t), H_2^{\mathcal{P}}(b,t)).$$

Moreover, $\mathrm{Res}_t(H_1^{\mathcal{P}}(x,t), H_2^{\mathcal{P}}(y,t)) = f(x,y)^{\mathrm{degree}(\varphi_{\mathcal{P}})}$ (see [14]). Therefore, since $(a,b) \in \mathcal{C}$ and $\ell_1(a) \neq 0$, one deduces that $\mathrm{Res}_t(H_1^{\mathcal{P}}(a,t), H_2^{\mathcal{P}}(b,t)) = 0$. Thus, $\gcd(H_1^{\mathcal{P}}(a,t), H_2^{\mathcal{P}}(b,t)) \neq 1$. So, one deduces that

$$\{(a,b) \in \mathcal{C} \mid \ell_1(a)\ell_2(b) \neq 0\} \subset \varphi_{\mathcal{P}}(\mathbb{K}),$$

and then

$$\mathcal{C} \setminus \varphi_{\mathcal{P}}(\mathbb{K}) \subset \{(a,b) \in \mathbb{K}^2 \,|\, f(a,b) = \ell_1(a) = \ell_2(b) = 0\}.$$

□

Remark. Theorem 1 follows directly from Lemma 1: if one of the denominators in the parametrization has degree less than the degree of its numerator, then the corresponding ℓ_i is a non-zero constant. Thus, $\mathcal{C} = \varphi_{\mathcal{P}}(\mathbb{K})$. □

The next theorem gives a complete characterization of normal parametrizations.

Theorem 2. Let

$$\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)} \right)$$

with

$$p_1(t) = \sum_{i=0}^{n} a_i t^i,\ q_1(t) = \sum_{i=0}^{m} b_i t^i,\ p_2(t) = \sum_{i=0}^{r} c_i t^i,\ q_2(t) = \sum_{i=0}^{s} d_i t^i,$$

where $a_n \neq 0, b_m \neq 0, c_r \neq 0, d_s \neq 0$, be a rational parametrization of an affine plane curve $\mathcal{C}$. Then, it holds that

(1) If $n > m$ or $r > s$ then $\mathcal{P}(t)$ is normal.

(2) If $n < m$ and $r < s$ then $\mathcal{P}(t)$ is normal if and only if

$$\gcd(H_1^{\mathcal{P}}(0,t), H_2^{\mathcal{P}}(0,t)) \neq 1.$$

Furthermore, if $\mathcal{P}(t)$ is not normal all points in $\mathcal{C}$ are generated by $\mathcal{P}(t)$ with the exception of $(0,0)$ which is a point on $\mathcal{C}$.

(3) If $n = m$ and $r = s$ then $\mathcal{P}(t)$ is normal if and only if

$$\gcd\left(H_1^{\mathcal{P}}\left(\frac{a_n}{b_m}, t\right), H_2^{\mathcal{P}}\left(\frac{c_r}{d_s}, t\right) \right) \neq 1.$$

Furthermore, if $\mathcal{P}(t)$ is not normal all points in $\mathcal{C}$ are generated by $\mathcal{P}(t)$ with the exception of $\left(\frac{a_n}{b_m}, \frac{c_r}{d_s}\right)$ which is a point on $\mathcal{C}$.

(4) If $n < m$ and $r = s$ then $\mathcal{P}(t)$ is normal if and only if

$$\gcd\left(H_1^{\mathcal{P}}(0,t), H_2^{\mathcal{P}}\left(\frac{c_r}{d_s}, t\right) \right) \neq 1.$$

Furthermore, if $\mathcal{P}(t)$ is not normal all points in $\mathcal{C}$ are generated by $\mathcal{P}(t)$ with the exception of $\left(0, \frac{c_r}{d_s}\right)$ which is a point on $\mathcal{C}$.

(5) If $n = m$ and $r < s$ then $\mathcal{P}(t)$ is normal if and only if

$$\gcd\left(H_1^{\mathcal{P}}\left(\frac{a_n}{b_m}, t\right), H_2^{\mathcal{P}}(0,t) \right) \neq 1.$$

Furthermore, if $\mathcal{P}(t)$ is not normal all points in $\mathcal{C}$ are generated by $\mathcal{P}(t)$ with the exception of $\left(\frac{a_n}{d_m}, 0\right)$ which is a point on $\mathcal{C}$.

Proof. Statement (1) is Theorem 1. For the remaining statements, let $\ell_1(x), \ell_2(y)$ be the leading coefficient w.r.t. t of $H_1^{\mathcal{P}}(x,t)$ and $H_2^{\mathcal{P}}(y,t)$, respectively, and let $\mathcal{Q}(t) = \mathcal{P}\left(\frac{1}{t}\right)$. Clearly

$$\mathcal{Q}(t) = \left(\frac{a_n + \cdots + a_0 t^n}{b_m + \cdots + b_0 t^m} t^{m-n}, \frac{c_r + \cdots + c_0 t^r}{d_s + \cdots + d_0 t^s} t^{s-r}\right).$$

In this situation, the proof is as follows:

(2) First observe that the denominators of $\mathcal{Q}(t)$ do not vanish at $t = 0$, since $b_m \neq 0, d_s \neq 0$, and $m - n > 0, s - r > 0$. Thus, the parametrization $\mathcal{Q}$ is defined for $t = 0$, and $\mathcal{Q}(0) = (0,0) \in \mathcal{C}$. Moreover, $\ell_1(x) = x, \ell_2(y) = y$. Therefore, applying Lemma 1, the only point that might not be generated by $\mathcal{P}(t)$ is the origen. Furthermore, by Lemma 1, $(0,0)$ is generated by $\mathcal{P}(t)$ if and only if $\gcd(H_1^{\mathcal{P}}(0,t), H_2^{\mathcal{P}}(0,t)) \neq 1$; that is $\mathcal{P}(t)$ is normal if and only if $\gcd(H_1^{\mathcal{P}}(0,t), H_2^{\mathcal{P}}(0,t)) \neq 1$.

(3) Since $m = n$, $r = s$, $b_m \neq 0, d_s \neq 0$, one has that $\mathcal{Q}(0) = \left(\frac{a_n}{b_m}, \frac{c_r}{d_s}\right) \in \mathcal{C}$. Moreover, $\ell_1(x) = b_m x - a_n, \ell_2(y) = d_s y - c_r$. Therefore, applying Lemma 1 as in the proof of statement (2), one gets statement (3).

(4) Since $m > n$, $r = s$, $b_m \neq 0, d_s \neq 0$, one has that $\mathcal{Q}(0) = \left(0, \frac{c_r}{d_s}\right) \in \mathcal{C}$. Moreover, $\ell_1(x) = x, \ell_2(y) = d_s y - c_r$. Therefore, applying Lemma 1 as in the proof of statement (2), one gets statement (4).

(5) Since $m = n$, $r < s$, $b_m \neq 0, d_s \neq 0$, one has that $\mathcal{Q}(0) = \left(\frac{a_n}{b_m}, 0\right) \in \mathcal{C}$. Moreover, $\ell_1(x) = b_m x - a_n, \ell_2(y) = y$. Therefore, applying Lemma 1 as in the proof of statement (2), one gets statement (5). $\square$

Example 2. We briefly give some examples of normal proper parametrizations for each of the cases discussed in Theorem 2.

(1) Take, for instance, the hyperbola $\left(t, \frac{1}{t}\right)$.

(2) See the cubic in Example 1.

(3) The parametrization

$$\mathcal{P}(t) = \left(\frac{t^2+1}{t^2+2t}, \frac{2t^2+1}{t^2+t}\right)$$

defines the cubic of equation

$$f(x,y) = 9x^2 + 2x - 8xy + 1 - 2y + 2y^2 - 2x^2y + xy^2.$$

For this parametrization

$$\gcd\left(H_1^{\mathcal{P}}\left(\frac{a_n}{b_m}, t\right), H_2^{\mathcal{P}}\left(\frac{c_r}{d_s}, t\right)\right) = 2t - 1,$$

and therefore $\mathcal{P}\left(\frac{1}{2}\right) = (1, 2)$.

(4) The parametrization

$$\mathcal{P}(t) = \left(\frac{t^2 - 1}{t^3 + t^2 + 2}, \frac{t^3 - 2}{t^3 - t^2 - 1}\right)$$

defines the curve of degree 6 of equation

$$f(x, y) = -3 + 24\,x + 9\,y - 61\,xy - 9\,y^2 + 50\,y^2x - 100\,x^2y + 36\,x^2 + 3\,y^3 - 13\,y^3x + 68\,x^3$$
$$+98\,y^2x^2 - 148\,x^3y + 101\,y^2x^3 - 29\,x^3y^3 - 34\,x^2y^3.$$

For this parametrization

$$\gcd\left(H_1^{\mathcal{P}}(0, t), H_2^{\mathcal{P}}\left(\frac{c_r}{d_s}, t\right)\right) = t^2 - 1,$$

and therefore $\mathcal{P}(1) = \mathcal{P}(-1) = (0, 1)$.

(5) Permute in (4) x and y. □

Corollary 3. Let $\mathcal{P}(t)$ be a rational parametrization of an affine plane curve $\mathcal{C}$. Then, at most one point of $\mathcal{C}$ is not generated by the parametrization. Moreover, if $p_i(t), q_i(t)$ are written as in Theorem 2, it holds that

(1) If $n > m$ or $r > s$ then there is no missing point in $\mathcal{C}$.

(2) If $n < m$ and $r < s$ then the only possible missing point in $\mathcal{C}$ is $(0, 0)$.

(3) If $n = m$ and $r = s$ then the only possible missing point in $\mathcal{C}$ is $\left(\frac{a_n}{b_m}, \frac{c_r}{d_s}\right)$.

(4) If $n < m$ and $r = s$ then the only possible missing point in $\mathcal{C}$ is $\left(0, \frac{c_r}{d_s}\right)$.

(5) If $n = m$ and $r < s$ then the only possible missing point in $\mathcal{C}$ is $\left(\frac{a_n}{d_m}, 0\right)$.

Corollary 3 motives the following definition.

Definition 3. Let $\mathcal{P}(t)$ be a rational parametrization in reduced form such that the degree of each numerator is not bigger than the degree of the corresponding denominator, i.e. $\deg_t(q_i(t)) \geq \deg_t(p_i(t)$ for $i = 1, 2$. Then, the only possible missing point of the parametrization is called the *critical point* of $\mathcal{P}(t)$.

Remark. Observe that the notion of critical point is not defined for parametrizations such that at least one of the numerators has bigger degree than its denominator. □

Theorem 2 provides the following algorithm to decide whether a given parametrization is normal (see also in Remark 1.6. [5]). For this purpose, we use the notation: if $p \in \mathbb{K}[t]$ and $k \in \mathbb{N}$, $\mathrm{coeff}(p(t), t^k)$ denotes the coefficient of the term t^k in $p(t)$.

Algorithm: NORMALITY-TEST.

Given a rational parametrization $\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}\right)$, in reduced form, the algorithm decide whether $\mathcal{P}(t)$ is normal.

1. Compute $n := \deg(p_1(t)), m := \deg(q_1(t)), r := \deg(p_2(t)), s := \deg(q_1(t))$.
2. If $n > m$ or $r > s$ then return "$\mathcal{P}(t)$ `is normal`"
3. $a := \mathrm{coeff}(p_1, t^m), b := \mathrm{coeff}(q_1, t^m), c := \mathrm{coeff}(p_2, t^s), d := \mathrm{coeff}(q_2, t^s)$.
4. Compute $M(t) := \gcd(aq_1 - bp_1, cq_2 - dp_2)$.
5. If $\deg_t(M) \geq 1$ then return "$\mathcal{P}(t)$ `is normal`" else return "$\mathcal{P}(t)$ `is not normal and the only point on` $\mathcal{C}$ `that is not reachable is` $\left(\frac{a}{b}, \frac{c}{d}\right)$".

Example 3. Let us consider the affine conic of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \;\; a \neq 0, b \neq 0$$

and the standart proper parametrization

$$\mathcal{P}(t) = \left(\frac{a\,(t^2-1)}{1+t^2}, 2\,\frac{bt}{1+t^2}\right).$$

Applying algorithm NORMALITY-TEST, in Step 4, one gets that

$$M(t) := \gcd(2a, -2bt) = 1.$$

Therefore, $\mathcal{P}(t)$ is not normal and the only point of the affine conic that is not reachable is $(a, 0)$, that is the critical point of the parametrization. □

We have already approached the problem of deciding whether a given parametrization is normal. In the last part of this section we deal with the problem of computing normal parametrizations of a rational curve over $\mathbb{K}$. We start with the following theorem that is due to Prof. Tomás Recio[1]. Geometrically, the idea is to take a projective parametrization that sends the infinity of $\mathbb{K}$ into a point of $\mathcal{C}$ at infinity.

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[1] Tomás Recio is Professor at Universidad de Cantabria (Spain), and he stated and proved construtively, in a pesonal communication (1994), this theorem.

Theorem 3 (T. Recio, 1994). Every rational affine curve over $\mathbb{K}$ can be properly and normally parametrized.

Proof. Let $\mathcal{C}$ be an affine rational curve over $\mathbb{K}$, and let $\mathcal{P}(t)$ be a proper parametrization, in reduced form, of $\mathcal{C}$ (note that by Lüroth's Theorem, this is always possible) that is not normal. Then, by Theorem 1, $\mathcal{P}(t)$ is not polynomial, and hence either q_1 or q_2 are non-constant. Let q_1 be non-constant; similarly if q_2 is non–constant. Then, we take $\alpha \in \mathbb{K}$ such that $q_1(\alpha) = 0$ but $p_1(\alpha) \neq 0$. This is always possible since p_1 and q_1 are relatively prime. Then, applying the linear change of parameter $\phi(t) = \frac{\alpha t+1}{t}$, the parametrization $\mathcal{P}(t)$ is transformed on a normal parametrization. Furthermore, since $\phi(t)$ is invertible, the reparametrization preserves the properness. $\square$

Remark. In Theorem 3. we have emphasized the fact that the normal parametrization is proper. If the parametrization $\mathcal{P}(t)$ in the proof has tracing index k (the tracing index measures the number of times that the parametrization traces the curve when the parameter takes values in $\mathbb{K}$, and therefore it gives a notion of how improper the parametrization is; see [14]) then, since the reparametrization function $\phi(t)$ is linear, the tracing index of the normal parametrization is also k (see Theorem 4 in [14]). $\square$

Combining algorithm NORMALITY-TEST and the constructive proof of Theorem 3. one can derive the following algorithm.

<u>Algorithm</u>: NORMAL-PARAMETRIZATION.

Given a rational parametrization $\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}\right)$, in reduced form, of an affine rational curve $\mathcal{C}$ the algorithm computes a normal parametrization of $\mathcal{C}$.

1. Apply algorithm NORMALITY-TEST to check whether $\mathcal{P}(t)$ is normal. If so, return $\mathcal{P}(t)$.
2. If $q_1(t)$ is not constant take $\alpha \in \mathbb{K}$ such that $q_1(\alpha) = 0$ but $p_1(\alpha) \neq 0$ else take $\alpha \in \mathbb{K}$ such that $q_2(\alpha) = 0$ but $p_2(\alpha) \neq 0$.
3. Return $\mathcal{P}\left(\frac{\alpha t+1}{t}\right)$.

Example 4. Applying algorithm NORMAL-PARAMETRIZATION to the proper parametrizations given in Example 3, one gets

$$\left(-\frac{a\left(2\,t^2 - 2\sqrt{-1}t - 1\right)}{2\sqrt{-1}t + 1}, 2\,\frac{b\left(\sqrt{-1}t + 1\right)t}{2\sqrt{-1}t + 1}\right)$$

that parametrize normally and properly the affine conics. $\square$

3 Normality Over Non-Algebraically Closed Fields.

In Section 2 we have studied the normality of parametrizations over an algebraically closed field. However in most applications, as plotting, one is basically interested in parametrizations over the reals. In this section we study the problem in a more general frame, namely fields that are not algebraically closed.

Thought out this section, we consider a subfield $\mathbb{L} \subseteq \mathbb{K}$ of $\mathbb{K}$. Therefore, $\mathbb{L}$ has characteristic zero but, in general, is not algebraically closed. In this situation, we consider affine rational curves $\mathcal{C}$ over $\mathbb{K}$ such that $\mathcal{C} \cap \mathbb{L}^2$ is also curve, that is, such that the ideal in $\mathbb{L}[x, y]$ of $\mathcal{C} \cap \mathbb{L}^2$ is one–dimensional, or equivalently that $\text{Card}(\mathcal{C} \cap \mathbb{L}^2) = \infty$ (see Lemma 2 in [1]). Rational curves of this type are precisely those rational curves that can be parametrized over $\mathbb{L}$ (see Lemma 2 in [1]). In this situation, we deal with the problem of finding a parametrization over $\mathbb{L}$ of $\mathcal{C}$ that is surjective over $\mathcal{C} \cap \mathbb{L}^2$.

There exist parametrization algorithms (see [6],[13]) and reparametrization algorithms (see [1],[2]) to generate parametrizations over $\mathbb{L}$, if $\mathcal{C}$ satisfies the above condition. Therefore, we assume w.l.o.g. that we are given an affine parametrization $\mathcal{P}(t)$ over $\mathbb{L}$; that is $\mathcal{P}(t) \in \mathbb{L}(t)^2$. Moreover, we observe that any parametrization of $\mathcal{C}$ over $\mathbb{L}$ can be computed as a reparametrization of a proper parametrization over $\mathbb{L}$ of $\mathcal{C}$. Indeed, if $\mathcal{Q}(t)$ is a parametrization over $\mathbb{L}$ of $\mathcal{C}$, and $\mathcal{P}(t)$ is a proper parametrization over $\mathbb{L}$ of $\mathcal{C}$, then $\mathcal{P}^{-1}(t)$ can be computed by elimination techniques, and hence there exists an element in $\mathbb{L}(x, y)$ that represents the inverse. Thus, taking $R(t) = \mathcal{P}^{-1}(\mathcal{Q}(t)) \in \mathbb{L}(t)$ one has that $\mathcal{Q}(t) = \mathcal{P}(R(t))$.

Therefore in the sequel, unless explicitly stated otherwise, we assume that $\mathcal{P}(t)$ is a proper parametrization over $\mathbb{L}$ of an affine curve $\mathcal{C}$. In this situation, $\mathcal{P}(t)$ induces the rational map

$$\begin{array}{cccc} \varphi_{\mathcal{P}}|_{\mathbb{L}}: & \mathbb{L} & \longrightarrow & \mathcal{C} \cap \mathbb{L}^2 \\ & t & \longmapsto & \mathcal{P}(t) \end{array}$$

Then, we will say that $\mathcal{P}(t)$ is $\mathbb{L}$–normal if $\varphi_{\mathcal{P}}|_{\mathbb{L}}(\mathbb{L}) = \mathcal{C} \cap \mathbb{L}^2$. More precisely, one has the following definition.

Definition 4. A parametrization $\mathcal{P}(t) \in \mathbb{L}(t)^2$, not necessarely proper, of an affine curve $\mathcal{C}$ is $\mathbb{L}$*–normal*, or *normal over* $\mathbb{L}$, if for all $P \in \mathcal{C} \cap \mathbb{L}^2$ there exists $t_0 \in \mathbb{L}$ such that $\mathcal{P}(t_0) = P$. If there exists an $\mathbb{L}$-normal parametrization of $\mathcal{C}$ we say that $\mathcal{C}$ can be $\mathbb{L}$*–normally parametrized.*

$\mathbb{L}$–Degenerations

In Section 2, the problem was that some points on the curve might not be reachable by the parametrization. In this section we have an additional difficulty; namely, a point on the curve, although reachable by the parametrization, might only be generated by parameter values in $\mathbb{K} \setminus \mathbb{L}$. For instance, the parametrization $\mathcal{P}(t) = (t^4 + 1, t^3 + 2t + 1)$, which is proper and normal over $\mathbb{C}$, is not $\mathbb{R}$–normal because the point $(5, 1)$, which is on the curve, is only reachable via $\mathcal{P}(t)$ taking $t = \pm i\sqrt{2}$. This remark motivates the next definition. For this purpose, we consider the fibre of the rational mapping $\varphi_{\mathcal{P}}$ by $\mathcal{P}(t)$ of points on $\mathcal{C}$. That is, for any point $P \in \mathcal{C}$, we consider the set

$$\mathcal{F}_{\mathcal{P}}(P) = \{t_0 \in \mathbb{K} \,|\, \mathcal{P}(t_0) = P\}.$$

Definition 5. A parametrization $\mathcal{P}(t) \in \mathbb{L}(t)^2$, not necessarely proper, of an affine curve $\mathcal{C}$ is $\mathbb{L}$–*degenerated* if there exists $P \in \mathcal{C} \cap \mathbb{L}^2$, such that $\mathcal{F}_{\mathcal{P}}(P) \neq \emptyset$ and $\mathcal{F}_{\mathcal{P}}(P) \subset \mathbb{K} \setminus \mathbb{L}$.

Definition 6. We define the *set of* $\mathbb{L}$–*degenerations* of a paramatrization $\mathcal{P}(t) \in \mathbb{L}(t)^2$, and we denote it by $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$, as the set

$$\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} = \{P \in \mathcal{C} \cap \mathbb{L}^2 \,|\, \mathcal{F}_{\mathcal{P}}(P) \neq \emptyset \text{ and } \mathcal{F}_{\mathcal{P}}(P) \subset \mathbb{K} \setminus \mathbb{L}\}.$$

Proposition 1. For every proper parametrization $\mathcal{P}(t) \in \mathbb{L}(t)$, $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ is a zero–dimensional algebraic set.

Proof. Let $f(x, y)$ be the implicit equation of the curve defined by $\mathcal{P}(t)$. Clearly $f(x, y) \in \mathbb{L}[x, y]$, since $f(x, y) = \mathrm{Res}_t(H_1^{\mathcal{P}}, H_2^{\mathcal{P}})$ (see e.g. Theorem 7 in [14]). Moreover, since $\mathcal{P}(t)$ is proper, we can compute its inverse $\mathcal{P}^{-1}$ and, as we have remarked at the begining of the section, it can be represented by an element $S(x, y) = A(x, y)/B(x, y) \in \mathbb{L}(x, y)$ where $\gcd(A, f) = \gcd(B, f) = 1 = \gcd(A, B)$. Then, almost all points $P \in \mathcal{C} \cap \mathbb{L}^2$ are generated by a parameter value in $\mathbb{L}$, namely $S(P)$. Hence, points in $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ are included in a finite set of points, and therefore it is an algebraic set of dimension 0. More precisely, if $\overline{q_1}(x, y)$ and $\overline{q_2}(x, y)$ denote the numerator of $q_1(A(x, y)/B(x, y))$ and $q_2(A(x, y), B(x, y))$, respectively, then

$$\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \subset \{(x, y) \in \mathbb{L}^2 \,/\, f(x, y) = 0, B(x, y)\overline{q_1}(x, y)\overline{q_2}(x, y) = 0\}. \qquad \square$$

Remark. If $\mathcal{P}(t)$ is not proper, Proposition 1 does not hold. For instance $\mathcal{D}_{\mathbb{R}}^{(t^2,t^4)} = \{(a, b) \,|\, a < 0, b = a^2\}$. $\square$

Taking into account that parametrizations are given in reduced form, it is clear that for every $(a, b) \in \mathcal{C} \cap \mathbb{L}^2$, $\{t_0 \in \mathbb{L} \,|\, \mathcal{P}(t_0) = (a, b)\}$ is the set of roots in $\mathbb{L}$ of $\gcd(H_1^{\mathcal{P}}(a, x), H_2^{\mathcal{P}}(b, x))$. From this result, and provided a method for deciding whether a univariate polynomial has a root over $\mathbb{L}$ (see Step 4 in algorithm $\mathbb{L}$–Degenerations), we can derive a method to compute the set of $\mathbb{L}$–degenerations of a parametrization.

Algorithm: $\mathbb{L}$–Degenerations

Given a rational proper parametrization in reduced form $\mathcal{P}(t) \in \mathbb{L}(t)^2$ of an affine rational curve $\mathcal{C}$, and provided a method for deciding whether a univariate polynomial has a root over $\mathbb{L}$, the algorithm computes the set $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$.

1. Compute the inverse of $\mathcal{P}(t)$. Let $A(x, y)/B(x, y)$ be a representative of the inverse over $\mathbb{L}$, such that $\gcd(A, f) = \gcd(B, f) = \gcd(A, B) = 1$ where f is the defining polynomial of $\mathcal{C}$.

2. For $i = 1, 2$ compute the numerator $\overline{q_i}(x, y)$ of $q_i(A(x, y)/B(x, y))$.

3. Compute the set $\mathcal{B}$ of intersection points of $\mathcal{C}$ and each of the curves defined by $B(x, y), \overline{q_1}(x, y), \overline{q_2}(x, y)$.

4. $\mathcal{D} := \emptyset$. For every $(a,b) \in \mathcal{B} \cap \mathbb{L}^2$ check whether $\gcd(H_1^{\mathcal{P}}(a,t), H_2^{\mathcal{P}}(b,t))$ has a root in $\mathbb{L}$. If not then $\mathcal{D} = \mathcal{D} \cup \{(a,b)\}$.

5. Return $\mathcal{D}$.

Remark. The computation of the inverse of the parametrization can be performed by means of Gröbner bases, but for the case of curve there exist special methods based on resultants and gcd's (see e.g. [11]). □

Example 5. Let $\mathbb{K} = \mathbb{C}$ and $\mathbb{L} = \mathbb{R}$ and $\mathcal{P}(t) = (t^4+1, t^3+2t+1)$. The inverse of the parametrization (Step 1) can be expressed as

$$\frac{-(3y-3+yx-x)}{(-11+2x-y^2+2y)}.$$

The polynomials $\overline{q_i}(x,y)$ are constant (Step 2). The set $\mathcal{B}$ (Step 3) is

$$\mathcal{B} = \{(-3, 1 \pm 4i), (1,1), (5,1)\}.$$

In Step 4, one has that

$$\gcd(H_1^{\mathcal{P}}(1,t), H_2^{\mathcal{P}}(1,t)) = t, \;\; \gcd(H_1^{\mathcal{P}}(5,t), H_2^{\mathcal{P}}(1,t)) = t^2+2.$$

Thus, $\mathcal{D}_{\mathbb{R}}^{\mathcal{P}} = \{(5,1)\}$. □

There are two types of behavior on the $\mathbb{L}$–degenerations in relation to the $\mathbb{L}$–degenerations. This motivates the following definition.

Definition 7. Let $\mathcal{P}(t) \in \mathbb{L}(t)^2$ be an $\mathbb{L}$–degenerated parametrization, not necessarely proper, of an affine curve $\mathcal{C}$. Then, we say that $\mathcal{P}(t)$ is $\mathbb{L}$–*weakly degenerated* if $\deg_t(p_i) \leq \deg_t(q_i)$ for $i = 1,2$ and $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ contains only the critical point of $\mathcal{P}(t)$. Otherwise, we say that $\mathcal{P}(t)$ is $\mathbb{L}$–*strongly degenerated.*

Remark. Observe that $\mathbb{L}$–degenerated parametrizations such that at least one of the numerators has bigger degree that its corresponding numerators are $\mathbb{L}$–strongly degenerated. In particular polynomial $\mathbb{L}$–degenerated parametrizations are $\mathbb{L}$–strongly degenerated. □

The proper parametrization in Example 5 is $\mathbb{R}$–strongly degenerated because it is polynomial. The next example shows that "pure" strong $\mathbb{L}$–degeneration may also happen even for proper parametrizations.

Example 6. Let $\mathbb{K} = \mathbb{C}$ and $\mathbb{L} = \mathbb{R}$ and

$$\mathcal{P}(t) = \left(\frac{t^2+1}{t^4+1}, \frac{t^2+1}{t^3+2t+1}\right).$$

This parametrization is proper and defines a curve of degree 6. Applying algorithm $\mathbb{L}$–DEGENERATIONS one gets

$$\mathcal{D}_{\mathbb{R}}^{\mathcal{P}} = \left\{(0,0), \left(-\frac{1}{5}, -1\right), \left(-\frac{1}{3}, 1\right)\right\}.$$

In fact

$$\mathcal{F}_{\mathcal{P}}((0,0)) = \{\pm i\},\ \mathcal{F}_{\mathcal{P}}\left(\left(-\frac{1}{5}, -1\right)\right) = \{\pm\sqrt{2}i\},\ \mathcal{F}_{\mathcal{P}}\left(\left(-\frac{1}{3}, 1\right)\right) = \left\{\frac{1}{2} \pm i\frac{\sqrt{7}}{2}\right\}.$$

Note that $\left(-\frac{1}{5}, -1\right), \left(-\frac{1}{3}, 1\right)$ are singularities on the curve with imaginary tangent cone, and compare to the geometric explanation to he behavior of the strong degeneration given in the introduction and in the next subsection. □

Strong $\mathbb{L}$-Degeneration

In this subsection we prove that, if a curve has an $\mathbb{L}$-strongly degenerated proper parametrization then the curve can not be $\mathbb{L}$-normally parametrized. The geometric intuitive explanation for the strong singularity behavior, at least for the real case, is that these degenerated points different to the critical point are the singularities of the curve which tangent cone is completely imaginary, and this property does not depend on the parametrization. We start this analysis with the following lemmas.

Lemma 2. Let $\mathcal{P}(t)$ be a rational parmetrization. Let $R(t) = \frac{M(t)}{N(t)}$ be a non–constant rational function in reduced form, and let $\mathcal{Q}(t) = \mathcal{P}(R(t))$. Then, it holds that

$$H_1^{\mathcal{Q}}(x,t) = N^{\max\{\deg(p_1),\deg(q_1)\}} H_1^{\mathcal{P}}(x,R),\ \ H_2^{\mathcal{Q}}(y,t) = N^{\max\{\deg(p_2),\deg(q_2)\}} H_2^{\mathcal{P}}(y,R).$$

Proof. We prove it for $i=1$, similarly for $i=2$. Let $p_1(t) = \sum_{i=0}^{n} a_i t^i$ and $q_1(t) = \sum_{i=0}^{m} b_i t^i$, with $a_n \neq 0, b_m \neq 0$, and $\overline{p_1}(t) = \sum_{i=0}^{n} a_i M^i N^{n-i}, \overline{q_1}(t) = \sum_{i=0}^{m} b_i M^i N^{m-i}$. Then

$$N^n p_1(R) = \overline{p_1}(t),\ \ N^m q_1(R) = \overline{q_1}(t).$$

Thus,

$$\frac{p_1(R)}{q_1(R)} = \frac{\overline{p_1}(t)}{\overline{q_1}(t)} N^{m-n}.$$

Now observe that $\gcd(N, \overline{p_1}) = \gcd(N, \overline{q_1}) = 1$, since otherwise one gets that M and N have a common root, which is impossible because $\gcd(M,N) = 1$. Also, let us see that $\gcd(\overline{p_1}, \overline{q_1}) = 1$. Indeed, let α be a common root of $\overline{p_1}, \overline{q_1}$. It holds that

$$\overline{p_1}(\alpha) = p_1(R(\alpha))N(\alpha)^n = 0,\ \overline{q_1}(\alpha) = q_1(R(\alpha))N(\alpha)^m = 0,$$

and since $N(\alpha) \neq 0$, one gets that $R(\alpha)$ is a common root of p_1, q_1, which is impossible because $\gcd(p_1, q_1) = 1$. Therefore, if $n = m$ then

$$H_1^{\mathcal{Q}}(x,t) = x\overline{q_1}(t) - \overline{p_1}(t) = xq_1(R)N^n - p_1(R)N^n = H_1^{\mathcal{P}}(x,R)N^n.$$

If $m > n$ then

$$H_1^{\mathcal{Q}}(x,t) = x\overline{q_1}(t) - \overline{p_1}(t)N^{m-n} = xq_1(R)N^m - p_1(R)N^m = H_1^{\mathcal{P}}(x,R)N^m.$$

Similarly if $m < n$. $\square$

Lemma 3. Let $\mathcal{P}(t) \in \mathbb{L}(t)^2$ be a proper parametrization of an affine curve $\mathcal{C}$. If there exists $(a_1, a_2) \in \mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ such that for some $i \in \{1, 2\}$

$$\deg_t(H_i^{\mathcal{P}}(a_i, t)) = \max\{\deg_t(p_i), \deg_t(q_i)\}$$

then any parametrization of $\mathcal{C}$ is $\mathbb{L}$–degenerated, and therefore $\mathcal{C}$ can not be $\mathbb{L}$–normally parametrized.

Proof. Since $\mathcal{P}(t)$ is proper over $\mathbb{L}$, any other rational parametrization over $\mathbb{L}$ of $\mathcal{C}$ can be obtained as a reparametrization over $\mathbb{L}$ of $\mathcal{P}(t)$. Therefore, it is enough to check that for any non–constant rational function $R(t) \in \mathbb{L}(t)$, $\mathcal{Q}(t) = \mathcal{P}(R(t))$ is $\mathbb{L}$–degenerated. Let $R(t) = \frac{M}{N}$ be in reduced form, and let $(a_1, a_2) \in \mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ be such that $k := deg_t(H_1^{\mathcal{P}}(a_1, t)) = \max\{\deg_t(p_1), \deg_t(q_1)\}$; similarly if the property holds for $i = 2$. Let also assume that $H_i^{\mathcal{P}}(a_i, t)$ factors over $\mathbb{K}$ as

$$H_1^{\mathcal{P}}(a_1, t) = \prod_{i=1}^{k}(\alpha_i t - \beta_i),\ \ H_2^{\mathcal{P}}(a_2, t) = \prod_{i=1}^{k'}(\alpha_i' t - \beta_i').$$

Then, if $\ell = \max\{\deg_t(p_2), \deg_t(q_2)\}$, by Lemma 2 one gets that

$$H_1^{\mathcal{Q}}(a_1, t) = N^k H_1^{\mathcal{P}}(a_1, R(t)) = \prod_{i=1}^{k}(\alpha_i M(t) - \beta_i N(t)),$$

and

$$H_2^{\mathcal{Q}}(a_2, t) = N^{\ell} H_2^{\mathcal{P}}(a_2, R(t)) = N^{\ell-k'} \prod_{i=1}^{k'}(\alpha_i' M(t) - \beta_i' N(t)).$$

Note that $\ell - k' \geq 0$. Moreover, observe that $\gcd(N, \alpha_i M - \beta_i N) = 1$ for every i, since otherwise it would imply that $\gcd(M, N) \neq 1$. Therefore,

$$\gcd(H_1^{\mathcal{Q}}(a_1, t), H_2^{\mathcal{Q}}(a_2, t)) = \gcd\left(\prod_{i=1}^{k}(\alpha_i M(t) - \beta_i N(t)), \prod_{i=1}^{k'}(\alpha_i' M(t) - \beta_i' N(t))\right).$$

Observe that $\gcd(H_1^{\mathcal{Q}}(a_1, t), H_2^{\mathcal{Q}}(a_2, t))$ is not trivial since $H_1^{\mathcal{P}}(a_1, t)$ and $H_2^{\mathcal{P}}(a_2, t))$ are coprime. This implies that $\mathcal{F}_{\mathcal{Q}}((a, b)) \neq \emptyset$. Now, let us see that $(a, b) \in \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$. Let us assume that there exists a root ρ of $\gcd(H_1^{\mathcal{Q}}(a_1, t), H_2^{\mathcal{Q}}(a_2, t))$ in $\mathbb{L}$. Then, there exists some i and j such that

$$\alpha_i M(\rho) - \beta_i N(\rho) = \alpha_j' M(\rho) - \beta_j' N(\rho) = 0.$$

As we have already observed $N(\rho) \neq 0$, thus

$$\frac{\beta_i}{\alpha_i} = R(\rho) = \frac{\beta'_j}{\alpha'_j}.$$

But this implies that $\frac{\beta_i}{\alpha_i} \in \mathbb{L}$, since $\rho \in \mathbb{L}$ and $R \in \mathbb{L}(t)$. Hence, one has that $\gcd(H_1^{\mathcal{P}}(a_1,t), H_2^{\mathcal{P}}(a_2,t))$ has a common root in $\mathbb{L}$, which is impossible because $(a_1,a_2) \in \mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$. Therefore, $(a_1, a_{)} \in \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$ and $\mathcal{Q}(t)$ is $\mathbb{L}$–degenerated. □

Remark. Observe that for all but at most one point on the curve it holds that $\deg_t(H_i^{\mathcal{P}}(a_i,t)) = \max\{\deg_t(p_i), \deg_t(q_i)\}$ for $i = 1, 2$. In fact, such a special case exists if and only if $\deg_t(p_i) \leq \deg_t(q_i)$ for $i = 1, 2$, and in this case it is the critical point of the parametrization.

The next corollaries are direct applications of Lemma 3.

Corollary 4. Let $\mathcal{P}(t) \in \mathbb{L}(t)^2$ be a proper parametrization of an affine curve $\mathcal{C}$. If $\mathrm{Card}(\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}) > 1$ then $\mathcal{C}$ can not be $\mathbb{L}$–normally parametrized.

Proof. It follows from Lemma 3, and from the fact that degrees w.r.t. t of $\mathcal{H}_i^{\mathcal{P}}$ for $i = 1, 2$ can only decrease for the critical point. □

Corollary 5. Let $\mathcal{P}(t) \in \mathbb{L}(t)^2$ be an $\mathbb{L}$–degenerated proper parametrization of an affine curve $\mathcal{C}$. If $\mathcal{P}(t)$ is not normal, then $\mathcal{C}$ can not be $\mathbb{L}$–normally parametrized.

Proof. If the parametrization is not normal then by Theorem 2 the only point that may decrease the degrees w.r.t. of of $\mathcal{H}_i^{\mathcal{P}}$ is not reachable by the parametrization, and hence it is not in $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$. Therefore, since $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \neq \emptyset$, Lemma 3 implies the result. □

The following theorem states the fundamental property of $\mathbb{L}$–strongly degenerated parametrizations.

Theorem 4. Let $\mathcal{P}(t)$ be an $\mathbb{L}$–strongly degenerated proper parametrization of an affine plane curve $\mathcal{C}$. Then, $\mathcal{C}$ can not be $\mathbb{L}$–normally parametrized.

Proof. We distinguish two cases.

(1) If $\deg_t(p_1) > \deg_t(q_1)$ or $deg_t(p_2) > deg_t(q_2)$, then for some $i \in \{1, 2\}$ the equality on the degrees in Lemma 3 holds for all points on $\mathcal{C}$. Thus, since $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \neq \emptyset$ the result follows from Lemma 3.

(2) Let $\deg_t(p_1) \leq \deg_t(q_1)$ and $deg_t(p_2) \leq deg_t(q_2)$ for $i = 1, 2$. Let assume that $\deg_t(p_1) < \deg_t(q_1)$ and $deg_t(p_2) < deg_t(q_2)$. Then the critical point is the origen. Since $\mathcal{P}(t)$ is $\mathbb{L}$–strongly degenerated, one has that $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \neq \emptyset$ and $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \neq \{(0, 0)\}$. Thus, there exists a point in $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ satisfying the condition on the degree in Lemma 3. Thus, the result follows from Lemma 3. Similarly for the other cases. □

Weak $\mathbb{L}$–Degeneration

In the previous subsection, we have seen that strong $\mathbb{L}$–degeneration can not be avoided. In this section, we prove that weak $\mathbb{L}$–degenerations can be resolved. We start with the following lemma.

Lemma 4. If $\mathcal{P}(t) \in \mathbb{L}(t)^2$ is a proper $\mathbb{L}$–weakly degenerated parametrization of a rational curve $\mathcal{C}$, then any other proper parametrization of $\mathcal{C}$ is either non $\mathbb{L}$–degenerated or it is $\mathbb{L}$–weakly degenerated.

Proof. Let $\mathcal{Q}(t)$ be a proper parametrization of $\mathcal{C}$, then there exists a linear rational function $R(t)$ over $\mathbb{L}$ such that $\mathcal{Q}(t) = \mathcal{P}(R(t))$. Let $R(t) = \frac{at+b}{ct+d}$, with $a, b, c, d \in \mathbb{L}$ such that $ad - bd \neq 0$, and let $P_{\mathcal{P}}, P_{\mathcal{Q}}$ be the critical points of $\mathcal{P}(t)$ and $\mathcal{Q}(t)$, respectively. We distinguish several cases:

(1) Let $c = 0$, then we may assume w.l.o.g. that $R(t) = at + b$, with $a \neq 0$. Note that degrees in $\mathcal{Q}(t)$ and $\mathcal{P}(t)$ are the same and that $P_{\mathcal{P}} = P_{\mathcal{Q}}$. Now, take $P \in [\mathcal{C} \cap \mathbb{L}^2] \setminus \{P_{\mathcal{Q}}\}$. Since $P \neq P_{\mathcal{P}}$ one has that $\mathcal{F}_{\mathcal{P}}(P) \neq \emptyset$, and since $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$ there exists $\alpha \in \mathbb{L}$ such that $\mathcal{P}(\alpha) = P$. Then, $\mathcal{Q}\left(\frac{\alpha-b}{a}\right) = P$ and $\frac{\alpha-b}{a} \in \mathbb{L}$. Thus, $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$. That is $\mathcal{D}_{\mathbb{L}}^{\mathcal{Q}} \subset \{P_{\mathcal{Q}}\}$.

(2) Let $c \neq 0$ and assume that a/c is not a root of $q_1(t)q_2(t)$. Then, the degree of the numerator and the denominator of each component of $\mathcal{Q}(t)$ are equal, and $P_{\mathcal{Q}} = \mathcal{P}(a/c)$. Now, take $P \in [\mathcal{C} \cap \mathbb{L}^2] \setminus \{P_{\mathcal{Q}}\}$. First note that if $P = P_{\mathcal{P}}$ then $\mathcal{Q}(-d/c) = P$ and hence $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$. Let $P \neq P_{\mathcal{P}}$, then $\mathcal{F}_{\mathcal{P}}(P) \neq \emptyset$, and $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$. Therefore, there exists $\alpha \in \mathbb{L}$ such that $\mathcal{P}(\alpha) = P$. Moreover, since $P \neq P_{\mathcal{Q}} = \mathcal{P}(a/c)$ then $\alpha \neq \frac{a}{c}$. Thus, $R^{-1}(\alpha)$ is defined and $R^{-1}(\alpha) \in \mathbb{L}$ (note that $R^{-1}(t) = -\frac{b-td}{a-ct}$). Therefore, $\mathcal{Q}(R^{-1}(\alpha)) = P$, and hence $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$. Summarizing $\mathcal{D}_{\mathbb{L}}^{\mathcal{Q}} \subset \{P_Q\}$.

(3) Let $c \neq 0$ and assume that a/c is a root of $q_1(t)q_2(t)$. Then, we prove that $\mathcal{Q}(t)$ is, in fact, $\mathbb{L}$–normal. First note that at least one of the denominators in $\mathcal{Q}(t)$ has smaller degree than its corresponding numerator. Thus, by Theorem 1, $\mathcal{Q}(t)$ is normal. Now, take $P \in \mathcal{C} \cap \mathbb{L}^2$. If $P \neq P_{\mathcal{P}}$, then as above there exists $\alpha \in \mathbb{L}$ such that $\mathcal{P}(\alpha) = P$. Since $\mathcal{P}(t)$ is not defined at a/c one has that $\alpha \neq a/c$. Thus, $R^{-1}(\alpha)$ is defined and $R^{-1}(\alpha) \in \mathbb{L}$. Therefore, $\mathcal{Q}(R^{-1}(\alpha)) = P$, and hence $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$. Let $P = P_{\mathcal{P}}$, then $\mathcal{Q}(-d/c) = P_{\mathcal{P}}$. That is, $P \notin \mathcal{D}_{\mathbb{L}}^{\mathcal{Q}}$. Summarizing $\mathcal{D}_{\mathbb{L}}^{\mathcal{Q}} = \emptyset$, and $\mathcal{Q}(t)$ is normal. Therefore, $\mathcal{Q}(t)$ is $\mathbb{L}$–normal. □

Theorem 5. Let $\mathcal{P}(t) \in \mathbb{L}(t)^2$ be a proper $\mathbb{L}$–weakly degenerated parametrization of a rational curve $\mathcal{C}$. Then it holds that:

(1) If $a \in \mathbb{L}$ is such that $\mathrm{Card}(\mathcal{F}_{\mathcal{P}}(\mathcal{P}(a))) = 1$, and $b, d \in \mathbb{L}$ is such that $ad - b \neq 0$, then

$$\mathcal{P}\left(\frac{at+b}{t+d}\right)$$

is a non $\mathbb{L}$–degenerated proper parametrization over $\mathbb{L}$ of $\mathcal{C}$.

(2) If $q_1(t)q_2(t)$ has a root α in $\mathbb{L}$, then

$$\mathcal{P}\left(\frac{\alpha t+1}{t}\right)$$

is an $\mathbb{L}$–normal parametrization of $\mathcal{C}$.

Proof. (2) has been proved in the step (3) of the proof of Lemma 4. Let us prove statement (1). First we observe that a always exists (see [14]). Let $\mathcal{Q}(t) = \mathcal{P}\left(\frac{at+b}{t+d}\right)$, and let $P_{\mathcal{P}}, P_{\mathcal{Q}}$ be the critical points of $\mathcal{P}(t)$ and $\mathcal{Q}(t)$, respectively. From Lemma 4, we know that $\mathcal{D}_{\mathbb{L}}^{\mathcal{Q}} \subset \{P_{\mathcal{Q}}\}$. We prove that $\mathcal{F}_{\mathcal{Q}}(P_{\mathcal{Q}}) = \emptyset$, which will imply that $\mathcal{D}_{\mathbb{L}}^{\mathcal{Q}} = \emptyset$ and hence $\mathcal{Q}(t)$ is non $\mathbb{L}$–degenerated (clearly proper). Since $q_1(a)q_2(a) \neq 0$ (otherwise the fibre has cardinality 0), the degree of the numerator and the denominator of each component of $\mathcal{Q}(t)$ are equal, and $P_{\mathcal{Q}} = \mathcal{P}(a)$. Moreover, $\mathcal{Q}(-d) = P_{\mathcal{P}}$. Therefore, $P_{\mathcal{P}} = \mathcal{Q}(-d) \neq \mathcal{P}(a) = P_{\mathcal{Q}}$, because $a \in \mathbb{L}$ and $P_{\mathcal{P}} \in \mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$. In this situation, let us assume that $\mathcal{F}_{\mathcal{Q}}(P_{\mathcal{Q}}) \neq \emptyset$. Take $\alpha \in \mathbb{K}$ such that $\mathcal{Q}(\alpha) = P_{\mathcal{Q}}$. Since $\alpha \neq -d$ because $\mathcal{Q}(-d) \neq P_{\mathcal{Q}}$, $R(\alpha)$ is defined and $\mathcal{P}(R(\alpha)) = P_{\mathcal{Q}}$. Now, since $\mathrm{Card}(\mathcal{F}_{\mathcal{P}}(\mathcal{P}(a))) = \mathrm{Card}(\mathcal{F}_{\mathcal{P}}(P_{\mathcal{Q}})) = 1$ one has that $R(\alpha) = a$, but this implies that $b = da$ which is impossible. $\square$

Remark. Taking into account [14], one may proceed as follows in order to determine the element a in statement (1) of Theorem 5. Let ℓ_i be the leading coefficient of $H_i^{\mathcal{P}}$, let $G_i(s,t) = q_i(s)p_i(t) - q_i(t)p_i(s)$ and let

$$T(s) = \mathrm{Res}_t\left(\frac{G_1}{G}, \frac{G_2}{G}\right),\ R_i = \mathrm{Res}_t\left(G_i, \frac{\partial G_i}{\partial t}\right).$$

Then, for every $a \in \mathbb{L}$ such that

$$q_1(a)q_2(a)\ell_1(a)\ell_2(a)T(a)q_1(a)q_2(a)R_1(a)R_2(a) \neq 0$$

it holds that $\mathrm{Card}(\mathcal{F}_{\mathcal{P}}(\mathcal{P}(a))) = 1$. $\square$

Thoerem 5 provides an algorithm to resolve weak $\mathbb{L}$–degenerations.

<u>**Algorithm:**</u> Resolution of $\mathbb{L}$–degenerations.
Given a rational proper $\mathbb{L}$–degenerated parametrization $\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}\right) \in \mathbb{L}(t)^2$, in reduced form, the algorithm decides whether $\mathcal{P}(t)$ can be reparametrized into a non $\mathbb{L}$-degenerated parametrization, and in the affirmative case it computes such a reparametrization.

1. Apply algorithm $\mathbb{L}$–degenerations to compute $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$.

2. If $\mathrm{Card}(\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}) = 0$ then return $\mathcal{P}(t)$.

3. Compute $n := deg_t(p_1); m := deg_t(q_1); r := deg_t(p_2); s := deg_t(q_2)$.
4. If $n > m$ or $r > s$ or $\mathrm{Card}(\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}) > 1$ then return "`any parametrization of the curve is` $\mathbb{L}$`-degenerated`".
5. Compute the critical point $P_{\mathcal{P}}$ of $\mathcal{P}(t)$.
6. If $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \neq \{P_{\mathcal{P}}\}$ then return "`any parametrization of the curve is` $\mathbb{L}$`-degenerated`".
7. Check whether $q_1(t)q_2(t)$ has a root in $\mathbb{L}$. In the affirmative case, compute a root $\alpha \in \mathbb{L}$ of q_1q_2 and return "$\mathcal{P}\left(\frac{\alpha t+1}{t}\right)$ `is an` $\mathbb{L}$`--normal parametrization of` $\mathcal{C}$".
8. Take $a \in \mathbb{L}$ as pointed in the remark to Theorem 5.
9. Return $\mathcal{P}\left(\frac{at+b}{t+d}\right)$.

We finish this subsection with some examples to illustrate algorithm RESOLUTION OF $\mathbb{L}$–DEGENERATIONS.

Example 7. We consider the proper parametrization

$$\mathcal{P}(t) = \left(\frac{t^2+1}{(t-2)^2\,t}, \frac{t^2+1}{(t-2)^2\,(t+1)}\right).$$

Applying algorithm $\mathbb{L}$–DEGENERATIONS one gets that $\mathcal{D}_{\mathbb{R}}^{\mathcal{P}} = \{(0,0)\}$, in fact, computing $\gcd(H_1^{\mathcal{P}}(0,t), H_2^{\mathcal{P}}(0,t))$ one obtains that $\mathcal{F}_{\mathcal{P}}((0,0)) = \{\pm i\}$. Therefore, $\mathcal{P}(t)$ is $\mathbb{R}$–weakly degenerated. Since 2 is a root of one of the denominators, we consider the parametrization

$$\mathcal{Q}(t) = \mathcal{P}\left(\frac{2t+1}{t}\right) = \left(\frac{(5\,t^2+4\,t+1)\,t}{2\,t+1}, \frac{(5\,t^2+4\,t+1)\,t}{3\,t+1}\right),$$

whic is $\mathbb{R}$–normal. In fact. applying algorithm $\mathbb{L}$–DEGENERATIONS to $\mathcal{Q}(t)$ one gets that $\mathcal{D}_{\mathbb{R}}^{\mathcal{P}} = \emptyset$, and computing $\gcd(H_1^{\mathcal{Q}}(0,t), H_2^{\mathcal{Q}}(0,t))$ one obtains that $\mathcal{F}_{\mathcal{P}}((0,0)) = \{0, -\frac{2}{5} \pm \frac{1}{5}i\}$. Thus, $\mathcal{Q}(t)$ is $\mathbb{R}$–normal. □

Example 8. We consider the proper parametrization

$$\mathcal{P}(t) = \left(\frac{t^2+1}{t^4+1}, \frac{t^2+1}{t^4+t+1}\right).$$

Applying algorithm $\mathbb{L}$–DEGENERATIONS one gets that $\mathcal{D}_{\mathbb{R}}^{\mathcal{P}} = \{(0,0)\}$. Therefore, $\mathcal{P}(t)$ is $\mathbb{R}$–weakly degenerated. Also, we observe that denominators in $\mathcal{P}(t)$ do not have real roots. We take $a = 1$ in Step 8 of algorithm Resolution of $\mathbb{L}$–degenerations. Note that $\mathcal{F}_{\mathcal{P}}(\mathcal{P}(1)) = \{1\}$. Therefore, the parametrization

$$\mathcal{Q}(t) = \mathcal{P}\left(\frac{t+1}{t+2}\right)$$

is not $\mathbb{R}$–degenerated. In fact, the critical point of $\mathcal{Q}(t)$ is $P_{\mathcal{Q}} = \left(1, \frac{2}{3}\right)$, and $\mathcal{F}_{\mathcal{Q}}(P_{\mathcal{Q}}) = \emptyset$. $\square$

$\mathbb{L}$–normality

Once we have analyzed the $\mathbb{L}$–degenerations, we proceed to study the $\mathbb{L}$–normality. That is, we deal with the problem of finding parametrizations that are $\mathbb{L}$–normal. For this purpose, and as a consequence of the discussion done before, we assume that the given proper parametrization is not $\mathbb{L}$–degenerated. We start with the following lemma.

Lemma 5. Let $\mathcal{P}(t) \in \mathbb{L}(t)^2$ be a parametrization, not necessarely proper, of an affine rational curve $\mathcal{C}$. If $\mathcal{P}(t)$ is not $\mathbb{L}$–degenerated, $\mathcal{P}(t)$ is normal if and only if $\mathcal{P}(t)$ is $\mathbb{L}$–normal.

Proof. Since $\mathcal{P}(t)$ is not $\mathbb{L}$–degenerated, for every $P \in \mathcal{C} \cap \mathbb{L}^2$ one has that either $\mathcal{F}_{\mathcal{P}}(P) = \emptyset$ or $\mathcal{F}_{\mathcal{P}}(P) \cap \mathbb{L} \neq \emptyset$.
Let $\mathcal{P}(t)$ be normal. Then, for every $P \in \mathcal{C}$, it holds that $\mathcal{F}_{\mathcal{P}}(P) \neq \emptyset$. Thus, for $P \in \mathcal{C} \cap \mathbb{L}^2$ one has that $\mathcal{F}_{\mathcal{P}}(P) \cap \mathbb{L} \neq \emptyset$; that is $\mathcal{P}(t)$ is $\mathbb{L}$–normal.
Conversely, let $\mathcal{P}(t)$ be $\mathbb{L}$–normal. Now, observe that taking into account Corollary 3, the only point on the curve that the parametrization might not reach is the critical point, that belongs to $\mathbb{L}^2$. Since $\mathcal{P}(t)$ is $\mathbb{L}$–normal, it is in fact reachable by $\mathcal{P}(t)$, and therefore the parametrization is normal. $\square$

Taking into account Lemma 5, one can extend trivially Theorem 2 to $\mathbb{L}$–normality as follows.

Theorem 6. Let

$$\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}\right)$$

with

$$p_1(t) = \sum_{i=0}^{n} a_i t^i,\ q_1(t) = \sum_{i=0}^{m} b_i t^i,\ p_2(t) = \sum_{i=0}^{r} c_i t^i,\ q_2(t) = \sum_{i=0}^{s} d_i t^i,$$

where $a_n \neq 0, b_m \neq 0, c_r \neq 0, d_s \neq 0$, be a rational parametrization in reduced form of an affine plane curve $\mathcal{C}$. If $\mathcal{P}(t)$ is not $\mathbb{L}$–degenerated it holds that

(1) If $n > m$ or $r > s$ then $\mathcal{P}(t)$ is $\mathbb{L}$–normal.

(2) If $n < m$, $r < s$ then $\mathcal{P}(t)$ is $\mathbb{L}$–normal iff $\gcd(H_1^{\mathcal{P}}(0,t), H_2^{\mathcal{P}}(0,t)) \neq 1$.

(3) If $n = m$, $r = s$ then $\mathcal{P}(t)$ is $\mathbb{L}$–normal iff $\gcd\left(H_1^{\mathcal{P}}\left(\frac{a_n}{b_m}, t\right), H_2^{\mathcal{P}}\left(\frac{c_r}{d_s}, t\right)\right) \neq 1$.

(4) If $n < m$, $r = s$ then $\mathcal{P}(t)$ is $\mathbb{L}$–normal iff $\gcd\left(H_1^{\mathcal{P}}(0, t), H_2^{\mathcal{P}}\left(\frac{c_r}{d_s}, t\right)\right) \neq 1$.

(5) If $n = m$, $r < s$ then $\mathcal{P}(t)$ is $\mathbb{L}$–normal iff $\gcd\left(H_1^{\mathcal{P}}\left(\frac{a_n}{b_m}, t\right), H_2^{\mathcal{P}}(0, t)\right) \neq 1$.

In this situation, the analogue to Corollary 3 is

Corollary 6. Let $\mathcal{P}(t)$ be a non $\mathbb{L}$–degenerated rational parametrization, not necessarely proper, of an affine plane curve $\mathcal{C}$. Then, either $\mathcal{P}(t)$ is $\mathbb{L}$–normal or the only point of $\mathcal{C} \cap \mathbb{L}^2$ that is not generated by the parametrization is the critical point of $\mathcal{P}(t)$.

In addition, from these results one can derive the following algorithm to decide the $\mathbb{L}$–normality of a proper parametrization.

Algorithm: $\mathbb{L}$–NORMALITY-TEST.
Given a rational proper parametrization $\mathcal{P}(t) = \left(\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)}\right)$, in reduced form, the algorithm decides whether $\mathcal{P}(t)$ is $\mathbb{L}$–normal.

1. Apply the algorithm $\mathbb{L}$–DEGENERATIONS to compute $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}}$.
2. If $\mathcal{D}_{\mathbb{L}}^{\mathcal{P}} \neq \emptyset$ then return "$\mathcal{P}(t)$ `is not` $\mathbb{L}$`-normal`".
3. Apply algorithm NORMALITY-TEST to $\mathcal{P}(t)$. If the answer is yes then return "$\mathcal{P}(t)$ `is` $\mathbb{L}$`-normal`" else return "$\mathcal{P}(t)$ `is not` $\mathbb{L}$`-normal`".

4 Real Normal Parametrizations.

In this section we show how the results in the previous section can be applied an extended to the case of rational real curves.

In [3] it is proved that any real parametrization that is not $\mathbb{R}$–degenerated can be $\mathbb{R}$–normally parametrized with tracing index 2 and the precise cahnge of parameters to achive this situation is given. Applying the results of the previous section, we can improve this theorem as follows. Statement (1) in Theorem 7 is a particular case of Theorem 2 in [3]. However, for reasons of completeness we give here an alternative simple proof.

Theorem 7. Let $\mathcal{P}(t)$ be a proper parametrization of a real curve $\mathcal{C}$ that is not $\mathbb{R}$–strongly degenerated and it is not $\mathbb{R}$–normal. Then, it holds that:

(1) If $\mathcal{P}(t)$ is not $\mathbb{R}$–degenerated, then $\mathcal{C}$ can be $\mathbb{R}$–normally parametrized as

$$\mathcal{P}\left(\frac{t}{t^2 - 1}\right).$$

(2) If $\mathcal{P}(t)$ is not $\mathbb{R}$–degenerated and $q_1(t)q_2(t)$ has a real root α, then $\mathcal{C}$ can be $\mathbb{R}$–normally and properly parametrized as

$$\mathcal{P}\left(\frac{\alpha t+1}{t}\right).$$

(3) If $\mathcal{P}(t)$ is $\mathbb{R}$–weakly degenerated, then $\mathcal{C}$ can be $\mathbb{R}$–normally parametrized as

$$\mathcal{P}\left(\frac{t^2+at-1}{t^2+t-1}\right),$$

where $a \in \mathbb{R}\setminus\{1\}$ such that $\mathrm{Card}(\mathcal{F}_{\mathcal{P}}(\mathcal{P}(a))) = 1$.

(4) If $\mathcal{P}(t)$ is $\mathbb{R}$–weakly degenerated and $q_1(t)q_2(t)$ has a real root α, then $\mathcal{C}$ can be $\mathbb{R}$–normally and properly parametrized as

$$\mathcal{P}\left(\frac{\alpha t+1}{t}\right).$$

Proof.

(1) Let $R(t) = \frac{t}{t^2-1}$ and $\mathcal{Q}(t) = \mathcal{P}(R(t))$. Since $\mathcal{P}(t)$ is not $\mathbb{R}$–normal and it is not $\mathbb{R}$–degenerated, by Theorem 6 (1), we know that the critical point $P_{\mathcal{P}}$ of $\mathcal{P}(t)$ is defined. Let $P \in \mathcal{C}\cap\mathbb{R}^2$. If $P \neq P_{\mathcal{P}}$, then by Corollary 6, one has that $\mathcal{F}_{\mathcal{P}}(P)\cap\mathbb{R}^2 \neq \emptyset$. Let $\alpha \in \mathbb{R}$ such that $\mathcal{P}(\alpha) = P$. Now, observe that the equation $R(t) = \alpha$ has always real solutions; namely

$$\beta = \alpha \ \text{ if } \ \alpha = 0, \ \text{ and } \ \beta = \frac{1 \pm \sqrt{1+4\alpha^2}}{2\alpha} \ \text{ if } \ \alpha \neq 0.$$

Thus, $\mathcal{Q}(\beta) = P$. On the other hand, it is easy to check that $\mathcal{Q}(1) = P_{\mathcal{P}}$. Summarizing $\mathcal{Q}(t)$ is $\mathbb{R}$–normal.

(2) Let $R(t) = \frac{\alpha t+1}{t}$ and $\mathcal{Q}(t) = \mathcal{P}(R(t))$. Then $\mathcal{Q}(t)$ is not $\mathbb{R}$–strongly degenerated, because otherwise by Lemma 3 it would imply that $\mathcal{P}(t) = \mathcal{Q}(R^{-1}(t))$ is $\mathbb{R}$–degenerated. Moreover, the numerator of one of the components of $\mathcal{Q}(t)$ has bigger degree than its denominator. Therefore, $\mathcal{Q}(t)$ is not $\mathbb{R}$–weakly degenerated. Thus, $\mathcal{Q}(t)$ is not $\mathbb{R}$–degenerated. Now the result follows from Theorem 6, statement (1).

(3) It follows from statement (1) and Theorem 5. statement (2).

(4) It follows from Theorem 5 statement (2). □

From Theorem 7 one deduces the following corollary.

Corollary 6. Let $\mathcal{C}$ be a rational real curve. Then, the following statements are equivalents

(1) $\mathcal{C}$ can be $\mathbb{R}$–normally parametrized.

(2) None proper parametrization of $\mathcal{C}$ is $\mathbb{R}$–strongly degenerated.

(3) There exists a proper parametrization of $\mathcal{C}$ that is not $\mathbb{R}$–strongly degenerated.

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