The Finite Element Approximation for Minimal Surfaces Subject to the Plateau Problems

H. Gu*

*RISC-Linz Institute,
Johannes Kepler University, Linz, Austria,
Email: hgu@risc.uni-linz.ac.at

Abstract

This paper talks about generating the minimal surfaces which are subject to the well-known Plateau problem. The differential form of the Plateau problem is defined at first and, its associated discrete schemes which reduced from the finite element methods could be practically solved by the numerical iteration methods like the Newton's iteration. The convergence property of the finite element solutions are proved by steps and some multi-grid algorithms have been implemented to speed up the computation. These new approximation methods will be applied to the project of generating the minimal surfaces on computer softwares later.

Keywords: Plateau problem, variational form, convexity, brouwer's fixed point theorem, finite element, maximum value principle, multigrid method.

1 Introduction

The study of minimal surfaces is a branch of differential geometry, because the methods of differential calculus are applied to geometrical problems. One of the oldest questions is: "What is the surface of smallest area spanned by a given contour?" Such question is nontrivial despite the fact that every physical soap film appears to know the answer. The question of the existence to the minimal surface within such a fixed contour is usually taken as the well-known Plateau problem. It has been proved by Rado, Douglas and Osserman in the 1930's that the existence holds when the contour is constructed by Jordan curves piecewisely.

From the point of view of local geometry, a minimal surface can also be taken as the surface which has the zero average mean curvature on each surface point, e.g., a saddle shape.

Previously unknown and certainly unexpected minimal surfaces were found by David Hoffman and his collaborators at GANG, the Center for Geometry, Analysis, Numerics, and Graphics at the University of Massachusetts in 1985 [5]. They first used their MESH computer graphics system to find these surfaces, and then later proved their existence with fully rigorous mathematics. This truly excited the minimal surface community and piqued their interest in computer graphics.

Solving such minimal surface problems in optimal geometry also requires specialized software systems because there is often no explicit parametrization of a desired minimal surface. Although in the past years, lots of softwares have been produced for displaying minimal surfaces, e.g, the "Surface Evolver" [1] by applying the evolution techniques, which worked out by the geometric center of the University of Minnesota, there is very few numerical analysis for how the approximation procedure works. One possibility of using the finite element method solving minimal surfaces has been proposed in [7] years ago but computing its discrete scheme owns a very high complexity. In this paper, we discuss the finite element solutions to the discrete Plateau problem based on its equivalent representations. The convergence properties and the speedup techniques for the whole numerical approaches will be proved and some typical experimental results have been demonstrated to show the numerical error coincident to the theoretical analysis.

In the next section we will firstly define the differential equations to the Plateau problems and extend their variational forms from the continuous Sobolev space to the finite element space. Some error resolutions for solving the linear finite element equations will be previewed in Section 3, and we will focus on the proving of the convergence property of the discrete solution to the Plateau problem mainly by Brouwer's fixed point theorem in Section 4. In section 5 we introduce two kinds of multigrid algorithms in order to speed up the nomal computation and in Section 6 we show some numerical experiments of generating the minimal surface on Maple software for testing these new approximating methods.

2 Preliminaries

Due to the fact that the minimal surfaces has the sufficient and necessary property of having zero average mean curvature at each surface point, we can define the Plateau problem in terms of a nonlinear partial differential equation, which satisfying a fixed boundary function.

Let us also assume that such minimal surface can be represented explicitly by term $z = u(x, y)$ in a 3-dimensional space with domain Ω and the boundary restriction

$$u = f(x, y), \ \ (x, y) \in \partial\Omega,$$

then $u(x, y)$ must solve the following system of differential equations:

$$\begin{cases} -div((1 + u_y^2)u_x, (1 + u_x^2)u_y) + 6u_{xy}u_x u_y = 0 \ \text{ in } \ \Omega, \\ \qquad\qquad u = f(x, y), \ \ \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is strictly convex. Here the convexity of the domain Ω is especially required for purpose of the numerical analysis with respect to the finite element method. And furthermore, the strictly convexity is sufficient to assume that the solution to the previous equation is unique if $f(x, y)$, which satisfies the bounded slope condition [9], is the restriction to $\partial\Omega$ of a function in the Sobolev space $W_q^2(\Omega)$, for some $q > 2$.

Set the nonlinear bi-argument operator

$$A(u, v) = \int_\Omega [(1 + u_y^2)u_x v_x + (1 + u_x^2)u_y v_y + 6u_{xy}u_x u_y v]dxdy,$$

where $u_x, u_{xy}...$ denote $\partial_x u, \partial_x(\partial_y u)...$, etc. Then the equation (2.1) is weakly equivalent to the following variational form:

Looking for $w \in H_0^1(\Omega) \cap H^2(\Omega) \cap C_1(\Omega)$, such that

$$A(w + \bar{f}, v) = 0, \quad \forall v \in H_0^1(\Omega) \cap H^2(\Omega) \cap C_1(\Omega), \tag{2.2}$$

where $\bar{f}$ is the "regular" extension of f from $\partial\Omega$ to Ω, especially, we could take $\bar{f} \in W_q^2(\Omega), q > 2$. (Obviously, $u = w + \bar{f}$ is the exact soultion according to the previous uniqueness assumption.)

We now consider the discrete scheme for (2.2). Let $\Omega_h \subset \Omega$ be divided by mesh T_h with size h. $S_h(\Omega_h)$ be the finite element space constructed by the interpolation operator I_h [2, 3]. It is also assumed that the mesh partition is regular and all the grid nodes on $\partial\Omega_h$ also located on $\partial\Omega$.

Then the discrete solution $w_h \in S_h(\Omega_h)$ is defined by

$$A(w_h + \bar{f}, v) = 0, \qquad \forall v \in S_h(\Omega_h). \tag{2.3}$$

and $u_h = w_h + \bar{f}$ will be taken as the finite element solution to the Plateau problems.

The above discrete scheme appears to be a bunch of polynomial equations so that it can be practically solved either by the Newton's iteration method [6, 15], or by

symbolic eliminations if possible. And furthermore, solving such schemes has lower complexity compared to the method proposed in [7].

For the continuous models, w is unique according to the assumption, One sees that $w^h = w|_{\Omega_h} \in H_0^1(\Omega_h) \cap H^2(\Omega_h) \cap C_1(\Omega_h)$ satisfying

$$A(w^h + \bar{f}, v) = 0, \qquad \forall v \in H_0^1(\Omega_h) \cap H^2(\Omega_h) \cap C_1(\Omega_h)$$

also must be unique. Then we could in fact consider the oringinal problem based on a certain convex polygon domain. For convenience, we still take w^h as w, Ω_h as Ω, etc., and will prove that w_h convergence to the real solution w when the mesh size h goes to zero in the following sections.

3 Some Error Resolutions for Solving Linear Finite Element Equations

Let $L(\cdot)$ be a linear operator defined by

$$L(u) = -div(a_{11}u_x + a_{21}u_y, a_{12}u_x + a_{22}u_y) + b_1u_x + b_2u_y + cu,$$

where $a_{ij}, b_i (i, j = 1, 2)$, and $c \geq 0$ are continuous functions satisfying

$$a_{11}v_x^2 + a_{22}v_y^2 + (a_{12} + a_{21})v_xv_y \geq c_0(\nabla v)^2 \quad \forall v \in S_h(\Omega)$$

for some $c_0 > 0$.

According to the maximum value principle, it can be proved that the following equation

$$A'(P_hu, v)(= (L(P_hu), v)) = (f, v) \quad \forall v \in S_h(\Omega) \tag{3.1}$$

where $(\cdot, \cdot)$ denote the inner product on $L_2(\Omega)$, will has a unique solution for the given condition f [10, 12, 13].

Namely, $P_h : H_0^1(\Omega) \to S_h(\Omega)$ is the Galerkin's projection operator which satisfies

$$A'(u - P_hu, v) = 0 \quad \forall v \in S_h(\Omega).$$

And we have the error estimation result [2, 11].

Theorem 3.1 *For $h << 1$, P_hu admits the following estimate*

$$\|u - P_hu\|_{1,\infty} \leq Ch\|u\|_{2,\infty}, \tag{3.2}$$

for $C > 0$.

Moreover, we introduce the discrete Green functions as, given $z \in \Omega$, the Green functions $g^z_{h,x}, g^z_{h,y} \in S_h(\Omega)$ satisfying

$$A'(v, g^z_{h,x}) = v_x(z), \quad A'(v, g^z_{h,y}) = v_y(z) \quad \forall v \in S_h(\Omega). \tag{3.3}$$

It has been proved that [14]

$$\|g^z_{h,x}\|_{1,1} \leq C|\log h|, \tag{3.4}$$

$$\|g^z_{h,y}\|_{1,1} \leq C|\log h|, \tag{3.5}$$

for constant $C > 0$.

4 The Approximation to the Discrete Plateau Problems

Let us firstly consider the discrete problem (2.3) defined on a linear finite element space $S^1_h(\Omega)$ [3, 11]. The following lemma shows the sufficient and necessary conditions of being the finite element solution to problem (2.3).

Lemma 4.1 *For any w, w_h and $v \in S^1_h(\Omega)$,*

$$A(w_h + \bar{f}, v) = A(w + \bar{f}, v) + A'(w + \bar{f}; w_h - w, v) + R(w + \bar{f}, w_h + \bar{f}, v) \tag{4.1}$$

where $A'(w + \bar{f}; \cdot, \cdot)$ is the bilinear operator defined from (3.1) with the coefficients $a_{ij}, b_i, c,\ i, j = 1, 2$, defined respectively as

$$a_{11} = 1 + u_y^2,\ a_{22} = 1 + u_x^2,$$

$$a_{12} = a_{21} = -u_y u_x,$$

$$b_1 = 3u_y u_{xy} - 3u_{yy} u_x,$$

$$b_2 = 3u_x u_{xy} - 3u_{xx} u_y,$$

and

$$c = 0,$$

$w_h \in S^1_h(\Omega)$ solves (2.3) if and only if

$$A'(w + \bar{f}; w - w_h, v) = R(w + \bar{f}, w_h + \bar{f}, v),\ \ \forall v \in S^1_h(\Omega). \tag{4.2}$$

Further more, if $\|\partial_{xy} w_h\|_{0,\infty} \leq K$, then the remainder R satisfying

$$R(w + \bar{f}, w_h + \bar{f}, v) \leq C(K)\|w - w_h\|^2_{1,\infty}\|v\|_{1,1}. \tag{4.3}$$

Proof: Set $G(t) = A(w + \bar{f} + t(w_h - w), v)$. then it follows from the identity

$$G(1) = G(0) + G_t(0) + \int_0^1 G_{tt}(t)(1-t)dt.$$

It is easy to compute that

$$G_t(0) = A'(w + \bar{f}; w_h - w, v)$$

where the coefficients of A' satisfying the given conditions from the lemma. And, by taking

$$R(w + \bar{f}, w_h + \bar{f}, v) = \int_0^1 G_{tt}(t)(1-t)dt.$$

using the condition $\|\partial_{xy} w_h\|_{0,\infty} \leq K$, Cauchy-Fredriches ineuality and a straightforward calculation, we get that

$$\begin{aligned} & |R(w + \bar{f}, w_h + \bar{f}, v)| \\ \leq & \max|G_{tt}(t)| \\ \leq & C(K)\|w - w_h\|_{1,\infty}^2 \|v\|_{1,1}. \end{aligned}$$

Finally, if w_h solves (2.3), then $G(1) = 0$ and complete the proof.

Let P_h be the Galerkin's projection with respect to the bilinear operator $A'(w + \bar{f}; \cdot, \cdot)$, we have the following theorem then:

Theorem 4.1 *For problems of the form (2.2), there exist a constant $C > 0$, such that for mesh T_h with its size $h << 1$, the corresponding finite element equation (2.3) defined on the linear finite element space $S_h^1(\Omega)$ contains a solution w_h satisfying*

$$\|w_h - P_h w\|_{1,\infty} \leq Ch,$$

and thus we have the following error estimation

$$\|w_h - w\|_{1,\infty} \leq Ch,$$

according to Theorem 3.1.

Proof: Define a nonlinear operator $\Phi : S_h^1(\Omega) \to S_h^1(\Omega)$ by

$$A'(w + \bar{f}, \Phi(v) - w, \phi) = R(w + \bar{f}, v + \bar{f}, \phi) \quad \forall \phi \in S_h^1(\Omega).$$

It can be proved according to the maximum value priciple and the fact that $S_h^1(\Omega)$ is of finite dimensional, Φ is well defined and a contiunous operator. According to the result of Theorem 3.1 we can also get that

$$\|w - P_h w\|_{1,\infty} \leq Ch.$$

Then define a set $B = \{v \in S_h(\Omega) : \|v - P_h w\|_{1,\infty} \leq Ch, C > 0\}$. Since $S_h^1(\Omega)$ is a linear finite element space, by the inverse estimation, there exist $C_1, C_2, C_3 > 0$, such that for all $v \in S_h^1(\Omega)$,

$$\begin{aligned}
&\|\partial_{xy}(v - w)\|_{0,\infty} \\
&\leq \|\partial_{xy}(v - I_h w)\|_{0,\infty} + \|\partial_{xy}(I_h w - w)\|_{0,\infty} \\
&\leq C_1 h^{-1}(\|v - P_h w\|_{1,\infty} + \|I_h w - P_h w\|_{1,\infty}) + C_2 \\
&\leq C_3,
\end{aligned}$$

thus we get $\|\partial_{xy} v\|_{0,\infty}$ is uniformly bounded.

Now we can prove that $\Phi(B) \subset B$ which means Φ is a contraction operator. In fact, when we substitute ϕ by the discrete Green functions $\phi = g_{h,x}^z$ and $\phi = g_{h,y}^z$ into

$$A'(w + \bar{f}; \Phi(v) - P_h w, \phi) = R(w + \bar{f}, v + \bar{f}, \phi),$$

according to Lemma 4.1, (3.4) and (3.5), for all $v \in B$, we get

$$\begin{aligned}
&\|\Phi(v) - P_h w\|_{1,\infty} \\
&\leq C_0 |\log h| \|w - v\|_{1,\infty}^2 \\
&\leq 2C_0 |\log h| (\|P_h w - v\|_{1,\infty}^2 + \|w - P_h w\|_{1,\infty}^2) \\
&\leq 2C_0 |\log h| (C^2 h^2 + C^2 h^2) \\
&\leq Ch,
\end{aligned}$$

where $C_0, C_1 > 0$ and $h << 1$.

By Brouwer's fixed point theorem, there exist a solution $w_h \in B$, such that $\Phi(w_h) = w_h$. And according to Lemma 4.1, w_h solves (2.3) and satisfies

$$\|w_h - P_h w\|_{1,\infty} \leq Ch.$$

Thus we complete the proof.

Based on the lower order finite element space like $S_h^1(\Omega)$, to generate the discrete system (2.3) is convenient but the uniqueness property of the solution u_h is at present not able to be proved. However we could prove that problem (2.3) will contains a unique solution if the finite element space is conforming, e.g.,

$$S_h(\Omega) \subset H_0^1(\Omega) \cap H^2(\Omega) \cap C_1(\Omega).$$

Theorem 4.2 *For problems of the form (3.2), there exist a constant $C > 0$, such that in the conforming finite element subspace $S_h(\Omega)$ with its mesh partition parameter $h << 1$, the finite element equation will has a unique solution u_h satisfying*

$$\|u_h\|_{2,\infty} \leq C,$$

and

$$\|u_h - u\|_{1,\infty} \leq Ch.$$

Proof: For the proof of the existence and convergence properties of u_h, those are basically the same as for proving Theorem 4.1, only need to change the holding condition for the inequality (4.3) into $\|w_h\|_{2,\infty} \leq K$ and later could prove $\|v\|_{2,\infty}$ is also uniformly bounded.

For uniqueness, we consider that u_h should still maintains the minimal area property over the domain Ω in the conforming finite element space $S_h(\Omega)$. For details, see [4]

$$u_h = \min_{w|_\Omega = f, w-f \in S_h(\Omega)} \{A^*(w) = \int_\Omega (1 + |\nabla w|^2)^{1/2}\}. \tag{4.4}$$

In fact we could prove that definition of (4.4) is weakly equivalent to the original problem (2.3) under the assumption of a conforming finite element space (for instance, constructed by Agyris mesh). By the fact that the operator A^* is strictly convex [7] and $S_h(\Omega)$ is of finite dimensional, it then follows that there exists a unique minimizing solution u_h.

5 The Multi-grid Approaches

In this section, we introduce the multigrid speedup techniques, based on the previous computational method.

Our basic assumption is, the finite element space should be a subspace belongs to $H^2(\Omega) \cap C_1(\Omega)$. We denote H be the maximal mesh size to the coarse grid partition and h the maximal mesh size to the refined grid partition (see Figure 1). And we can set for instance, $0 < H^2 < h << H$.

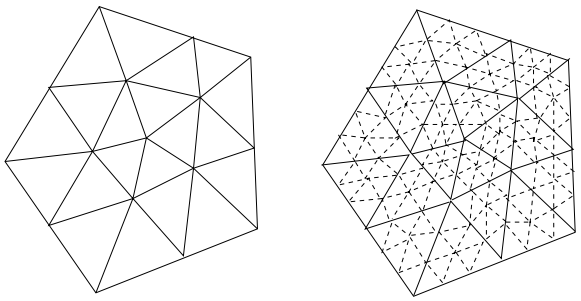

Figure 1: The coarse and refined grid partition

The multi-grid algorithm, roughly speaking, is to use the coarse grid approximation on finite element space $S_H(\Omega)$ as an initial result for the recollection on the refined space $S_h(\Omega)$ later.

The multi-grid algorithm 1.

1. Solve $w_H \in S_H(\Omega)$ such that

$$A(w_H + \bar{f}, v) = 0, \ \ \forall v \in S_H(\Omega).$$

2. Solve $e_h \in S_h(\Omega)$ from the linear equation

$$A'(w_H + \bar{f}; e_h, v) = -A(w_H + \bar{f}, v), \ \ \forall v \in S_h(\Omega).$$

3. Set $u^h = w_H + \bar{f} + e_h$.

Theorem 5.1 *Assume that u^h is the solution obtained by the multi-grid algorithm 1, If $h << 1$, then by definition of the coarse solution $u_H = w_H + \bar{f}$, we could get*

$$\|u_h - u^h\|_1 \leq C\|u_h - u_H\|^2_{1,\infty}. \tag{5.1}$$

Consequently,

$$\|u - u^h\|_1 \leq C(h + H^2) \tag{5.2}$$

for the constant $C > 0$.

Proof: From the algorithm, (4.1) and (4.3), we have

$$\begin{aligned}
&A'(w_H + \bar{f}; u_h - u^h, v) \\
&= A'(w_H + \bar{f}; u_h - u_H, v) - A'(w_H + \bar{f}; e_h, v) \\
&= A'(u_H; u_h - u_H, v) + A(w_H + \bar{f}, v) \\
&= -R(u_H, u_h, v) \\
&\leq C\|u_H - u_h\|^2_{1,\infty}\|v\|_{1,1}.
\end{aligned}$$

Since when h is sufficiently small, the bilinear operator $A'(u_H; \cdot, \cdot)$ will be well-posed, then we can use the sufficient and necessary condition [16]

$$\begin{aligned}
\|u^h - u_h\|_1 &\leq C\text{sup}_{v \in S_h(\Omega)} \frac{A'(u_H; u_h - u^h, v)}{\|v\|_1} \\
&\leq C\text{sup}_{v \in S_h(\Omega)} \frac{C\|u_H - u_h\|^2_{1,\infty}\|v\|_{1,1}}{\|v\|_1} \\
&\leq C\|u_H - u_h\|^2_{1,\infty}
\end{aligned}$$

By applying Theorem 4.2, it can be obtained that, for two different grid partition sizes, there holds

$$\|u - u_h\|_{1,\infty} \leq Ch,$$
$$\|u - u_H\|_{1,\infty} \leq CH.$$

Thus we can prove (5.2) by the triangular inequalities.

In fact, the multi-grid algorithm 1 can be simplified by eliminating the lower differential order term from the operator A'.

If we define

$$A'_s(u;\cdot,\cdot) = A'(u;\cdot,\cdot) - N(u;\cdot,\cdot)$$

where

$$N(u;\cdot,v) = \int_\Omega (b_1(u)\partial_x(\cdot)v + b_2(u)\partial_y(\cdot)v)$$

and the coefficients $b_1(u), b_2(u)$ follows the definitions in Lemma 4.1, then we could have

The multi-grid algorithm 2.

1. Solve $w_H \in S_H(\Omega)$ such that

$$A(w_H + \bar{f}, v) = 0, \ \ \forall v \in S_H(\Omega).$$

2. Solve $e_h \in S_h(\Omega)$ from the linear equation

$$A'_s(w_H + \bar{f}; e_h, v) = -A(w_H + \bar{f}, v), \ \ \forall v \in S_h(\Omega).$$

3. Set $u_s^h = w_H + \bar{f} + e_h$.

Theorem 5.2 *Assume that u^h is the solution obtained by multi-grid algorithm 2, we could still get*

$$\|u_h - u_s^h\|_1 \le C(\|u_h - u_H\|_{1,\infty}^2 + \|u - u_H\|_{0,\infty}). \tag{5.3}$$

Proof: By definitions and the facts from Lemma 4.1 we could obtain that

$$\begin{aligned}
&A'_s(w_H + \bar{f}; u_h - u_s^h, v)\\
&= A'_s(w_H + \bar{f}; u_h - u_H, v) - A'_s(w_H + \bar{f}; e_h, v)\\
&= A'(u_H; u_h - u_H, v) + A(w_H + \bar{f}, v) - N(u_H; u_h - u_H, v)\\
&= -R(u_H, u_h, v) - N(u_H; u_h - u_H, v).
\end{aligned}$$

Since

$$\begin{aligned}
&N(u; u_h - u_H, v)\\
&= \int_\Omega (b_1\partial_x(u_h - u_H)v + b_2\partial_y(u_h - u_H)v)\\
&= -\int_\Omega (b_1(u_h - u_H)v_x + b_2(u_h - u_H)v_y + 2(u_{xy}^2 - u_{xx}u_{yy})(u_h - u_H)v),
\end{aligned}$$

and $\|u_H\|_{2,\infty}$ is uniformly bounded according to Theorem 4.2, we have

$$\begin{aligned}
&|N(u_H; u_h - u_H, v)|\\
&\le \|b_1(u_H)\|_{0,\infty}\|u_h - u_H\|_{0,\infty}\|v\|_{1,1} + \|b_2(u_H)\|_{0,\infty}\|u_h - u_H\|_{0,\infty}\|v\|_{1,1}\\
&\quad + 2\|u_{H,xy}^2 - u_{H,xx}u_{H,yy}\|_{0,\infty}\|u_h - u_H\|_{0,\infty}\|v\|_{0,1}\\
&\le 6\|u_H\|_{2,\infty}^2\|u_h - u_H\|_{0,\infty}\|v\|_{1,1}\\
&\le C\|u_h - u_H\|_{0,\infty}\|v\|_{1,1}.
\end{aligned}$$

By the fact that the bilinear operator $A_s'(u_H;\cdot,\cdot)$ is positive definite, we could still get that

$$\begin{aligned}\|u_s^h - u_h\|_1 &\leq C\text{sup}_{v\in S_h(\Omega)}\frac{A_s'(u_H;u_h-u^h,v)}{\|v\|_1}\\ &\leq C\text{sup}_{v\in S_h(\Omega)}\frac{\|u_H-u_h\|_{1,\infty}^2\|v\|_{1,1}+\|u_h-u^H\|_{0,\infty}\|v\|_{1,1}}{\|v\|_1}\\ &\leq C(\|u_H - u_h\|_{1,\infty}^2 + \|u_h - u^H\|_{0,\infty}).\end{aligned}$$

Thus complete the proof.

Remark:

Actually to generate the discrete scheme based on the higher order finite element space regarding the assumption from this section, is not convenient in practical. However, we might find that the multi-grid algorithm 2 still makes sense if the finite element space is defined linear, since we only use the first order derivative function of u_H in the discrete schemes. Although we cound not prove that multigrid-algorithm 2 is also valid under the linear finite element space, (even if we know priorily that the discrete system has a unique solution,) we could still find it works on some special cases from the experiments in the next section.

6 Examples of Generating Minimal Surfaces

To illustrate the features of the approximation methods proposed in this paper, we now show two model examples.

Example 1.

For $\Omega = [0,1]\times[0,1]$, we use the bilinear finite element interpolation with mesh size $h = 1/16$. The border curves are the restriction of smooth function $\bar{f} = 0.25 - (x-0.5)^2$ (see Figure 2). Then the finite element solution after sufficient iteration steps is displayed in Figure 3.

Example 2.

This trival example shows the convergence property for the approximations. By given the initial border function the restriction of $xy(1-x)(1-y)$ on the same domain, we know the exact solution to this Plateau problem should be a plane $z = 0$.

Figure 4 shows the maxium error between the numerical solution and the exact one will not be greater than $2 * 10^{-4}$ if we use the same grid partition as the last example. And based on the relatively coarser grid partition by 8×8 and 4×4 meshes, we get the maximum errors are less than $8 * 10^{-4}$ and $4 * 10^{-3}$, respectively, which coincides with the theoretical estimations in the above section.

Figure 5 indicates the result solved by the multi-grid algorithm 2 from the last section, whose initial iteration is based on the coarse grid 8×8 on the linear finite element space $S_H^1(\Omega)$. We can obviously find the multigrid solution u_s^h has the same efficiency as that solved directly from the refined grid partition (Figure 4), but it only costs 1/4 computing time as the standard method.

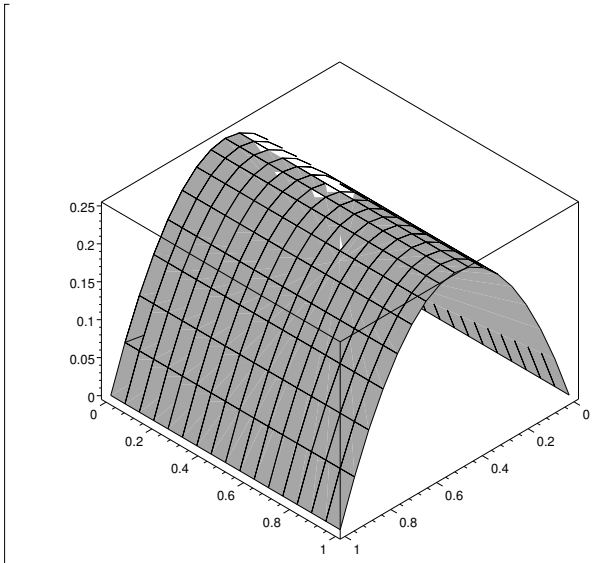

Figure 2: The initial shape with fixed border

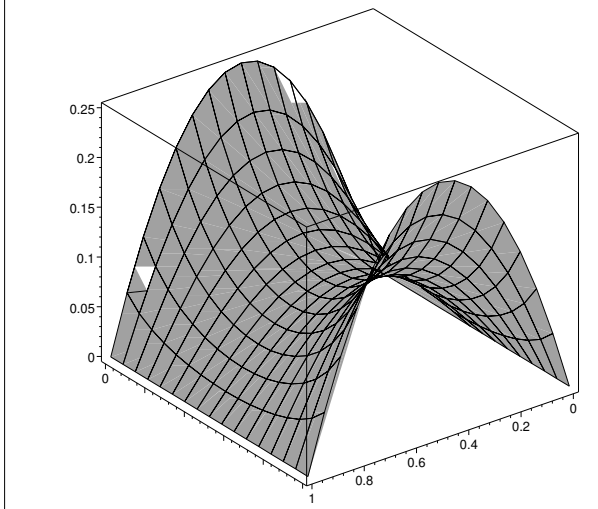

Figure 3: The approximated minimal surface

Those experimental datas for the computations are obtained from the Solaris operating system computer galaxy.risc.uni-linz.ac.at with 1152M memory.

7 Conclusion and Remarks

In this paper, we discussed the finite element methods for solving the Plateau problems on convex domains, and give the error analysis to all approachs. Such kind of discrete schemes have relatively lower complexity and we have also inplemented some multigrid approachs which could speed up the computations. There also exists possibilities of applying the parallel algorithms [16, 17] which fitted for sloving problems on the

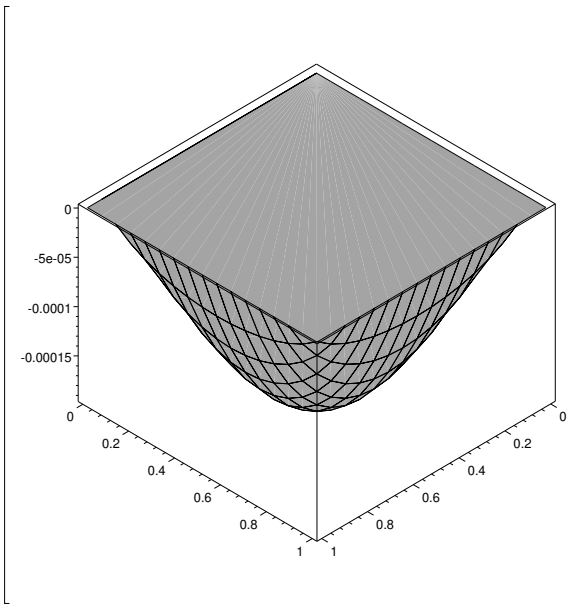

Figure 4: The shape of the discrete sulution which also indicate the error

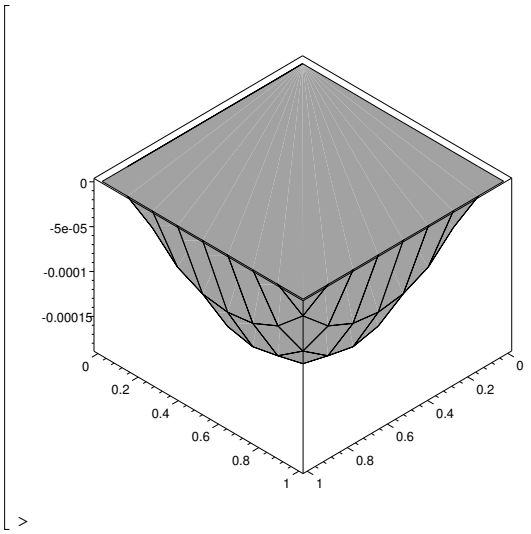

Figure 5: The shape of the solution solved by the multi-grid algorithm 2

large scale domains, however, the works for implementing the parallel algorithms is still in process. For the final generation of the minimal surface, we carried out all the numerical examples on the Maple software.

References

[1] K.Brakke, *"The surface evolver. Experiment"*, Math. 1 (1992), no. 2, 141–165.

[2] S.Brenner and R. Scott, *"The Mathematical Theory of Finite Element Methods"*, Springer-Verlag, 1994.

[3] P.Ciarlet, *"The Finite Element Method for Elliptic Problems"*, North-Holland, 1978.

[4] H.Gu, *"The Software Generating of Minamal Surface Subject to the Plateau Problems"*, Ph.D. thesis inprocess, RISC-Linz, 2002.

[5] D.Hoffman, *"Natural minimal surfaces. Via theory and computation"*, Science Television, NewYork; American Mathematical Society, Providence, RI, 1990.

[6] H.Matthies and G.Strang, *"The solution of nonlinear finite element equations"*, Internat. J. Numer. Methods Engrg. 14 (1979), no. 11, 1613–1626.

[7] C.Johnson and V.Thomeé, *"Error estimates for a finite element approximation of a minimal surface"*, Math. of Comp. Vol. 29, Num. 130, 1975 343-349.

[8] M.Marion and J.Xu, *"Error estimates on a new nonlinear Galerkin method based on two-grid finite elements"*, SIAM J. Numer. Anal. 32 (1995), no.4, 1170–1184.

[9] C.Morry, *"Multiple Intergrals in the Calculus of Variations, Die Grundlehren der math"*, Wissenschften, Band 130, Springer-Verlag, New York, 1966.

[10] M.Protter, H.Weinerger, *"Maximum Principles in Differential Equations"*, Corrected reprint of the 1967 original. Springer-Verlag, New York, 1984.

[11] R.Rannacher and R.Scott, *"Some optional error estimates for piecewise linear finite element approximation"*, Math. Comp. 54(1982, 437-445.

[12] A.Schatz, *"An observation concerning Ritz-Galerkin Methods with indefinite bilinear forms"*, Math. Comp. 28(1974), 959-962.

[13] A.Schatz, and J.Wang, *"Some new error estimates for Ritz-Galerkin methods with minimal regularity assumptions"*, Math. Comp. 65 (1996), no. 213, 19–27.

[14] V.Thomeé, J.Xu and N.Zhang, *"Supperconvergence of the gradient in piecewise linear finite element approximation to a parabolic problem"*, SIAM J. Numer. Anal., 26(1989), 553-573.

[15] J.Xu, *"Two-Grid Discretization Techniques for Linear and Nonlinear Partial Differential Equation"*, SIAM J. Numer. Anal., 1996.

[16] J.Xu and A.Zhou, *"Local and parallel finite element algorithms based on two-grid discretizations"*, Math. Comp., 69(2000), 881-909.

[17] J.Xu and A.Zhou, *"Local and parallel finite element algorithms based on two-grid discretizations for nonlinear problems"*, Adv. Comput. Math., 14(2001), 393-327.