VERIFYING THE JUMPING CHAMPION CONJECTURE

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ABSTRACT. In this paper, we present the results of computational verification of jumping champion conjecture made by Odlyzko et. al. in [ORW99]. They have given heuristics and computational results for two intervals of size 10^7, claiming that 210 takes over as jumping champion from 30 somewhere between 10^{425} and 10^{426}. In one of those two intervals, starting from 10^{450}, 210 was not a dominant gap. This motivated us to verify the conjecture computationally. We considered intervals of size 10^7 starting from $10^{410}, 10^{420}, \ldots, 10^{550}$ and in many of them 210 did not appear as a dominant gap. Therefore we continued our experiments with intervals of size 10^8. We present the results of these experiments. We used computer algebra system MATHEMATICA® (MATHEMATICA® is a product of Wolfram Research, Inc.) for the computation.

1. INTRODUCTION

Let p_n and p_{n+1} denote the consecutive prime numbers, then $d_n = p_{n+1} - p_n$ denotes the gap between these two prime numbers. We use the term "prime gap" or "gap" for d_n. An integer D is called a *jumping champion* if it is the most frequently occurring difference between consecutive primes $\leq x$ for some positive integer x, we denote it by $D(x)$. In other words, the most common prime gap till a given number x is called jumping champion. This term was coined by John Conway. There are many interesting conjectures on prime gaps, some of them can be found in **A**8 in [Guy81], and for more comprehensive information check [Wei98].

Conjecture: 210 displaces 30 as jumping champion at about $x = 10^{425}$. See [ORW99] for conjecture and further details.

The simple computations will show that 6 takes over as the jumping champion at $x = 947$. Earlier with extensive computations Harley in [Har94] have independently confirmed that 30 takes over as jumping champion from 6 at around $x = 1.7427 \cdot 10^{35}$. Harley has also showed why the prime factorials are likely candidates for the jumping champion. Our main goal was to check the dominant gap in the chosen intervals.

2. PROBLEM DESCRIPTION

In [ORW99], the conjecture has been showed to be true using heuristics developed using the strong form of Hardy-Littlewood k-tuple conjecture.

It is impossible for even the state of the art hardware with the best implementation of fastest prime generation algorithm to enumerate all primes up to 10^{450} to determine

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x	*next prime*(x)	Time taken in seconds PARI	Maple-6	MATHEMATICA® 4.0
10^{410}	$10^{410} + 67$	3.58	15.17	4.59
10^{420}	$10^{420} + 1261$	45.45	190.44	43.57
10^{430}	$10^{430} + 1587$	60.54	266.58	57.31
10^{440}	$10^{440} + 709$	29.58	122.84	27.00
10^{450}	$10^{450} + 2269$	99.50	393.31	83.52
10^{460}	$10^{460} + 111$	7.19	25.52	5.61
10^{470}	$10^{470} + 829$	41.91	180.96	40.81
10^{480}	$10^{480} + 2941$	148.93	676.12	129.35
10^{490}	$10^{490} + 379$	22.82	87.32	16.02
10^{500}	$10^{500} + 961$	56.59	261.12	48.21
10^{510}	$10^{510} + 2523$	154.23	720.62	136.34
10^{520}	$10^{520} + 343$	24.38	101.47	17.39
10^{530}	$10^{530} + 913$	64.25	292.38	58.83
10^{540}	$10^{540} + 2227$	160.02	688.89	133.01
10^{550}	$10^{550} + 2589$	197.54	828.05	169.03

TABLE 1. Time taken in seconds by *nextprime* command to generate the next prime after number x

jumping champion. So to find out most frequently occurring gap, we checked the intervals of same size.

We chose the intervals of size 10^7 starting from 10^{410}, 10^{420}, … and so on up to 10^{550} and intervals starting from 10^{425}, 10^{426}, … and so on up to 10^{435}. We also checked five intervals of size 10^8, namely starting from 10^{430}, 10^{435}, 10^{455}, 10^{475}, 10^{495} to check whether the change of size of interval has some effect on dominant gap results. In computations involving 10^7 size intervals, we did not get 210 as most common prime gap in most of the intervals. More precisely, 210 was not a dominant gap around the predicted takeover point. This was the main reason why we continued with our experiments with larger intervals.

An extensive computation like this needs best software. We compared the performance of three systems MATHEMATICA® 4.0, Maple 6 and PARI 2.2.1, using [Wol99, BBB+00, CGG+91]. We give, in Table 1, comparison of time taken by these systems for computing prime greater than given number for different sizes of input. MATHEMATICA® and Maple give *probable* prime, PARI documentation does not state it clearly whether they give probable prime or provable prime. Based on this comparison we chose MATHEMATICA® for further computations, which also helped us in transformation of results into explanatory graphs. We used nextprime function to generate the probable prime greater than the given number.

x	$P(d, x, 10^7)$ where $d =$					
	30	42	60	90	180	210
10^{410}	44	32	46	47	18	41
10^{420}	44	29	33	32	31	34
10^{430}	43	30	41	28	30	36
10^{440}	49	28	29	28	20	29
10^{450}	26	24	30	29	26	34
10^{460}	25	44	29	29	28	30
10^{470}	24	30	34	34	17	29
10^{480}	41	36	22	25	21	22
10^{490}	32	22	22	29	24	28
10^{500}	38	26	31	31	27	24
10^{510}	31	29	28	21	20	29
10^{520}	20	14	23	28	14	20
10^{530}	22	23	23	19	23	30
10^{540}	21	11	16	28	23	21
10^{550}	17	18	23	32	24	09
Sum	Sum of $P(d, x, 10^7)$ from $x = 10^{430}$ onwards					
	389	335	351	361	297	341
Total sum	477	396	430	440	346	416

TABLE 2. The number of times d appear as gap between consecutive primes in the interval of length 10^7 starting from x

3. RESULTS

We give the results in different parts, first the results from computations involving 10^7 size intervals, then 10^8 size intervals, and last some miscellaneous observations made during extensive computation. Let us denote by $P(d, x, l)$ the number of times d appears as a gap between consecutive primes in the interval $[x, x + l]$.

3.1. **Intervals of size 10^7.** Table 2 lists the values of $P(d, x, l)$ where $l = 10^7, x = 10^i$, for $i = 410, 420, \ldots, 550$. We chose values of d for which the sum of all $P(d, x, 10^7)$ for our chosen values of x is considerably more than other values of d. The last but one row gives the sum of the values in each column starting from row 10^{430}. The last row gives the number of times d has appeared as gap between consecutive primes in the chosen intervals starting from 10^{410}. If 210 should take over as jumping champion at around 10^{425} then it should dominate as prime gap over all other values of d. The rows giving the sums are added just to facilitate the comparison. As one can see from the Table 2, 30 dominates 210 till 10^{440}. This motivated us to continue with our computations.

We can see the comparison how each d value changes with the size of input better in graph 1. This graph compares the number of times d appears as gap between consecutive primes for $d = 30, 90$ and 210. We chose these values of d as $d = 30$ is jumping champion at least up to 10^{425}, $d = 210$ is the number in question and $P(d, x, l)$ for $d = 90$ is significant.

3.2. **Intervals of size 10^7 near 10^{425}.** Table 3 lists the values of $P(d, x, l)$ where $l = 10^7, x = 10^i$, for $i = 425, 426, \ldots, 435$. By the conjecture, if $d = 210$ takes over as jumping

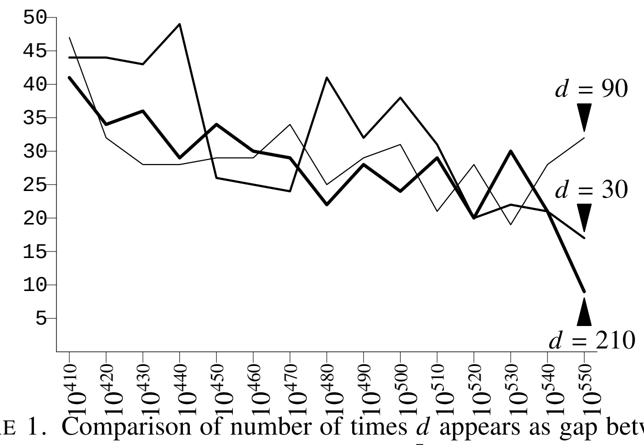

FIGURE 1. Comparison of number of times d appears as gap between consecutive primes in the interval of length 10^7 starting from x on X-axis

x	$P(d, x, 10^7)$ where $d =$							
	6	18	30	60	90	180	210	420
10^{425}	21	24	56	37	26	38	41	32
10^{426}	31	23	37	42	34	34	35	23
10^{427}	25	23	30	37	30	26	30	29
10^{428}	28	34	39	35	34	35	34	30
10^{429}	16	27	33	50	32	28	31	33
10^{430}	28	28	43	41	28	30	36	25
10^{431}	23	28	35	30	34	40	40	23
10^{432}	33	32	27	25	22	24	33	26
10^{433}	38	30	39	36	40	36	35	24
10^{434}	26	33	35	40	29	43	27	27
10^{435}	29	39	36	35	39	32	37	24

TABLE 3. The number of times d appear as gap between consecutive primes in the interval of length 10^7 starting from x

champion at 10^{425}, then it should occur as dominating gap from this point onwards. But, in the intervals starting from 10^{425} and 10^{426}, $d = 30$ clearly dominates $d = 210$. Moreover, in all these computations shown in Table 3, $d = 30$ dominates $d = 210$ eight times. Also 60 appears significantly near 10^{425}.

3.3. **Intervals of size 10^8.** We did not get any clear takeover behavior with the intervals of size 10^7, so we continued the experiments with intervals of size 10^8. The aim was to find out whether bigger interval gives some dominant gap, a candidate for the jumping champion. We considered five intervals of size 10^8 starting from 10^{430}, 10^{435}, 10^{455}, 10^{475} and 10^{495}, respectively. This choice of intervals was based on the observations gathered from the intervals of size 10^7. Intervals starting from 10^{430} and 10^{435} were chosen to check the dominant gap near the conjectured takeover point 10^{425}. Other intervals were chosen to check how many times 210 dominates 30.

x	$P(d, x, 10^8)$ where d =							
	30	42	60	90	120	150	210	420
10^{430}	366	282	339	325	312	313	336	298
10^{435}	360	285	352	343	296	310	323	260
10^{455}	272	273	297	303	306	287	325	270
10^{475}	267	263	257	269	262	259	280	284
10^{495}	271	209	238	231	217	241	252	202

TABLE 4. The number of times d appear as gap between consecutive primes in the interval of length 10^8 starting from x.

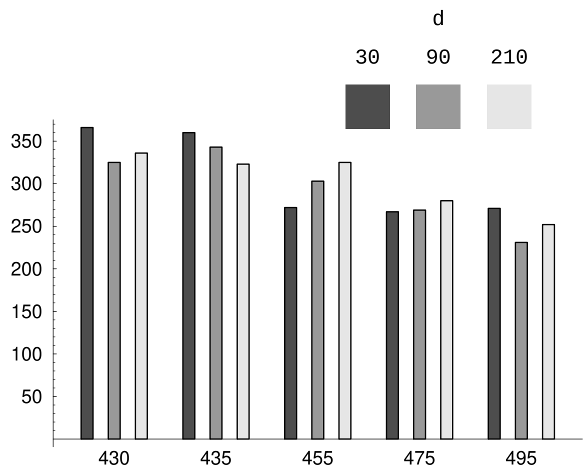

FIGURE 2. Comparison of number of times d appears as gap between consecutive primes, for $d = 30, 90, 210,$ in the interval of length 10^8 starting from x^i on X-axis

Table 4 lists the values of $P(d, x, l)$ where $l = 10^8, x = 10^{430}, 10^{435}, 10^{455}, 10^{475}$ and 10^{495}, and for the values of d, $(d = 30, 42, 60, 90, 120, 150, 210, 420)$. We present $P(d, x, l)$ for these d values as they occur more often than other values of d.

We point out the main observations and results.

- 30 dominates 210 in the intervals starting from 10^{430} and 10^{435}. This suggests, maybe, the takeover point is not near 10^{425}.
- Figure 3 shows the number of times d appears as gap between consecutive primes in the interval of size 10^8, starting from 10^{475}. All the even numbers from 2 to 3446 appear as gap between consecutive primes at least once in this interval. From Figure 3 it is clear that, almost all the numbers below 1000 appear as gap for more than 50 times, and most of the numbers below 6000 appear as gap for more than 4 times. This suggests that there can be a better method to generate consecutive primes with given gap than mentioned in [Cut01]. Also the size of the consecutive

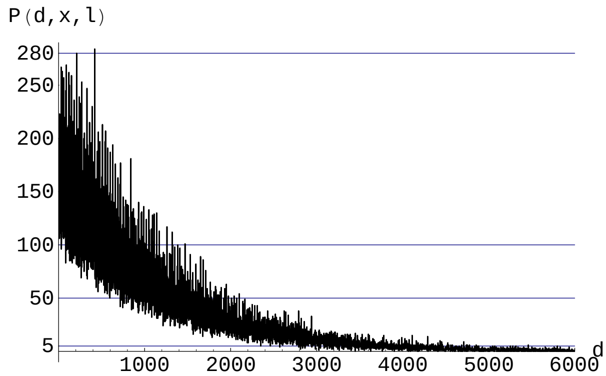

FIGURE 3. The number of times d appears as gap between consecutive primes in the interval of length 10^8 starting from 10^{475} for $d = 2, 4, , \ldots, 6000$

primes is smaller than the one generated in [Cut01] for gap larger than 2200. All the five intervals of size 10^8, which we checked, depict the same behavior.
- For certain small intervals, the computations showed that some gaps occur unusually often.Therefore we are checking those intervals for possible sequences of consecutive primes. We have not yet found any arithmetic progression of consecutive primes of length greater than 10, the longest known sequence given by [DFL+02].

3.4. **Big prime gaps.** We came across big gaps between consecutive primes during these experiments. We have already shown in Figure 3 that even numbers up to 6000 appear as gap quite often. In Table 5 we list some gaps of size bigger than 10000 in the increasing order and the prime after which gap occurs. We do not think that these are the first occurrences of these gaps as by [Wol01], the gaps of size 12000 should first occur around 10^{50}.

3.5. **Comparison of time required on different machines.** During these extensive computations we used *MATHEMATICA®* 4.0 running on different hardware and operating systems. In Table 6, we list the time required to do similar computation on different hardware and operating systems. We used Intel processors, Celeron or GenuineIntel. We have given the average time where average is taken over more than ten computations. This comparison points out that the computations took more time on Microsoft Windows 2000 with *MATHEMATICA®* 4.0 than on Linux. We give here these details for the benefit of researchers who have access to different machines and/or who wants to make choice of best platform to start the computations with *MATHEMATICA®*.

4. CONCLUSION

No strong computational evidence was found substantiating the conjecture in the intervals chosen near 10^{425}, and also in some other intervals. This shows that either the takeover point between 30 and 210 as jumping champions is greater than 10^{425} or some other number takes over as jumping champion from 30 for some small interval.

Gap (d)	Following prime	Gap (d)	Following prime	Gap (d)	Following prime
10016	$10^{530} + 2476327$	10022	$10^{530} + 5327647$	10056	$10^{500} + 874941$
10218	$10^{550} + 7274493$	10224	$10^{475} + 16611043$	10286	$10^{550} + 8880283$
10298	$10^{510} + 422691$	10434	$10^{510} + 321427$	10440	$10^{550} + 5607093$
10442	$10^{475} + 48656167$	10500	$10^{500} + 3357759$	10548	$10^{420} + 4889733$
10662	$10^{475} + 44360979$	10830	$10^{410} + 1181407$	10890	$10^{530} + 5275783$
10920	$10^{550} + 3142549$	11058	$10^{475} + 90282073$	11072	$10^{490} + 9646279$
11098	$10^{510} + 610491$	11212	$10^{475} + 30588861$	11246	$10^{475} + 71201311$
11274	$10^{540} + 8462997$	11300	$10^{510} + 9697429$	11430	$10^{475} + 99806517$
11618	$10^{420} + 6009073$	12026	$10^{480} + 4698193$	12030	$10^{475} + 61533597$
12762	$10^{520} + 82921$	15504	$10^{540} + 8726037$	15582	$10^{550} + 1800169$

TABLE 5. Big gaps (> 10000) between consecutive primes and the prime after which the gap occurs

CPU speed in MHz	RAM in MB	Platform	Time taken in minutes
500	256	Debian GNU/Linux	655
600	384	Windows 2000	605
733	256	Debian GNU/Linux	451
866	256	Debian GNU/Linux	380

TABLE 6. Comparison of time taken by machines with different speeds and operating systems running MATHEMATICA® 4.0 to list all the primes in the interval of size 10^6 starting from 10^{475}

5. FUTURE WORK

Harley [Har94] found out the exact point where 30 takes over as jumping champion from 6 by extensive computations using the strong form of Hardy-Littlewood k-tuple conjecture. We are in the process of carrying out similar computations for finding approximately the number where 210 takes over as jumping champion from 30. We will also continue with our search for long arithmetic progressions of consecutive prime numbers in the saved data.

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