Appendix

Pappus' Theorem

- Textbook formulation:

Let ABC and A1B1C1 be two lines and P = AB1 ⋂ A1B, Q = AC1 ⋂ A1C, S = BC1 ⋂ B1C. Then P, Q, and S are collinear.

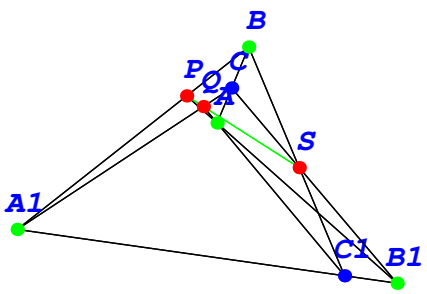
- Input to the system:

```
Proposition["Pappus", any[A, B, A1, B1, C, C1, P, Q, S],
 point[A, B, A1, B1] ∧
   pon[C, line[A, B]] ∧
   pon[C1, line[A1, B1]] ∧
   inter[P, line[A, B1], line[A1, B]] ∧
   inter[Q, line[A, C1], line[A1, C]] ∧
   inter[S, line[B, C1], line[B1, C]]
  ⇒ collinear[P, Q, S]]
```

- Input to the system:

```
Simplify[Proposition["Pappus"], by -> GraphicSimplifier]
```

- Output generated automatically by the system



Points **P Q** and **S** are collinear for this configuration of the points

- End of output

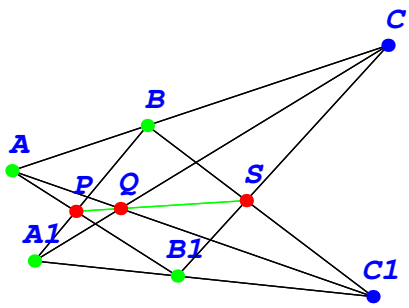
- Input to the system:

```
KnowledgeBase["Pappus_g", any[A, B, A1, B1, C, C1],
 {{A, {10, 120}}, {B, {100, 150}}, {A1, {25, 60}},
  {B1, {120, 50}}, {C, {260, 200}}, {C1, {250, 45}}}]
```

- Input to the system:

```
Simplify[Proposition["Pappus"],
 by -> GraphicSimplifier, using → KnowledgeBase["Pappus_g"]]
```

- Output generated automatically by the system



Points **P** **Q** and **S** are collinear for this configuration of the points

- End of output

- Input to the system:

```
Prove[Proposition["Pappus"],
 by → GeometryProver, ProverOptions → {Method -> "Wu"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Pappus))

$$\forall_{A,B,A1,B1,C,C1,P,Q,S} (\text{point}[A, B, A1, B1] \wedge \text{pon}[C, \text{line}[A, B]] \wedge \text{pon}[C1, \text{line}[A1, B1]] \wedge \text{inter}[P, \text{line}[A, B1], \text{line}[A1, B]] \wedge \text{inter}[Q, \text{line}[A, C1], \text{line}[A1, C]] \wedge \text{inter}[S, \text{line}[B, C1], \text{line}[B1, C]] \Rightarrow \text{collinear}[P, Q, S])$$

with no assumptions.

To prove the above statement we shall use Wu's method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point **A**, and points { } and {**B**, **C**} on the two axes.

Using this coordinate system we have the following points:

$$\{\{A, 0, 0\}, \{B, 0, u_1\}, \{A1, u_2, u_3\}, \{B1, u_4, u_5\}, \{C, 0, u_6\}, \{C1, u_7, x_1\}, \{P, x_2, x_3\}, \{Q, x_4, x_5\}, \{S, x_6, x_7\}\}$$

The algebraic form of the given construction is:

(1)

$$\forall_{x_1, x_2, x_3, x_4, x_5, x_6, x_7} (u_3 u_4 + -u_2 u_5 + -u_3 u_7 + u_5 u_7 + u_2 x_1 + -u_4 x_1 == 0 \wedge u_5 x_2 + -u_4 x_3 == 0 \wedge -u_1 u_2 + u_1 x_2 + -u_3 x_2 + u_2 x_3 == 0 \wedge x_1 x_4 + -u_7 x_5 == 0 \wedge -u_2 u_6 + -u_3 x_4 + u_6 x_4 + u_2 x_5 == 0 \wedge u_1 u_7 + -u_1 x_6 + x_1 x_6 + -u_7 x_7 == 0 \wedge -u_4 u_6 + -u_5 x_6 + u_6 x_6 + u_4 x_7 == 0 \Rightarrow x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 == 0)$$

First we compute the characteristic set of the hypothesis polynomials.We use successive steps, introducing polynomials for each point in a separate step.

Characteristics of the characteristic set:

Time needed to compute the characteristic set: `1.82 Seconds`.

During the computation the following factors were removed:

The maximum number of monomials in a polynomial during the computation was: `28`.

The set of initials of the polynomials in the characteristic set is:

The index set of the polynomials in the characteristic set is:

```
{{6, 1, 1}, {4, 2, 1}, {4, 3, 1},
 {8, 4, 1}, {10, 5, 1}, {12, 6, 1}, {18, 7, 1}}
```

The characteristic set is:

Next we compute the pseudoremainder of the conclusion polynomial $x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7$ with respect to the obtained characteristic set..

The remainder is: `0`.

Time needed to compute the remainder: `1.19 Seconds`.

The maximum number of monomials in a polynomial during the computation was: `166`.

As the obtained remainder is 0 the statement is generically true..

□

- End of inserted notebook
- Input to the system:

```
Prove[Proposition["Pappus"], by → GeometryProver,
 ProverOptions → {Method -> "GroebnerProver", Refutation → True}]
```

```
- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Pappus))

$$\forall_{A,B,A1,B1,C,C1,P,Q,S} (\text{point}[A, B, A1, B1] \wedge \text{pon}[C, \text{line}[A, B]] \wedge \text{pon}[C1, \text{line}[A1, B1]] \wedge \text{inter}[P, \text{line}[A, B1], \text{line}[A1, B]] \wedge \text{inter}[Q, \text{line}[A, C1], \text{line}[A1, C]] \wedge \text{inter}[S, \text{line}[B, C1], \text{line}[B1, C]] \Rightarrow \text{collinear}[P, Q, S])$$

with no assumptions.

To prove the above statement we shall use the Gröbner basis method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point A, and points $\{\}$ and $\{B, C\}$ on the two axes.

Using this coordinate system we have the following points:

$$\{\{A, 0, 0\}, \{B, 0, u_1\}, \{A1, u_2, u_3\}, \{B1, u_4, u_5\}, \{C, 0, u_6\}, \{C1, u_7, x_1\}, \{P, x_2, x_3\}, \{Q, x_4, x_5\}, \{S, x_6, x_7\}\}$$

The algebraic form of the given construction is:

(1)

$$\forall_{x_1,x_2,x_3,x_4,x_5,x_6,x_7} (u_3 u_4 + -u_2 u_5 + -u_3 u_7 + u_5 u_7 + u_2 x_1 + -u_4 x_1 == 0 \wedge u_5 x_2 + -u_4 x_3 == 0 \wedge -u_1 u_2 + u_1 x_2 + -u_3 x_2 + u_2 x_3 == 0 \wedge x_1 x_4 + -u_7 x_5 == 0 \wedge -u_2 u_6 + -u_3 x_4 + u_6 x_4 + u_2 x_5 == 0 \wedge u_1 u_7 + -u_1 x_6 + x_1 x_6 + -u_7 x_7 == 0 \wedge -u_4 u_6 + -u_5 x_6 + u_6 x_6 + u_4 x_7 == 0 \Rightarrow x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 == 0)$$

This problem is equivalent to:

(2)

$$\neg \left(\exists_{x_1,x_2,x_3,x_4,x_5,x_6,x_7} (u_3 u_4 + -u_2 u_5 + -u_3 u_7 + u_5 u_7 + u_2 x_1 + -u_4 x_1 == 0 \wedge u_5 x_2 + -u_4 x_3 == 0 \wedge -u_1 u_2 + u_1 x_2 + -u_3 x_2 + u_2 x_3 == 0 \wedge x_1 x_4 + -u_7 x_5 == 0 \wedge -u_2 u_6 + -u_3 x_4 + u_6 x_4 + u_2 x_5 == 0 \wedge u_1 u_7 + -u_1 x_6 + x_1 x_6 + -u_7 x_7 == 0 \wedge -u_4 u_6 + -u_5 x_6 + u_6 x_6 + u_4 x_7 == 0 \wedge x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 \neq 0) \right)$$

To remove the last inequality, we use the well–knownRabinowitsch trick. Let v_0 be a new variable. Then the problem becomes:

(3)

$$\neg\left(\exists_{x_1,x_2,x_3,x_4,x_5,x_6,x_7,v_0}\ (u_3\,u_4 + -u_2\,u_5 + -u_3\,u_7 + u_5\,u_7 + u_2\,x_1 + -u_4\,x_1 == 0 \wedge u_5\,x_2 + -u_4\,x_3 == 0 \wedge -u_1\,u_2 + u_1\,x_2 + -u_3\,x_2 + u_2\,x_3 == 0 \wedge x_1\,x_4 + -u_7\,x_5 == 0 \wedge -u_2\,u_6 + -u_3\,x_4 + u_6\,x_4 + u_2\,x_5 == 0 \wedge u_1\,u_7 + -u_1\,x_6 + x_1\,x_6 + -u_7\,x_7 == 0 \wedge -u_4\,u_6 + -u_5\,x_6 + u_6\,x_6 + u_4\,x_7 == 0 \wedge 1 + -v_0\,(x_3\,x_4 + -x_2\,x_5 + -x_3\,x_6 + x_5\,x_6 + x_2\,x_7 + -x_4\,x_7) == 0)\right)$$

To prove this statement we have to compute the Gröbner bases of the above polynomials.

The polynomials of the Gröbner bases are: {1}

As the obtained Gröbner bases is {1} the statement is generically true.

Statistics:

The length of the Gröbner bases is 1.

Time needed to compute the Gröbner bases: 0.42 Seconds.

□

- End of inserted notebook
- Input to the system:

```
Prove[Proposition["Pappus"],
 by → GeometryProver, ProverOptions → {Method → "Area"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Pappus))

$$\forall_{A,B,A1,B1,C,C1,P,Q,S}\ (\text{point}[A, B, A1, B1] \wedge \text{pon}[C, \text{line}[A, B]] \wedge \text{pon}[C1, \text{line}[A1, B1]] \wedge \text{inter}[P, \text{line}[A, B1], \text{line}[A1, B]] \wedge \text{inter}[Q, \text{line}[A, C1], \text{line}[A1, C]] \wedge \text{inter}[S, \text{line}[B, C1], \text{line}[B1, C]] \Rightarrow \text{collinear}[P, Q, S])$$

with no assumptions.

We shall use the area method.

First we have to transform the problem into the internal form for the area method. We have to prove, that constructions

$\{A, B, A1, B1\}$ free points
$C\,A \parallel A\,B$ and $\frac{\overline{A\,C}}{\overline{A\,B}} = r_C$ with ndg. $\{A \neq B\}$
$C1\,A1 \parallel A1\,B1$ and $\frac{\overline{A1\,C1}}{\overline{A1\,B1}} = r_{C1}$ with ndg. $\{A1 \neq B1\}$
$P = A\,B1 \bigcap A1\,B$ with ndg. $\{A \neq B1,\ A1 \neq B,\ A\,B1 \not\parallel A1\,B\}$
$Q = A\,C1 \bigcap A1\,C$ with ndg. $\{A \neq C1,\ A1 \neq C,\ A\,C1 \not\parallel A1\,C\}$
$S = B\,C1 \bigcap B1\,C$ with ndg. $\{B \neq C1,\ B1 \neq C,\ B\,C1 \not\parallel B1\,C\}$

imply that points {**P**, **Q**, **S**} are collinear

Explanations

First we have to verify the non−degenerateconditions for each construction, then to prove our statement.

Step 1

Step 2

We have to verify the non−degenerateconditions for

Q = **A C1** ⋂ **A1 C** with ndg. {**A** ≠ **C1**, **A1** ≠ **C**, **A C1** ⫻ **A1 C**}

That is, constructions

{**A**, **B**, **A1**, **B1**} free points
C A ∥ **A B** and $\frac{\overline{A\,C}}{\overline{A\,B}}$ = r_C with ndg. {**A** ≠ **B**}
C1 A1 ∥ **A1 B1** and $\frac{\overline{A1\,C1}}{\overline{A1\,B1}}$ = r_{C1} with ndg. {**A1** ≠ **B1**}
P = **A B1** ⋂ **A1 B** with ndg. {**A** ≠ **B1**, **A1** ≠ **B**, **A B1** ⫻ **A1 B**}

do not imply

A C1 ∥ **A1 C**

We use the elimination method described in the above explanations.

Eliminating point **C1**:

Eliminating point **C**:

As the last expression contains no constructed points we have to rewrite the expression in area coordinates of the free points:

As the expression is still not 0, we try to use the possible relations between areas and lengths of segments

Non−degeneratecondition is verified.

Step 3

Step 4

As all the non−degenerateconditions are verified now we have to prove

{**A**, **B**, **A1**, **B1**} free points
C A ∥ **A B** and $\frac{\overline{A\,C}}{\overline{A\,B}}$ = r_C with ndg. {**A** ≠ **B**}
C1 A1 ∥ **A1 B1** and $\frac{\overline{A1\,C1}}{\overline{A1\,B1}}$ = r_{C1} with ndg. {**A1** ≠ **B1**}
P = **A B1** ⋂ **A1 B** with ndg. {**A** ≠ **B1**, **A1** ≠ **B**, **A B1** ⫻ **A1 B**}
Q = **A C1** ⋂ **A1 C** with ndg. {**A** ≠ **C1**, **A1** ≠ **C**, **A C1** ⫻ **A1 C**}
S = **B C1** ⋂ **B1 C** with ndg. {**B** ≠ **C1**, **B1** ≠ **C**, **B C1** ⫻ **B1 C**}
⇒points {**P**, **Q**, **S**} are collinear

Eliminating point **S**:

Eliminating point **Q**:

Eliminating point **P**:

Eliminating point **C1**:

We eliminate point **C1** using construction:
C1 A1 ∥ **A1 B1** and $\frac{\overline{A1\,C1}}{\overline{A1\,B1}}$ = r_{C1} with ndg. {**A1** ≠ **B1**}

After ordering the points we obtain:

The simplified expression is:

$$\begin{aligned}&(((-1)+r_{C1})\,r_{C1}\bullet S_{\{A1,B1,C\}}\\&\quad(\bullet S_{\{A,A1,C\}}\bullet S_{\{A,B,B1\}}\bullet S_{\{B,A1,B1\}}+\bullet S_{\{A,B,A1\}}\\&\qquad(\bullet S_{\{A,B,B1\}}\bullet S_{\{A1,B1,C\}}+-\bullet S_{\{A,A1,B1\}}\bullet S_{\{B,B1,C\}})))/\\&((\bullet S_{\{A,A1,C\}}+r_{C1}\bullet S_{\{A1,B1,C\}})(\bullet S_{\{A,B,A1\}}+-\bullet S_{\{B,A1,B1\}})\\&\quad(((-1)+r_{C1})\bullet S_{\{A1,B1,C\}}+\bullet S_{\{B,B1,C\}}))\end{aligned}$$

Eliminating point C:

We eliminate point C using construction:
$C\,A \parallel A\,B$ and $\frac{\overline{A\,C}}{\overline{A\,B}}=r_C$ with ndg. $\{A \neq B\}$

$$\begin{aligned}&(((-1)+r_{C1})\,r_{C1}\,((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{A1,B1,A\}}+\\&\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{A1,B1,B\}})\\&\quad(((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{A,A1,A\}}+\\&\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{A,A1,B\}})\bullet S_{\{A,B,B1\}}\bullet S_{\{B,A1,B1\}}+\\&\qquad\bullet S_{\{A,B,A1\}}\,(\bullet S_{\{A,B,B1\}}\,((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{A1,B1,A\}}+\\&\qquad\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{A1,B1,B\}})+\\&\qquad\quad-\bullet S_{\{A,A1,B1\}}\,((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{B,B1,A\}}+\\&\qquad\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{B,B1,B\}}))))/\\&((((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{A,A1,A\}}+(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{A,A1,B\}}+\\&\qquad r_{C1}\,((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{A1,B1,A\}}+\\&\qquad\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{A1,B1,B\}}))\\&\quad(\bullet S_{\{A,B,A1\}}+-\bullet S_{\{B,A1,B1\}})\\&\quad(((-1)+r_{C1})\,((-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{A1,B1,A\}}+\\&\qquad\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{A1,B1,B\}})+\\&\qquad(-r_C+\bullet R_{\{A,B,A,B\}})\bullet S_{\{B,B1,A\}}+\\&\qquad(r_C+\bullet R_{\{A,A,A,B\}})\bullet S_{\{B,B1,B\}}))\end{aligned}$$

After ordering the points we obtain:

$$\begin{aligned}&(((-1)+r_{C1})\,r_{C1}\,((1+-r_C)\bullet S_{\{A,A1,B1\}}+r_C\bullet S_{\{B,A1,B1\}})\\&\quad(-r_C\bullet S_{\{A,B,A1\}}\bullet S_{\{A,B,B1\}}\bullet S_{\{B,A1,B1\}}+\\&\qquad\bullet S_{\{A,B,A1\}}\,(-(1+-r_C)\bullet S_{\{A,A1,B1\}}\bullet S_{\{A,B,B1\}}+\\&\qquad\quad\bullet S_{\{A,B,B1\}}\,((1+-r_C)\bullet S_{\{A,A1,B1\}}+r_C\bullet S_{\{B,A1,B1\}}))))/\\&((\bullet S_{\{A,B,A1\}}+-\bullet S_{\{B,A1,B1\}})\,((1+-r_C)\bullet S_{\{A,B,B1\}}+\\&\qquad((-1)+r_{C1})\,((1+-r_C)\bullet S_{\{A,A1,B1\}}+r_C\bullet S_{\{B,A1,B1\}}))\\&\quad(-r_C\bullet S_{\{A,B,A1\}}+\\&\qquad r_{C1}\,((1+-r_C)\bullet S_{\{A,A1,B1\}}+r_C\bullet S_{\{B,A1,B1\}})))\end{aligned}$$

The simplified expression is:

$$0$$

The initial problem was proved, as we obtained 0 == 0.

□

- End of inserted notebook