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Two computational techniques for singularity resolution

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Abstract

This paper presents two efficient computational techniques in algebraic geometry. The first one allows the elimination of redundancies in the representation of quasi-projective varieties by atlases of affine charts. The second simplifies the computations with exponentiated ideals by attaching rational weights to the generators, applying Hironaka's theory of idealistic exponents. As the main application, we used these techniques to speed up Villamayor's algorithm for resolving hypersurface singularities in any dimension.

1. Introduction

Many ideas in algebraic geometry are constructive, to a certain extent. But try to compute nontrivial examples on the computer, and often you will find yourself in a situation where the constructive methods become infeasible because of the extreme size or complexity of the data describing the objects you have to work with. For instance, due to the local nature of several computations, we have to determine and maintain suitable neighborhoods of points or algebraic subsets of a variety in which the corresponding computations are realizable. The number as well as the complexity of such neighborhoods might increase into extreme heights, so that we lose the possibility to carry out the calculations unless we devise effective methods for simplification.

The results of the paper apply to two such kind of problems: First, in section 3, we present a method to simplify the representation of quasi-projective varieties given by atlases of affine charts. Then in section 4 we recall Hironaka's theory of idealistic exponents to define effective computational techniques for ideals generated by polynomials with formal rational exponents.

The motivation for us to search for the simplification methods was provided by our implementation of Villamayor's algorithm for resolution of singularities of hypersurfaces over fields of characteristic zero. The original implementation of the algorithm was paralyzed by inefficiency problems like the one discussed above, while the optimized version is able to compute a range of examples (see section 5).

Resolution of singularities is an important result in algebraic geometry with numerous applications in: elimination of base points of a linear system in a regular projective variety, principalization of ideals in a smooth scheme, domination of birational models, parametrization of algebraic varieties, analysis and classification of algebraic sets, etc. In a brief historical account on the classical approach to resolution of singularities we would like to mention the first rigorous (analytic) proof for resolution of surface singularities by Walker (1935), the foundations of the algebraic approach established by Zariski (1939, 1942, 1944), the non-constructive solution of the general case in characteristic zero by Hironaka (1964), the first results in positive characteristic by Abhyankar (1965) and the first constructive proofs in characteristic zero by Villamayor (1989, 1992); Bierstone and Milman (1991, 1997). The interested reader might also wish to consult the following papers on the topic: (Hironaka, 1984) presenting a constructive characteristic free proof of resolution of surface singularities, followed by a generalization for non-algebraically closed base fields in (Cossart, 1981) and a conceptualization and globalization in context of taut resolution in (Hauser, 2000); (Hauser, 1998) providing a survey on the difficulties of resolution of singularities, especially in positive characteristic; (Encinas and Hauser, 2000) presenting a simplified proof for constructive resolution of singularities after Villamayor and Bierstone-Milman; (de Jong, 1996) with a characteristic free approach via alterations.

In this paper we present generalized and corrected results of our contribution to ISSAC 2000 (Bodnár and Schicho, 2000c), where we discussed three improvements of Villamayor's algorithm for resolution of singularities. This time we omit the description of the third improvement, which describes a sophisticated representation of resolution problems, because we think it is so closely related to the algorithm that it is not really interesting for a larger audience. For additional information on our implementation we refer to (Bodnár and Schicho, 2000a,b), in particular (Bodnár, 2000) contains the collected results discussed in all details. The software is available at `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup`.

In algebraic geometry we often have to deal with quasi-projective varieties. For instance, the most elementary nontrivial birational morphisms, the blowing ups define such varieties (see section 2). We could represent them via their (multi-)graded coordinate rings like in (Bayer *et al.*, 1993; Greuel *et al.*, 1998) but that is far too inefficient for many practical computations. In (Bodnár and Schicho, 2000a,b) we use atlases of affine charts to describe quasi-projective varieties.

Unfortunately, the chart representation also carries significant redundancy, which originates in the presence of isomorphic subsets (chart overlaps). Let the charts of the atlas of a nonsingular quasi-projective variety be indexed by natural numbers. A relevant point of a chart with a given index is one that lies in no chart with smaller index. Now it is easy to see that a chart with no relevant point is redundant, while to detect such a chart efficiently (without examining chart change maps explicitly) is not at all trivial.

In the resolution algorithm, multiple treatment of singularities that lie in overlaps of several charts causes computational redundancy. Obviously, such redundant resolutions should be avoided for the sake of efficiency. However, the charts that contain isomorphic singularities might also carry unique information, thus a careful separation of relevant and redundant subsets has to be done in each chart.

In section 3 we present a surprisingly simple method that copes with both kind of problems, requiring minimal bookkeeping.

Given some nonsingular ambient variety, the most natural way of working with its subschemes is to represent them by ideals (or sheaves of ideals) in the coordinate ring (structure sheaf of rings) of the variety. In many cases, for example to compute intersections of subschemes, we have to add ideals. This addition might require the equilibration of orders of the summands by means of rising their ideals to appropriate powers (e.g. coefficient ideal computation in section 4). This way we are able to use standard commutative algebra machinery in a straightforward way on the result, however, we might run into serious performance problems as soon as we intend to do actual computations.

In section 4 we develop an efficient representation and several computational methods for ideals, allowing formal rational powers to be assigned to the generators. Using the generalized representation, we are able to get rid of the cumbersome powering, transferring the equilibration exclusively to the rational exponents. Our work is based on Hironaka's results on idealistic exponents of singularities (Hironaka, 1977).

We would also like to mention that using the theory of section 4 we were able to incorporate special resolution strategies, like the "good points strategy" suggested in (Encinas and Villamayor, 1998) with origins in (Abhyankar, 1988), into our implementation of Villamayor's algorithm very easily.

Using the new techniques various geometric subtasks like computing the singular locus and testing its emptiness, or computing the proper transform of an algebraic set along a blowing up, can be performed much more efficiently than with ordinary ideals. Many of the computational subtasks of the resolution algorithm, including the above mentioned ones, are realized by Gröbner basis computation: Buchberger (1965, 1985); Becker and Weispfenning (1993).

2. Preliminary definitions

Henceforth k denotes an algebraically closed computable field of characteristic zero. The purpose of the first restriction is theoretical, not computational; in fact, all algorithms discussed here use only field operations. The second restriction can be removed in section 3, but it is essential in section 4.

Let $R = k[x_1, \ldots, x_n]$, $S = k[x_0, \ldots, x_n]$ with grading $\deg(a) = 0$ for $a \in k$,

$\deg(x_i) = 1$. The n-dimensional *affine space* over k is denoted as $\mathbb{A}^n$, while the n-dimensional projective space as $\mathbb{P}^n$. Let J be an ideal in R, $\mathbf{Z}(J)$ denotes the zero set of J in $\mathbb{A}^n$. A set $X \subseteq \mathbb{A}^n$ is called an *affine algebraic set* if there exists an ideal $J \subseteq R$ such that $X = \mathbf{Z}(J)$. For an affine algebraic set X, $\mathbf{I}(X)$ denotes the unique radical vanishing ideal of X in R. The dimension of X is the *Krull dimension* of its coordinate ring $k[X] = R/\mathbf{I}(X)$. Analogous definitions can be given for projective algebraic sets, defined by homogeneous vanishing ideals in S. We consider the *Zariski topology* on $\mathbb{A}^n$ and $\mathbb{P}^n$, where the open sets are the complements of algebraic sets. We call an open subset of an affine algebraic set a *quasi-affine algebraic set* and an open subset of a projective algebraic set a *quasi-projective algebraic set*. An algebraic set is called *irreducible*, or a *variety*, if it is not the union of two of its proper closed subsets.

Identifying $\mathbb{A}^n$ with a principal open subset of $\mathbb{P}^n$ we can talk about the *projective closure* of affine algebraic sets. A homogeneous defining ideal of the projective closure of the affine algebraic set can be computed through the *homogenization* of the polynomials in a Gröbner basis of the affine ideal, with respect to a degree compatible term order (see Eisenbud (1994) page 363). Therefore affine algebraic sets are quasi-projective.

Let X, Y be algebraic sets (quasi-affine or quasi-projective). A continuous map $\varphi : X \to Y$ is called a *morphism*, if for all open $V \subseteq Y$ and for all regular $f : V \to k$ the map $f \circ \varphi : \varphi^{-1}(V) \to k$ is also regular on $\varphi^{-1}(V)$. A morphism of affine varieties is a map which can be given by means of polynomials. A morphism is called *proper* iff it is universally closed and separated (see Hartshorne (1977), page 100). We remark that blowing ups (defined below) are proper morphisms and that composition of proper morphisms is again proper. A morphism $\varphi : X \to Y$ of varieties is *birational* if there exists an open dense subset $V \subseteq Y$, with $U = \varphi^{-1}(V)$, and a map $\pi : V \to U$, which can be given by means of rational functions, such that $\pi \circ \varphi|_U = \mathrm{id}_U$ and $\varphi|_U \circ \pi = \mathrm{id}_V$.

Let X be an affine algebraic set of dimension r defined by $f_1, \ldots, f_k \in R$. X is *singular at a point* $a \in \mathbb{A}^n$ if the Jacobian matrix of $f_1, \ldots, f_k$ at a is of rank smaller than $n - r$. The set of singular points of X is called the *singular locus* of X. The singular locus is a closed algebraic set, which can be defined in X by the ideal $\langle \partial_{x_i} f | i = 1, \ldots, n\rangle$ if X is the hypersurface defined by $f \in R$.

At a nonsingular point a of an algebraic set W of dimension n, $f_1, \ldots, f_n$ regular functions on some open $U \subseteq W$ containing a form a *regular system of parameters* iff they all vanish at a, and $df_1, \ldots, df_n$ generate the module of differential forms Ω_U. For a nonsingular n-dimensional affine subset U of W, we call $p_1, \ldots, p_n \in k[U]$ a *global system of regular parameters* if for all $a \in U$, $p_1 - p_1(a), \ldots, p_n - p_n(a)$ form a system of regular parameters at a.

Let W be a projective algebraic set with $\overline{x}_0, \ldots, \overline{x}_n$ generators for $k[W]$. Let $Z \subset W$ be an algebraic set defined by the homogeneous ideal $\langle g_0, \ldots, g_s\rangle \subseteq k[W]$ with generators of the same degree d. The *blowing up* of W along Z is defined as a proper birational morphism $\pi : W' \to W$, where $W' \subset W \times \mathbb{P}^s$ is defined by the ideal

$$\{f \in k[W][y_0, \ldots, y_s] \mid f(tg_0, \ldots, tg_s) = 0 \text{ in } k[W][t]\}.$$

The morphism π is the projection to W (on the first $n+1$ coordinates). The preimage of Z is called the *exceptional divisor* of the blowing up. We denote the class of y_i in $k[W']$ by $\overline{y}_i$. The ring $k[W']$ is doubly graded with $\deg(a) = 0$ for $a \in k$, $\deg(\overline{x}_i) = 1$, $\deg(\overline{y}_j) = d$ and $\deg'(a) = 0$ for $a \in k[W]$, $\deg'(\overline{y}_j) = 1$. If W is an affine algebraic set the definition works by dropping the $\deg(\cdot)$-grading and the requirement that Z is defined by a homogeneous ideal.

The product $\mathbb{P}^n \times \mathbb{P}^s$ of projective spaces can be identified with a variety of $\mathbb{P}^{ns+n+s}$ by sending $(a_0 : \ldots : a_n) \times (b_0 : \ldots : b_s) \mapsto (a_0b_0 : a_0b_1 : \ldots : a_nb_s)$. This map is well-defined, injective and is called the *Segre embedding*. Thus we can see that blowing ups of quasi-projective algebraic sets are quasi-projective.

Given a variety Y, a *desingularization* of Y is a nonsingular variety X together with a proper, birational morphism $\pi : X \to Y$.

An algebraic set X contained in a nonsingular variety W has *normal crossings* iff for all points $a \in W$, there exists a regular system of parameters, such that each irreducible component of X is generated by a subset of this parameter system locally at a. Let W be a nonsingular variety, and $X \subset W$ be an algebraic set of codimension one with respect to W. We call the pair (W, X) a *hypersurface*. Let $\pi : W' \to W$ be a proper birational morphism with inverse defined on the open subset $U \subseteq W$. We call the closure of $\pi^{-1}(X \cap U)$ in W' the *proper transform* of X along π.

Given a hypersurface (W, X), an *embedded desingularization* is a nonsingular variety W' together with a proper, birational morphism $\pi : W' \to W$, such that the proper transform of X is nonsingular and the *total transform* $\pi^{-1}(X)$ has normal crossings.

When X is irreducible, an embedded desingularization of (W, X) provides a desingularization of X: let X' be the proper transform of X, then $\pi|_{X'} : X' \to X$ is a desingularization of X. The Villamayor-algorithm computes an embedded desingularization of a hypersurface by constructing a suitable sequence of blowing ups, whose composition provides the desingularization map.

It is essential in computational algebraic geometry to find an appropriate "machine representation" for the objects of our interest, which additionally allow easy manipulation of the objects. To achieve this goal, we describe an n-dimensional quasi-projective variety W by a finite list of n-dimensional affine charts $(U_i \mid i = 1, \ldots, m)$, where U_i is described by the following data:

– A list of affine variables $\mathrm{VAR}_i = (x_1, \ldots, x_{m_i})$, $m_i \geq n$. These are the generators of the polynomial ring $k[\mathrm{VAR}_i]$.
– A finite list $\mathrm{EQS}_i \subset k[\mathrm{VAR}_i]$. The chart is equal to the zero set $\mathbf{Z}(\mathrm{EQS}_i)$ in $\mathbb{A}^{m_i}$. (We have $k[U_i] = k[\mathrm{VAR}_i]/\langle \mathrm{EQS}_i \rangle$.)
– A list of lists of rational functions from $k(U_i)$,

$$\mathrm{MAP}_i = (\mathrm{MAP}_{i,j} \mid \mathrm{MAP}_{i,j} = (g_{j,1}/h_{j,1}, \ldots, g_{j,m_j}/h_{j,m_j}), j = 1, \ldots, m),$$

$g_{j,k}, h_{j,k} \in k[U_i]$. These lists represent the *chart change maps* from U_i to U_j as ring homomorphisms from $k[U_j]$ to $k[U_{i,j}]$ given by their values on the generators of $k[U_j]$, where $U_{i,j} \subseteq U_i$ is the domain of definition for $\mathrm{MAP}_{i,j}$.

We require that the MAP entries represent patching dominant birational maps, i.e., the map given by $\mathrm{MAP}_{i,j}$ restricts to an isomorphism on $U_{i,j}$ with inverse $\mathrm{MAP}_{j,i}$ on $U_{j,i}$. In particular $\mathrm{MAP}_{i,i}$ has to represent the identity on U_i. Patching means that $\mathrm{MAP}_{i,k}$ represents the same map as $\mathrm{MAP}_{j,k} \circ \mathrm{MAP}_{i,j}$ over $U_{i,j} \cap U_{i,k}$ for all $1 \leq i, j, k \leq m$.

In this context, we refer to closed points of the charts as *chart representatives* of closed points of W, which are called *geometric points*. Actually, having W given in this representation, we can think of its points as classes of chart representatives that are glued together by the chart change maps: $a_i \in U_{i,j}$ and $a_j \in U_{j,i}$ belong to the same class (or represent the same geometric point) iff a_j is the image of a_i under $\mathrm{MAP}_{i,j}$. For $a_i \in U_i$, $\overline{a}_i$ denotes the geometric point represented by a_i.

The above description is called the representation of W by an *atlas* of affine charts.

3. Elimination of chart redundancies

From now on W denotes a nonsingular quasi-projective variety. The representation of W by an atlas of affine charts has its pros and cons from the computational point of view. An advantage is that the charts are affine, thus the computations in their coordinate rings are straightforward, i.e., there is no need to complicate operations (like blowing up) to make them compatible with multiple gradings. Another important fact is that many problems in algebraic geometry are of local nature, offering the utilization of the "divide and conquer" principle by separating local subproblems into different charts.

On the other hand, this representation has its inherent redundancy, because chart representatives of geometric points may appear in many charts. For instance, in the desingularization problem this would lead to multiple resolution of isomorphic singularities over different charts. In this section we present a simple but efficient method to cope with this drawback.

We introduce a new data entry in the description of an affine chart U_i given in section 2:

- A finite list $\mathrm{FOCUS}_i \subset k[U_i]$.

Let W be represented by $(U_1, \ldots, U_m)$. We postulate that for all geometric points $\overline{a} \in W$ there is a unique U_i such that the corresponding chart representative $a \in \mathbf{Z}(\mathrm{FOCUS}_i)$.

In simple words, we want to achieve that the FOCUS entries of the charts of W form a decomposition of W, which differs from a partition only in that we allow empty components to be present. First we show how to compute the

FOCUS entries for a given atlas, then we describe how to transform them under basic operations without explicit examination of the chart change maps.

Theorem 3.1: *Let* $(U_1, \dots, U_m)$ *represent* W. *Let* FOCUS_1 *be the empty list, and for* $i = 2, \dots, m$, *let* $\mathrm{FOCUS}_i = (\prod_{g_{j,k}/h_{j,k} \in \mathrm{MAP}_{i,j}} h_{j,k} \mid j = 1, \dots i-1)$. *This setup fulfills the postulate of* FOCUS *entries.*

Proof: Let us remark that the sets $U_{i,j}$ are exactly the intersections of the domains of definition of the functions in $\mathrm{MAP}_{i,j}$.

We have that for $a_i \in \mathbf{Z}(\mathrm{FOCUS}_i)$ there exists no $a_j \in \mathbf{Z}(\mathrm{FOCUS}_j)$ for $j < i$ with $\overline{a}_i = \overline{a}_j$. For $i = 1$ the claim is trivial. For $i > 1$, if there were a_i and a_j with the above properties, because $\mathbf{Z}(\mathrm{FOCUS}_i) \subseteq U_i \setminus U_{i,j}$, a_i would fall outside the set of isomorphic points between U_i and U_j, making it impossible for a_i to represent the same geometric point as any $a \in U_j$.

To see that the points of the FOCUS entries exhaust W, let us assume that there exists a geometric point $\overline{a} \in W$ for which every chart representative $a_i \notin \mathbf{Z}(\mathrm{FOCUS}_i)$. Let us take the smallest index i for which a_i is a representative for $\overline{a}$. Then we can be sure that $a_i \in U_i \cap \bigcap_{j<i} (U_i \setminus U_{i,j})$, but this is just $\mathbf{Z}(\mathrm{FOCUS}_i)$. □

Now we describe two fundamental operations on varieties which are, for instance, also elementary steps of the resolution algorithm in (Bodnár and Schicho, 2000a). The *Cover* operation replaces the input chart U by the complements of the zero sets of $f_1, \dots, f_k$. It is permitted only in the case when $\mathbf{Z}(f_1, \dots, f_k) = \emptyset$. The operation is described by the *Cover* algorithm. For the sake of simplicity, and because it can be guessed, we omit the manipulation of the list MAP.

Algorithm *Cover*$(U, (f_1, \dots, f_k) \subset k[U])$;
    $M := ()$;
    for $j = 1$ to k do
        $z :=$ a new variable;
        $\mathrm{VAR}_j :=$ append z to VAR;
        $\mathrm{EQS}_j :=$ append $zf_j - 1$ to EQS;
        $\mathrm{FOCUS}_j :=$ append M to FOCUS;
        $M :=$ append f_j to M;
        $U_j := (\mathrm{VAR}_j, \mathrm{EQS}_j, \mathrm{FOCUS}_j)$;
    return $(U_1, \dots, U_k)$;

Theorem 3.2: *Let* W *be represented by an atlas* $\mathcal{A}$ *which fulfills the postulate of* FOCUS *entries,* $U \in \mathcal{A}$. *Let* $f_1, \dots, f_k \in k[U]$ *be such that* $\mathbf{Z}(f_1, \dots, f_k) = \emptyset$, *and let* $\mathcal{A}'$ *be the atlas obtained by replacing* U *by its cover at* $(f_1, \dots, f_k)$. *Then* $\mathcal{A}'$ *represents* W *and fulfills the postulate of* FOCUS *entries.*

Proof: The proof is a consequence of the following statements, which are easy to check: (i) U_i represents the complement of $\mathbf{Z}(f_i)$; (ii) $\mathbf{Z}(\mathrm{FOCUS}_i)$ corresponds to the intersection of $\mathbf{Z}(\mathrm{FOCUS})$ with $(\mathbf{Z}(f_1) \cap \dots \cap \mathbf{Z}(f_{i-1})) \setminus \mathbf{Z}(f_i)$. □

EXAMPLE 3.3: Let us describe $\mathbb{A}^3$ with the affine chart: $U_0 = (\mathrm{VAR}_0 = (x_1, x_2, x_3)$, $\mathrm{EQS}_0 = ()$, $\mathrm{FOCUS}_0 = ())$. In figure 1 we can see the result of the function call $(U_1, U_2, U_3, U_4) = Cover((U_0), (x_1, x_2, x_3, x_1 - x_2 - 2x_3 + 2))$. If we follow the steps of the algorithm we can easily generate the charts $U_1, \ldots, U_4$: At each step we introduce a new variable, say x_4, and the new dependency $x_4 f - 1$ which describes that f, the image of the corresponding polynomial from the second argument of *Cover* under the natural inclusion of $k[x_1, x_2, x_3]$ into $k[x_1, x_2, x_3, x_4]$, is invertible over the new chart with inverse x_4. Finally we update the FOCUS entry. For example, $U_4 = (\mathrm{VAR}_3 = (x_1, x_2, x_3, x_4)$, $\mathrm{EQS}_4 = (x_4(x_1 - x_2 - 2x_3 + 2) - 1)$, $\mathrm{FOCUS}_4 = (x_1, x_2, x_3))$.

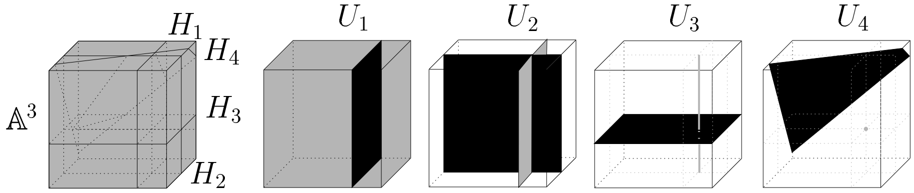

Figure 1: FOCUS in *Cover* operation (FOCUS: grey, missing subsets: black)

From now on, for the sake of simpler presentation, we consider only those blowing ups of W whose centers can locally, in a chart U of W, be described by global regular parameters. Our decision is also justified by the fact that the resolution algorithm computes exactly this kind of centers.

The blowing up of W along a given center in the chart-representation is performed by the *BlowUp* algorithm. Its input is a chart U and a list of global regular parameters $(p_1, \ldots, p_r)$, defining the blowing up center. We also assume that $\mathbf{Z}(p_1, \ldots, p_r) \neq \emptyset$ (otherwise the blowing up is trivial). The algorithm replaces U by charts with coordinate ring $k[U][\frac{p_1}{p_i}, \ldots, \frac{p_r}{p_i}]$. It is well-known that these charts cover the blown up set.

Algorithm $BlowUp(U,\ (p_1, \ldots, p_r) \subset k[U])$;
  for $i = 1$ to r do
    $\mathrm{VAR}_i := \mathrm{VAR}$; $\mathrm{EQS}_i := \mathrm{EQS}$; $\mathrm{FOCUS}_i := \mathrm{FOCUS}$;
    for $j = 1$ to r, $j \neq i$ do
      $z :=$ a new variable;
      $\mathrm{VAR}_i :=$ append z to VAR_i;
      $\mathrm{EQS}_i :=$ append $zp_i - p_j$ to EQS_i;
      if $j < i$ then
        $\mathrm{FOCUS}_i :=$ append z to FOCUS_i;
  return $(U_1, \ldots, U_r)$;

As the *BlowUp* operation, unlike *Cover*, does effect in a geometrical change, it is not true that blown up charts coming from an atlas automatically patch together. In the case of Villamayor's algorithm, the consistent patching is ensured by a theoretical result (see Villamayor (1989)).

In the consistent case, we have a center which intersects a subset of charts in the zero set of a set of global parameters. Obviously, the blowing up operation is trivial on the complement of this subset.

THEOREM 3.4: *Let W be a nonsingular variety, and let Z be a nonsingular subvariety. Let $\mathcal{A}$ be an atlas such that, for each chart intersecting Z, the intersection is the zero set of a set of global parameters. Let $\mathcal{A}'$ be the atlas obtained by replacing all charts of $\mathcal{A}$ that intersect Z by their* BlowUp *along the corresponding set of global parameters. Then $\mathcal{A}'$ represents the blowup of W along Z. Moreover, if $\mathcal{A}$ fulfills the postulate of* FOCUS *entries, so does $\mathcal{A}'$.*

Proof: The proof is a consequence of the following facts. The blowup variety of an open subset $W_0 \subset W$ is the inverse image of W_0 with respect to the blowing down map. This reduces the question to affine W, i.e., to the case of charts. Let $z_{i,j}$, be the j-th new coordinate in the chart U_i of the blowup variety. Then the sets $\mathbf{Z}(z_{i,1}, \ldots, z_{i,i-1}) \subset U_i$ represent a partition of the blowup variety into disjoint subsets. □

In our improved version of Villamayor's resolution algorithm, we use FOCUS in two ways.

(i) After a chart is created by *Cover* or *BlowUp*, we test whether $\mathbf{Z}$(FOCUS) is empty. If yes, we simply drop the chart; it does not add any new geometric points.

(ii) Before a chart is blown up by a *BlowUp* operation, say at $p_1 = \cdots = p_r = 0$, we check whether $\mathbf{Z}(\text{FOCUS} \cup \{p_1, \ldots, p_r\})$ is empty. If yes, we have already computed the blowing up in a different chart, and we want to avoid to do it again.

This can be achieved with a trick: instead of blowing up the chart, we do a *Cover* operation with respect to the tuple $(p_1, \ldots, p_r, 1)$ (the order is important!). The output charts $U_1, \ldots, U_r$ have empty intersection with the center Z, so they remain unchanged in the subsequent geometric blowing up. The last chart U_{r+1} is actually isomorphic to U, but its FOCUS has empty zero set; therefore we drop it as in (i). The advantage is that we have replaced a *Blowup* operation by a computationally cheaper *Cover* operation.

4. Powered ideals

The aim of this section is to provide an efficient representation of classes of affine algebraic sets that pose equivalent problems from the viewpoint of resolution of singularities. The results allow straightforward generalization to subschemes of quasi-projective varieties via the representation given at the end of section 2.

The cornerstone of our approach is Hironaka's theory of idealistic exponents of singularities presented in (Hironaka, 1977). We redefine the notion of idealistic exponents, allowing positive rational numbers to be attached to generators of ideals, acting as formal powers. Then we present several techniques to work with such ideals, in particular, we give a few lemmas which can be used to pass to an often significantly simpler representative of the class.

To motivate the introduction of powered ideals, we would like to present an important step in constructive resolution of singularities. It is called *coefficient ideal* computation, and it performs the transition of the search for the next blowing up center into lower dimension. Let U be an affine chart, let $f \in k[U]$ generate a nonzero proper ideal and let $b = \mathrm{ord}(f)$ be the maximal order of f at points in U. Let p be a global regular parameter, defining a hypersurface V of *maximal contact* for f (for the definition we refer to Encinas and Villamayor (2000), or, in context of powered ideals, to definition 4.22 in this section). The partial derivatives with respect to a fixed global system of regular parameters containing p are defined as in (Bodnár and Schicho, 2000a). Then the coefficient ideal of f with respect to p is defined as

$$\mathrm{Coeff}(f, b, V) := \langle (\partial_p^i f)^{\frac{b!}{b-i}} \mid i = 0, \ldots, b-1 \rangle k[V].$$

The only reason for this powering is to give different generators of the ideal different weights, in order to have the singularities of the coefficient ideal "approximate" the singularities of J in V. On the other hand, it is impossible to do actual computations, e.g. Gröbner basis, with ideals generated by nontrivial polynomials risen to the power 5! and higher. We must also note that even the resolution of a singular point on a surface requires two consecutive coefficient ideal computations, which leads to exponents with accumulated factorials (e.g. in case of a 4-fold point: 4!!).

Using powered ideals we can describe the coefficient ideal much simpler where the factorial expressions do not even appear in the formal exponents attached to the generators.

In this section U denotes a nonsingular affine chart with structure as in section 2, $F = \{(f_1, b_1), \ldots, (f_k, b_k)\}$, $G = \{(g_1, c_1), \ldots, (g_m, c_m)\}$ are finite sets of ordered pairs from $k[U] \times \mathbb{Q}_+$.

Definition 4.1: (cf. Definition 1 on page 54 in Hironaka (1977)) A *powered ideal* over U is an equivalence class of finite sets of pairs from $k[U] \times \mathbb{Q}_+$ under the equivalence relation $\sim$ defined as: $F \sim G$ if and only if for any ring homomorphism $\varphi : k[U] \to A$ we have $v_A(F) = v_A(G)$ with $v_A(F) := \min_i(v(\varphi(f_i))/b_i)$, where A is a valuation ring and v is its valuation.

Remark 4.2: The relation $\sim$ of definition 4.1 is indeed an equivalence, because it is based on equality.

We denote the powered ideal of the representative F by $\langle F \rangle$ or simply by the representative F itself.

DEFINITION 4.3: The *order* of F at $a \in U$ is $\mathrm{ord}_a(F) := \min_i(\mathrm{ord}_a(f_i)/b_i)$, where $\mathrm{ord}_a(f_i)$ is the maximal d such that $f_i \in \mathfrak{m}^d$, $\mathfrak{m}$ being the maximal ideal of $\mathcal{O}_{U,a}$; in case $f_i = 0$, $\mathrm{ord}_a(f_i) = \infty$.

LEMMA 4.4: *The* $\mathrm{ord}_a : k[U] \times \mathbb{Q}_+ \to \mathbb{Q}_{\geq 0} \cup \infty$ *function is compatible with* $\sim$.

Proof: The $\mathrm{ord}_a : k[U] \to \mathbb{Z}_{\geq 0} \cup \infty$ function gives rise to a valuation of the field of fractions of $\mathcal{O}_{U,a}$, hence the ord_a function takes the same value on equivalent representatives. □

DEFINITION 4.5: The *zero set* of F is defined as $\mathbf{Z}(F) := \{a \in U \mid \mathrm{ord}_a(F) > 0\}$. A point $a \in U$ is a *singularity* of F if $\mathrm{ord}_a(F) \geq 1$. The *singular locus* of F is defined as $\mathrm{Sing}(F) := \{a \in U \mid \mathrm{ord}_a(F) \geq 1\}$.

REMARK 4.6: (i) The $\mathbf{Z}$ and Sing functions of definition 4.5 are compatible with $\sim$, because of lemma 4.4. (ii) The Sing and ord_a functions are effectively computable using partial differentiation with respect to a global system of regular parameters over U and the method of Gröbner bases.

LEMMA 4.7: *We have* $\{(f,b)\} \sim \{(f^l, bl)\}$ *for* $l \in \mathbb{N}_+$.

Proof: Apply the property $v(xy) = v(x) + v(y)$ of valuations to the situation of definition 4.1, getting $v(\varphi(f))/b = v(\varphi(f))l/bl = v(\varphi(f^l))/bl$. □

DEFINITION 4.8: The *sum* of F and G, denoted as $F + G$, is defined as the set theoretic union of F and G.

LEMMA 4.9: *If* $F \sim G$ *and* $F' \sim G'$, *we also have* $F + F' \sim G + G'$.

Proof: We have for any ring homomorphism $\varphi : k[U] \to A$ as in definition 4.1 $v_A(F) = v_A(G)$ and $v_A(F') = v_A(G')$, but then we also have $v_A(F + F') = \min(v_A(F), v_A(F')) = \min(v_A(G), v_A(G')) = v_A(G + G')$. □

REMARK 4.10: (i) For any $a \in U$, $\mathrm{ord}_a(F + G) = \min(\mathrm{ord}_a(F), \mathrm{ord}_a(G))$. (ii) $\mathrm{Sing}(F + G) = \mathrm{Sing}(F) \cap \mathrm{Sing}(G)$.

DEFINITION 4.11: The blowing up $\pi : W \to U$ with center Z is *permissible* for F iff Z is regular and $Z \subseteq \mathrm{Sing}(F)$.

In the classical approach to the resolution of singularities it is crucial to have coefficient ideal computation and transformation of ideals along blowing ups commute. For this we have to take different transformations for the base and the coefficient ideals.

DEFINITION 4.12: Let $\pi : W \to U$ be a permissible blowing up for F with center Z, let $\langle h \rangle := \mathbf{I}(\pi^{-1}(Z))$ and let us assume for F: $b_i = r_i/q_i$, $r_i, q_i \in \mathbb{Z}_+$. The *controlled transform* of F along π is $\mathrm{T}(F) := \{(h^{-r_i} f_i^{q_i} k[W], r_i)\}_i$.

Remark 4.13: (i) When $\mathrm{ord}(F) = 1$, the controlled transform is called *weak transform* (originally introduced on page 209 in Hironaka (1964)). The name *controlled transform* is introduced in Encinas and Hauser (2000) according to our best knowledge. In context of the paragraph before the definition, we have to take the weak transform of the base ideal and the controlled transform of the coefficient ideal for commutation. (ii) For $F \sim G$ and a permissible blowing up as in definition 4.12, one can show $\mathrm{T}(F) \sim \mathrm{T}(G)$; however, we will not use this fact here.

Definition 4.14: Let $J := \{f_1, \ldots, f_k\} \subset k[U]$, $b \in \mathbb{Z}_+$. The *expansion* of the pair (J, b) as a generating set of a powered ideal is $\mathrm{E}(J, b) := \{(f_i, b)\}_i$.

Lemma 4.15: *Let $J := \{f_1, \ldots, f_k\}$, $J' := \{f'_1, \ldots, f'_l\}$ be finite sets of polynomials from $k[U]$ and let $b \in \mathbb{Z}_+$. If $\langle J\rangle = \langle J'\rangle$, we have $\mathrm{E}(J, b) \sim \mathrm{E}(J', b)$.*

Proof: Let us consider a ring homomorphism $\varphi : k[U] \to A$ as in definition 4.1. We have $f_i = \sum_{j=1}^{l} h'_j f'_j$ and $f'_j = \sum_{i=1}^{k} h_i f_i$, $(h'_j, h_i \in k[U])$, so $v(\varphi(f_i)) \geq \min_j(v(\varphi(h'_j)) + v(\varphi(f'_j)))$. Because A is a valuation ring $v(f) \geq 0$ for all $f \in A$, thus $v(\varphi(f_i)) \geq \min_j(v(\varphi(f'_j)))$ and by the same token $v(\varphi(f'_j)) \geq \min_i(v(\varphi(f_i)))$; therefore we get $v_A(\mathrm{E}(J, b)) = v_A(\mathrm{E}(J', b))$. □

Remark 4.16: In the context of definition 4.14, we have: (i) The pair (J, b) is a representative of an idealistic exponent on U in the sense of Definition 3 on page 56 in Hironaka (1977). (ii) $\mathrm{ord}_a(J) = \mathrm{ord}_a(\mathrm{E}(J, b)) \cdot b$ at any $a \in U$, for any $b \in \mathbb{Z}_+$. (iii) By defining $\mathrm{Sing}(J, b) := \{a \in U \mid \mathrm{ord}_a(J) \geq b\}$, we have $\mathrm{Sing}(J, b) = \mathrm{Sing}(\mathrm{E}(J, b))$. (iv) A blowing up $\pi : W \to U$ with center Z is permissible for (J, b) iff Z is regular and $Z \subseteq \mathrm{Sing}(J, b)$. (v) In the above blowing up, with $\langle h\rangle$ being the ideal for $\pi^{-1}(Z)$, we define $\mathrm{T}(J, b) = (\langle h\rangle^{-b} J k[W], b)$.

Lemma 4.17: *For (J, b) as in definition 4.14 a blowing up is permissible for $\mathrm{E}(J, b)$ iff it is permissible for (J, b), moreover $\mathrm{T}(\mathrm{E}(J, b)) \sim \mathrm{E}(\mathrm{T}(J, b))$.*

Proof: Because $\mathrm{Sing}(J, b) = \mathrm{Sing}(\mathrm{E}(J, b))$ (see remark 4.16(iii)) the first part is immediate. Because of permissibility h^b is a factor of every element of $Jk[W]$; therefore $Jk[W] \subseteq \langle h\rangle^b$. Knowing this, we may compute $\langle h\rangle^{-b} Jk[W]$ by division on the generators as in definition 4.12. □

Definition 4.18: A sequence of blowing ups $\pi_1, \ldots, \pi_l$ is permissible for F, iff π_i is permissible for F_{i-1}, where $F_0 = F$, $F_i = \mathrm{T}(F_{i-1})$ along π_i, $i = 1, \ldots, l$.

The subtle notion of the coefficient ideal is exemplified by the fact that its construction for powered ideals is not compatible with the equivalence relation $\sim$. In order to make the theory work we have to define a stronger equivalence relation, $\approx$, whose classes are bigger than the ones of $\sim$.

DEFINITION 4.19: (cf. the definition of $\approx$ on page 104 in Hironaka (1977)) Two powered ideals generated by F, G are *strongly equivalent*, denoted as $\langle F\rangle \approx \langle G\rangle$ or sometimes simply $F \approx G$, iff for any $d \geq 0$ we have for F and G considered in $k[U \times \mathbb{A}^d] \times \mathbb{Q}_+$ that $\mathrm{Sing}(F) = \mathrm{Sing}(G)$ and a sequence of blowing ups is permissible for F iff it is permissible for G.

REMARK 4.20: (i) Note that Hironaka's definition of strong local equivalence on page 67 in Hironaka (1977) slightly differs from his definition of strong equivalence, which is the analogue of definition 4.19. (ii) The relation $\approx$ is indeed an equivalence. (iii) If $F \approx G$ then for a permissible blowing up $\mathrm{T}(F) \approx \mathrm{T}(G)$. (iv) As $F \sim G$ implies equality of orders of transforms of F and G along any blowing up at any point of the blown up variety (which yields identical singular loci at any stage) we have $F \approx G$.

LEMMA 4.21: *If $F \approx G$ and $F' \approx G'$, we also have $F + F' \approx G + G'$.*

Proof: Using the notation of definition 4.19, for any d a sequence of blowing ups is permissible for $F + F'$ iff it is permissible for F and for F', because $\mathrm{Sing}(F + F') = \mathrm{Sing}(F) \cap \mathrm{Sing}(F')$. This is true also for the transforms of F and F' after any permissible blowing up sequence (this fact comes easily from $\mathrm{T}(F + F') = \mathrm{T}(F) + \mathrm{T}(F')$). Then by the assumption we get $\mathrm{Sing}(F + F') = \mathrm{Sing}(G + G')$, and blowing up sequences are permissible for $F + F'$ iff they are permissible for $G + G'$. □

DEFINITION 4.22: We say that a hypersurface with equation p has maximal contact for F, iff $F \approx F + \{(p, 1)\}$.

REMARK 4.23: Obviously, the property of maximal contact depends only on the strong equivalence class of the powered ideal.

THEOREM 4.24: *For $F \approx G$ we have $\mathrm{ord}_a(F) = \mathrm{ord}_a(G)$ for any $a \in \mathrm{Sing}(F)$.*

Proof: By lemma 4.9 and lemma 4.7 we can choose a representative $F' := \{(f_i^{q_i b/r_i}, b)\}_i$ for $\langle F\rangle$ (likewise a G' for $\langle G\rangle$), where we assume $b_i = r_i/q_i$, $b := \mathrm{lcm}_i(r_i)$. Such representatives are in one to one correspondence with Hironaka's representatives of idealistic exponents via $I : F' \mapsto (\langle\{f_i^{q_i b/r_i}\}_i\rangle, b)$, with the property $\mathrm{ord}_a(F') = \mathrm{ord}_a(I(F'))$ for any $a \in U$. We also have by remark 4.20(iv) $F' \approx G'$.

Now we can apply Proposition 8 on page 68 in Hironaka (1977) for our particular case, where, using Hironaka's terminology, $I(F')$ has strong local equivalence to $I(G')$ and vice versa via the identity on U. We get that the order of F' and G', hence of F and G, is the same at any point of the common singular locus of F and G. □

Proposition 4.25: *For a pair (J, b), if a hypersurface has maximal contact to (J, b) (in the sense of Bodnár and Schicho (2000a)) then it also has maximal contact to* $\mathrm{E}(J, b)$ *(in the sense of definition 4.22).*

Proof: Suppose that p defines a hypersurface of maximal contact for (J, b). Then p is contained in the ideal generated by the partial derivatives of elements in J of order up to $b-1$, hence it is contained in $\mathbf{I}(\mathrm{Sing}(\mathrm{E}(J, b)))$; therefore it does not change the strong equivalence class. □

Definition 4.26: Let V be a hypersurface of maximal contact for F defined by $p \in k[U]$. The *coefficient ideal* of F with respect to V, denoted as $\mathrm{Coeff}(F, V)$, is generated by

$$\{(\partial_p^j f_{i,j}, b_i - j) \mid j = 0, \ldots, \lceil b_i \rceil - 1,\ i = 1, \ldots, k\} \ \subset k[V] \times \mathbb{Q}_+,$$

where ∂_p denotes the partial derivative into the direction of p in a system of regular parameters containing p.

Theorem 4.27: *In the context of definition 4.26, if $F \approx G$,* $\mathrm{Coeff}(F, V) \approx \mathrm{Coeff}(G, V)$.

Proof: Suppose that F and G are expansions, i.e., $b_1 = \cdots = b_k \in \mathbb{Z}_+$ and $c_1 = \cdots = c_m \in \mathbb{Z}_+$. By the proof of Theorem 5 on page 111 in Hironaka (1977), $\mathrm{Coeff}(F, V)$ has strong local equivalence with F via the embedding $V \subset U$, in the sense of Hironaka. The same holds for G. It follows that $\mathrm{Coeff}(F, V)$ and $\mathrm{Coeff}(G, V)$ are in strong local equivalence via the identity on V, which implies strong equivalence in the sense of definition 4.19.

In the general case, an obvious modification in the proof of Hironaka's lemma 5.4 on page 112 shows equality of singular loci, and a similar modification in Hironaka's note 5.5 on page 113 shows equality of the singular loci after any sequence of permissible transformations. □

Remark 4.28: Let (J, b) be a pair as in definition 4.14, V a regular hypersurface for $\mathrm{E}(J, b)$ as in definition 4.26, then we obviously have $\mathrm{Coeff}(\mathrm{E}(J, b), V) \approx \mathrm{E}(\mathrm{Coeff}(J, b, V), b!)$.

The following theorem has already been proved in (Villamayor, 1989), in fact, constructive resolution of singularities is presented there in context of idealistic spaces. Similar constructions also appear in (Bierstone and Milman, 1997) (see the notion of infinitesimal presentation on page 241 with the equivalence relation to be factored out in Definition 4.6 on page 242), which allow the computational benefits of techniques presented here to be exploited also in the algorithm of Bierstone and Milman.

In the proof of the next theorem we would like only to recapitulate the main points of the compatibility of Villamayor's algorithm with powered ideals, giving a relative argument to the correctness proof of the resolution algorithm with ordinary ideals.

Theorem 4.29: *Let $\mathcal{A}$ be Villamayor's algorithm described in Bodnár and Schicho (2000a). Let $\mathcal{A}'$ the algorithm obtained from $\mathcal{A}$ by replacing any operation on pairs by its corresponding operation on powered ideals. Then the blowing up centers and singular loci of $\mathcal{A}'$ are the same as those in $\mathcal{A}$. Especially, the same resolution is computed by both algorithms.*

Proof: The operations occurring in $\mathcal{A}$ are: computing singular locus, computing a hypersurface of maximal contact, computing transforms along blowing up, computing order, computing coefficient object, adding equations of exceptional divisors to the ideal. The fact that proper and monomial parts transform into proper and monomial parts of a weighted object (see the proof of lemma 3.3 in Bodnár and Schicho (2000a)) dispenses us of dealing with the operations of extracting proper part and monomial part of a transformed weighted object. In the second case where a weighted object may arise—namely as coefficient object—this extraction is trivial.

First, we show that any of these operations is well-defined when passing to classes of strong equivalence. For the singular locus, this is obvious from definition 4.19. For computing a hypersurface of maximal contact, we have already noted this in remark 4.23. For computing transforms, this is obvious from definition 4.19, as the blowing up must be permissible (see remark 4.20). For computing order, this is theorem 4.24. For computing coefficient objects, this is theorem 4.27. For adding fixed equations, we may use lemma 4.21.

Second and last, we show that any of these operations commute with expansion, up to strong equivalence. For the singular locus, this was noted in remark 4.16. For computing a hypersurface of maximal contact, this is proposition 4.25. For computing transforms, this is lemma 4.17. For computing order, this was also noted in remark 4.16. For computing coefficient objects, this is remark 4.28. For adding fixed equations, this follows easily from lemma 4.9. □

5. Conclusion

The ideas presented in this paper provide efficient computational techniques in algebraic geometry. Section 3 describes a simple method that allows us to detect redundant charts in an atlas of a quasi-projective variety, so that after the elimination of redundancies the distinguished subsets of the charts form a partition of the variety. The computation required to set up and maintain the data of these subsets is minimal and can be easily incorporated into standard operations like blowing up and subchart covering.

The results of the tests on the implementation of Villamayor's resolution algorithm showed that the number of charts in the atlas of the computed variety dropped significantly when we applied the method of section 3. The used processor time dropped proportionally, moreover, the benefit increased with the complexity of the resolution. Table 1 shows that with very simple examples we might find no charts that we can eliminate, thus the computations with FOCUS actually increase the time of the resolution. On the other hand, with nontrivial examples the speedup was over a factor of four. The tests were made using the results of section 4 to extend the computable set of examples.

Equation	n_1	t_1	n_2	t_2
$xy - zw$	8	3.4	8	3.8
$x^2y + y^2z + z^2x$	24	12.7	13	11.6
$x^2y^2z^2 - x^4 + y^4$	24	50.0	9	21.1
$(x^2 + y^2 - 1)^2z^2 - z^4$	33	50.8	16	25.2
$x^2 - yz^2$	371	199.7	203	112.7
$x^6 + y^6 - xyz$	344	1116.9	136	268.7
$(x^2 + y^2 + z^2 - 1)^2(x^2 + (y - 1)^2 - 1)^2$	18	225.5	9	173.2
$x^2y^2z^2 - x^4 + y^4 - z^4$	?	>70000.0	14	250.5
$x^4 - x^2y(z - 1)$	?	>100000.0	219	1162.6
$(x^2 + y^2 + z^2 - 1)^2w^2 - w^4$	48	83.6	23	43.1

Table 1: Notation: $n_1(n_2)$ the number of charts in the atlas without (with) the FOCUS used; $t_1(t_2)$ the used processor time in seconds respectively.

In some cases the computations without the FOCUS took so much time and memory that we had to stop the process. This phenomenon is due to the fact that some charts, which could have been eliminated, contained singularities whose redundant resolution had to be done. Since Gröbner basis computation is a primitive operation in this process, the presence of the new generators of these charts penalized this elementary step of the resolution much more than the presence of these chart would have worsened it if the Gröbner basis computations hadn't got more difficult.

Equation	t_1	t_2
$x^4 - x^2 + y^2z^2$	6672	6641.1
$x^2y^2z^2 - x^4 + y^4$	63.0	21.1
$(x^2 + y^2 - 1)^2z^2 - z^4$	1760.8	25.2
$x^2 + y^3z^3$	1398.2	187.6
$x^6 + y^6 - xyz$	1579.9	268.7
$(x^2 + y^2 + z^2 - 1)^2(x^2 + (y - 1)^2 - 1)^2$	?	173.2
$x^2y^2z^2 - x^4 + y^4 - z^4$	?	250.5
$x^4 - x^2y(z - 1)$	?	1162.6
$x^6 - y^4 + xyz^3$	?	59322.0
$(x^2 + y^2 + z^2 - 1)^2w^2 - w^4$	?	43.1

Table 2: Notation: $t_1(t_2)$ the used processor time in seconds with ordinary (powered) ideals.

Section 4 presents an efficient way of representing powers of ideals by allowing positive rational numbers to be attached to the generators that act as formal exponents. The results are based on Hironaka's theory of idealistic exponents, and include several computational methods for the new objects.

We are interested in powered ideals mainly because of the speedup we can get by using them in Villamayor's resolution algorithm. As we move to simpler representatives of a powered ideal the underlying computer algebra machinery becomes more effective while the program still resolves every ideal of the class, including the original one, with the computed sequence of blowing ups.

Table 2 shows some performance data of the Villamayor-algorithm using ordinary and powered ideals. Question marks denote computations when the computer algebra system refused to work with polynomials generated for intermediate resolution problems.

Unfortunately, the inherent complexity of Villamayor's algorithm still strikes hard. We can compute several demonstrative examples, similar to the ones in the above tables, but we are currently not able to compute full resolutions for middle sized and research problems. To extend the scope of the resolution algorithm to such problems, it is necessary to find much more effective invariants than the currently known ones. This can be the topic of future research.

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