The Butterfly Theorem

- Textbook formulation:

A, B, C and *D* are four points on circle (*O*). *E* is the intersection of *AC* and *BD*. Through *E* draw a line perpendicular to *OE*, meeting *AD* at *F* and *BC* at *G*. Show that $\overline{FE} = \overline{GE}$.

- Remark:

Using the default coordinate system the algebraic methods are very slow.

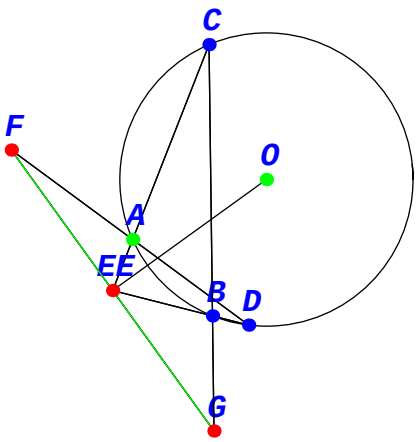
- Input to the system:

```
Proposition["Butterfly", any[O, A, B, C, D, EE, F, G], point[O, A] ∧
   pon[C, circle[O, A]] ∧
   pon[B, circle[O, A]] ∧
   pon[D, circle[O, A]] ∧
   inter[EE, line[A, C], line[B, D]] ∧ inter[F, line[D, A],
    tline[EE, EE, O]] ∧ inter[G, line[B, C], line[EE, F]]
  ⇒ distequal[EE, F, G, EE]]
```

- Input to the system:

```
Simplify[Proposition["Butterfly"], by -> GraphicSimplifier]
```

- Output generated automatically by the system



$|EE\,F| \cong |G\,EE|$ for this configuration of the points

- End of output

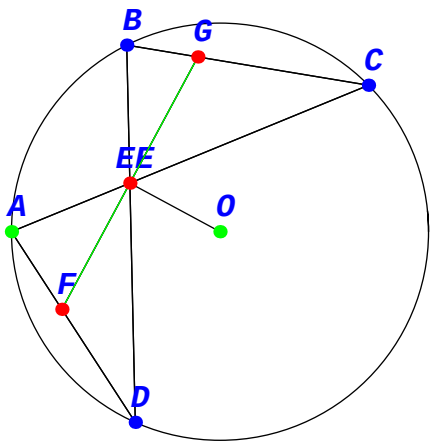
- Input to the system:

```
KnowledgeBase["Butterfly_g", any[O, A, B, C, D], {{O, {100, 100}},
  {A, {0, 100}}, {B, {55, 190}}, {C, {171, 170}}, {D, {60, 10}}}]
```

- Input to the system:

```
Simplify[Proposition["Butterfly"],
 by -> GraphicSimplifier, using → KnowledgeBase["Butterfly_g"]]
```

- Output generated automatically by the system



$|EE\,F| \cong |G\,EE|$ for this configuration of the points

- End of output

- Input to the system:

```
Prove[Proposition["Butterfly"], by → GeometryProver,
 ProverOptions → {Method -> "Wu", Iterate → False}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Butterfly))

$$\forall_{O,A,B,C,D,EE,F,G} \big(\text{point}[O, A] \wedge \text{pon}[C, \text{circle}[O, A]] \wedge \text{pon}[B, \text{circle}[O, A]] \wedge \text{pon}[D, \text{circle}[O, A]] \wedge \text{inter}[EE, \text{line}[A, C], \text{line}[B, D]] \wedge \text{inter}[F, \text{line}[D, A], \text{tline}[EE, EE, O]] \wedge \text{inter}[G, \text{line}[B, C], \text{line}[EE, F]] \Rightarrow \text{distequal}[EE, F, G, EE]\big)$$

with no assumptions.

To prove the above statement we shall use Wu's method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point EE, and points $\{O\}$ and $\{F, G\}$ on the two axes.

Using this coordinate system we have the following points:

$$\{\{EE, 0, 0\}, \{O, u_1, 0\}, \{A, u_2, u_3\}, \{B, u_4, x_1\}, \{F, 0, x_2\}, \{G, 0, x_3\}, \{C, x_4, x_5\}, \{D, x_6, x_7\}\}$$

The algebraic form of the given construction is:

(1)

$$\forall_{v_1, v_2, x_1, x_2, x_3, x_4, x_5, x_6, x_7} ((-1) + u_4\, v_2 + -v_2\, x_6 == 0 \bigwedge$$
$$(-1) + u_2\, v_1 + -v_1\, x_4 == 0 \bigwedge$$
$$2\, u_1\, u_2 + -u_2^2 + -u_3^2 + -2\, u_1\, x_4 + x_4^2 + x_5^2 == 0 \bigwedge$$
$$2\, u_1\, u_2 + -u_2^2 + -u_3^2 + -2\, u_1\, u_4 + u_4^2 + x_1^2 == 0 \bigwedge$$
$$2\, u_1\, u_2 + -u_2^2 + -u_3^2 + -2\, u_1\, x_6 + x_6^2 + x_7^2 == 0 \bigwedge$$
$$u_3\, x_4 + -u_2\, x_5 == 0 \bigwedge x_1\, x_6 + -u_4\, x_7 == 0 \bigwedge$$
$$-u_2\, x_2 + -u_3\, x_6 + x_2\, x_6 + u_2\, x_7 == 0 \bigwedge$$
$$u_4\, x_3 + x_1\, x_4 + -x_3\, x_4 + -u_4\, x_5 == 0 \Rightarrow x_2^2 + -x_3^2 == 0)$$

First we compute the characteristic set of the hypothesis polynomials.

Characteristics of the characteristic set:

Time needed to compute the characteristic set: `15.36 Seconds`.

During the computation the following factors were removed:

The maximum number of monomials in a polynomial during the computation was: `155`.

The set of initials of the polynomials in the characteristic set is:

The index set of the polynomials in the characteristic set is:

```
{{5, 1, 1}, {8, 2, 1}, {6, 3, 2}, {11, 4, 1},
 {11, 5, 1}, {5, 6, 1}, {5, 7, 1}, {7, 8, 1}, {7, 9, 1}}
```

The characteristic set is:

Next we compute the pseudoremainder of the conclusion polynomial $x_2^2 + -x_3^2$ with respect to the obtained characteristic set.

The remainder is: `0`

Time needed to compute the remainder: `0.3 Seconds`.

The maximum number of monomials in a polynomial during the computation was: `32`.

As the obtained remainder is 0 the statement is generically true.

□

- End of inserted notebook
- Input to the system:

```
Prove[Proposition["Butterfly"], by → GeometryProver,
 ProverOptions → {Method -> "GroebnerProver", Refutation → False}]
```

```
- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Butterfly))

$$\forall_{O,A,B,C,D,EE,F,G} (\text{point}[O, A] \wedge \text{pon}[C, \text{circle}[O, A]] \wedge \text{pon}[B, \text{circle}[O, A]] \wedge \text{pon}[D, \text{circle}[O, A]] \wedge \text{inter}[EE, \text{line}[A, C], \text{line}[B, D]] \wedge \text{inter}[F, \text{line}[D, A], \text{tline}[EE, EE, O]] \wedge \text{inter}[G, \text{line}[B, C], \text{line}[EE, F]] \Rightarrow \text{distequal}[EE, F, G, EE])$$

with no assumptions.

To prove the above statement we shall use the Gröbner bases method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point EE, and points $\{O\}$ and $\{F, G\}$ on the two axes.

Using this coordinate system we have the following points:

$$\{\{EE, 0, 0\}, \{O, u_1, 0\}, \{A, u_2, u_3\}, \{B, u_4, x_1\}, \{F, 0, x_2\}, \{G, 0, x_3\}, \{C, x_4, x_5\}, \{D, x_6, x_7\}\}$$

The algebraic form of the given construction is:

(1)

$$\forall_{v_1,v_2,x_1,x_2,x_3,x_4,x_5,x_6,x_7} ((-1) + u_4\, v_2 + -v_2\, x_6 == 0 \bigwedge (-1) + u_2\, v_1 + -v_1\, x_4 == 0 \bigwedge 2\, u_1\, u_2 + -u_2^2 + -u_3^2 + -2\, u_1\, x_4 + x_4^2 + x_5^2 == 0 \bigwedge 2\, u_1\, u_2 + -u_2^2 + -u_3^2 + -2\, u_1\, u_4 + u_4^2 + x_1^2 == 0 \bigwedge 2\, u_1\, u_2 + -u_2^2 + -u_3^2 + -2\, u_1\, x_6 + x_6^2 + x_7^2 == 0 \bigwedge u_3\, x_4 + -u_2\, x_5 == 0 \bigwedge x_1\, x_6 + -u_4\, x_7 == 0 \bigwedge -u_2\, x_2 + -u_3\, x_6 + x_2\, x_6 + u_2\, x_7 == 0 \bigwedge u_4\, x_3 + x_1\, x_4 + -x_3\, x_4 + -u_4\, x_5 == 0 \Rightarrow x_2^2 + -x_3^2 == 0)$$

We have to compute the Gröbner bases of the hypothesis polynomials (GB).

We compute the normal form of the conclusion polynomial modulo GB.

As the obtained normal form equals to 0 the statement is generically true.

Statistics:

The length of the Gröbner bases is 9

Time needed to compute the Gröbner bases: 0.27 Seconds.

Time needed to compute the normal form of the conclusion polynomial modulo GB: 0.32 Seconds.

□

- End of inserted notebook

- Input to the system:

```
Prove[Proposition["Butterfly"],
 by → GeometryProver, ProverOptions → {Method → "Area"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Butterfly))

$$\forall_{O,A,B,C,D,EE,F,G} \; (\text{point}[O, A] \wedge \text{pon}[C, \text{circle}[O, A]] \wedge \text{pon}[B, \text{circle}[O, A]] \wedge \text{pon}[D, \text{circle}[O, A]] \wedge \text{inter}[EE, \text{line}[A, C], \text{line}[B, D]] \wedge \text{inter}[F, \text{line}[D, A], \text{tline}[EE, EE, O]] \wedge \text{inter}[G, \text{line}[B, C], \text{line}[EE, F]] \Rightarrow \text{distequal}[EE, F, G, EE])$$

with no assumptions.

We shall use the area method.

First we have to transform the problem into the internal form for the area method. We have to prove, that constructions

points $\{A, C, B, D\}$ are on a circle with center O
$EE = A\,C \bigcap B\,D$ with ndg. $\{A \neq C, B \neq D, A\,C \between B\,D\}$
$EE\ \beta_F \perp EE\ O$ and $\frac{EE\ \beta_F}{EE\ O} = r_9$ with ndg. $\{EE \neq O\}$
$F = D\,A \bigcap EE\ \beta_F$ with ndg. $\{D \neq A, EE \neq \beta_F, A\,D \between \beta_F\ EE\}$
$G = B\,C \bigcap EE\ F$ with ndg. $\{B \neq C, EE \neq F, B\,C \between EE\ F\}$

imply that $|EE\ F| = |G\ EE|$

Explanations

First we have to verify the non−degenerateconditions for each construction, then to prove our statement.

Step 1

Step 2

Step 3

We have to verify the non−degenerateconditions for

$G = B\,C \bigcap EE\ F$ with ndg. $\{B \neq C, EE \neq F, B\,C \between EE\ F\}$

That is, constructions

points $\{A, C, B, D\}$ are on a circle with center O
$EE = A\,C \bigcap B\,D$ with ndg. $\{A \neq C, B \neq D, A\,C \between B\,D\}$
$EE\ \beta_F \perp EE\ O$ and $\frac{EE\ \beta_F}{EE\ O} = r_9$ with ndg. $\{EE \neq O\}$
$F = D\,A \bigcap EE\ \beta_F$ with ndg. $\{D \neq A, EE \neq \beta_F, A\,D \between \beta_F\ EE\}$

do not imply

$B\,C \parallel EE\ F$

We use the elimination method described in the above explanations.

Eliminating point F:

Eliminating point β_F:

Eliminating point EE:

All the remained points are on a circle, so we have to rewrite the expression using angles:

As the expression is still not 0, we try to use the possible relations between areas and lengths of segments

Non−degeneratecondition is verified.

Step 4

As all the non−degenerateconditions are verified now we have to prove

points $\{A, C, B, D\}$ are on a circle with center O
$EE = AC \cap BD$ with ndg. $\{A \neq C, B \neq D, AC \nparallel BD\}$
$EE\ \beta_F \perp EE\ O$ and $\frac{EE\ \beta_F}{EE\ O} = r_9$ with ndg. $\{EE \neq O\}$
$F = DA \cap EE\ \beta_F$ with ndg. $\{D \neq A, EE \neq \beta_F, AD \nparallel \beta_F\ EE\}$
$G = BC \cap EE\ F$ with ndg. $\{B \neq C, EE \neq F, BC \nparallel EE\ F\}$
$\Rightarrow |\overline{EE\ F}| = |\overline{G\ EE}|$

Eliminating point G:

Eliminating point F:

Eliminating point β_F:

Eliminating point EE:

We eliminate point EE using construction:
$EE = AC \cap BD$ with ndg. $\{A \neq C, B \neq D, AC \nparallel BD\}$

After ordering the points we obtain:

The simplified expression is:

All the remained points are on a circle, so we have to rewrite the expression using angles:

Rewriting

Ordering

Simplifying

$$
\begin{aligned}
&\big(2 \cdot L_{\{A,B\}} \cdot L_{\{B,D\}}^{\,2} \cdot L_{\{C,D\}} \\
&\quad \big((\cdot L_{\{A,B\}}^{\,3} \cdot L_{\{A,C\}} \cdot L_{\{C,B\}}^{\,2} \cdot L_{\{C,D\}} + -\cdot L_{\{A,D\}}^{\,2} \cdot L_{\{C,B\}} \\
&\qquad (\cdot L_{\{A,C\}} \cdot L_{\{A,D\}} + \cdot L_{\{B,D\}} \cdot L_{\{C,B\}}) \cdot L_{\{C,D\}}^{\,2} + \\
&\qquad -\cdot L_{\{A,B\}}^{\,2} \cdot L_{\{C,B\}} \ (\cdot L_{\{A,C\}}^{\,2} \cdot L_{\{B,D\}} \\
&\qquad\quad (2 \operatorname{Cos}[\cdot A_C + -\cdot A_D] \cdot L_{\{B,D\}} + 3 \cdot L_{\{C,B\}}) + \\
&\qquad\quad \cdot L_{\{B,D\}} \ (-2 \operatorname{Cos}[\cdot A_C + -\cdot A_D] \cdot L_{\{B,D\}} + \cdot L_{\{C,B\}}) \\
&\qquad\quad \cdot L_{\{C,D\}}^{\,2} + \cdot L_{\{A,C\}} \cdot L_{\{A,D\}} \ (\cdot L_{\{C,B\}}^{\,2} + 2 \cdot L_{\{C,D\}}^{\,2})) + \\
&\qquad \cdot L_{\{A,B\}} \cdot L_{\{C,D\}} \ (3 \cdot L_{\{A,C\}}^{\,2} \cdot L_{\{A,D\}} \cdot L_{\{B,D\}} \cdot L_{\{C,B\}} + \\
&\qquad\quad \cdot L_{\{A,D\}} \cdot L_{\{B,D\}} \cdot L_{\{C,B\}} \\
&\qquad\quad (-2 \operatorname{Cos}[\cdot A_C + -\cdot A_D] \cdot L_{\{B,D\}} \cdot L_{\{C,B\}} + \cdot L_{\{C,B\}}^{\,2} + \\
&\qquad\quad\quad \cdot L_{\{C,D\}}^{\,2}) + \cdot L_{\{A,C\}} \ (4 \cdot L_{\{B,D\}}^{\,2} \cdot L_{\{C,B\}}^{\,2} +
\end{aligned}
$$

$$\bullet L_{\{A,D\}}^{\,2}\,(2\,\bullet L_{\{C,B\}}^{\,2}+\bullet L_{\{C,D\}}^{\,2})))\,/$$
$$(\bullet L_{\{A,B\}}\,(2\,\mathrm{Cos}[\bullet A_C+-\bullet A_D]\,\bullet L_{\{B,D\}}+\bullet L_{\{C,B\}})+$$
$$-\bullet L_{\{A,D\}}\,\bullet L_{\{C,D\}})^2+$$
$$(-\bullet L_{\{A,D\}}^{\,2}\,\bullet L_{\{C,B\}}\,(-\bullet L_{\{A,B\}}^{\,4}\,\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}\,\bullet L_{\{C,D\}}^{\,2}+$$
$$\bullet L_{\{A,C\}}\,(\bullet L_{\{A,C\}}^{\,2}\,\bullet L_{\{A,D\}}\,\bullet L_{\{B,D\}}^{\,2}+$$
$$2\,\bullet L_{\{A,C\}}\,\bullet L_{\{B,D\}}\,(-\bullet L_{\{A,D\}}^{\,2}+\bullet L_{\{B,D\}}^{\,2})\,\bullet L_{\{C,B\}}+$$
$$-\bullet L_{\{A,D\}}^{\,3}\,\bullet L_{\{C,B\}}^{\,2})\,\bullet L_{\{C,D\}}^{\,2}+$$
$$-\bullet L_{\{A,B\}}\,\bullet L_{\{C,D\}}\,(3\,\bullet L_{\{A,C\}}^{\,3}\,\bullet L_{\{B,D\}}^{\,2}\,\bullet L_{\{C,B\}}+$$
$$-2\,\bullet L_{\{A,D\}}\,\bullet L_{\{B,D\}}\,(\bullet L_{\{A,D\}}^{\,2}+\bullet L_{\{B,D\}}^{\,2})\,\bullet L_{\{C,B\}}^{\,2}+$$
$$-\bullet L_{\{A,C\}}^{\,2}\,\bullet L_{\{A,D\}}\,\bullet L_{\{B,D\}}\,(2\,\mathrm{Cos}[\bullet A_C+-\bullet A_D]$$
$$\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}+2\,\bullet L_{\{C,B\}}^{\,2}+-\bullet L_{\{C,D\}}^{\,2})+$$
$$-\bullet L_{\{A,C\}}\,\bullet L_{\{A,D\}}^{\,2}\,\bullet L_{\{C,B\}}\,(5\,\bullet L_{\{B,D\}}^{\,2}+$$
$$2\,\bullet L_{\{C,B\}}^{\,2}+\bullet L_{\{C,D\}}^{\,2}))+\bullet L_{\{A,B\}}^{\,3}\,\bullet L_{\{C,D\}}$$
$$(\bullet L_{\{A,C\}}\,(3\,\bullet L_{\{B,D\}}^{\,2}\,\bullet L_{\{C,B\}}+\bullet L_{\{C,B\}}^{\,3})+$$
$$\bullet L_{\{A,D\}}\,\bullet L_{\{B,D\}}\,(-2\,\mathrm{Cos}[\bullet A_C+-\bullet A_D]\,\bullet L_{\{B,D\}}$$
$$\bullet L_{\{C,B\}}+3\,\bullet L_{\{C,B\}}^{\,2}+\bullet L_{\{C,D\}}^{\,2}))+$$
$$\bullet L_{\{A,B\}}^{\,2}\,(\bullet L_{\{A,C\}}^{\,2}\,\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}\,\bullet L_{\{C,D\}}^{\,2}+$$
$$-\bullet L_{\{A,C\}}\,\bullet L_{\{A,D\}}\,(\bullet L_{\{C,B\}}^{\,4}+2\,\bullet L_{\{C,B\}}^{\,2}\,\bullet L_{\{C,D\}}^{\,2}+$$
$$\bullet L_{\{B,D\}}^{\,2}\,(3\,\bullet L_{\{C,B\}}^{\,2}+\bullet L_{\{C,D\}}^{\,2}))+$$
$$-\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}\,(2\,\bullet L_{\{B,D\}}^{\,2}\,\bullet L_{\{C,D\}}^{\,2}+$$
$$\bullet L_{\{A,D\}}^{\,2}\,(-2\,\mathrm{Cos}[\bullet A_C+-\bullet A_D]\,\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}+$$
$$2\,\bullet L_{\{C,B\}}^{\,2}+3\,\bullet L_{\{C,D\}}^{\,2})))))\,/$$
$$(\bullet L_{\{A,B\}}^{\,2}\,\bullet L_{\{C,B\}}\,\bullet L_{\{C,D\}}+\bullet L_{\{B,D\}}\,(\bullet L_{\{A,C\}}\,\bullet L_{\{A,D\}}+$$
$$2\,\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}})\,\bullet L_{\{C,D\}}+-\bullet L_{\{A,B\}}$$
$$(3\,\bullet L_{\{A,C\}}\,\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}+\bullet L_{\{A,D\}}\,(-2\,\mathrm{Cos}[\bullet A_C+$$
$$-\bullet A_D]\,\bullet L_{\{B,D\}}\,\bullet L_{\{C,B\}}+\bullet L_{\{C,D\}}^{\,2})))^2))\,/$$
$$(\bullet L_{\{A,C\}}\,(\bullet L_{\{A,B\}}\,\bullet L_{\{C,B\}}+-\bullet L_{\{A,D\}}\,\bullet L_{\{C,D\}})^2)$$

As the expression is still not 0, we try to use the possible relations between areas and lengths of segments

Rewriting

Ordering

$$(-2\,\mathrm{Csc}[\bullet A_{\alpha O}+-\bullet A_{\alpha U}]\,\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_{\alpha V}]$$
$$\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_D]\,\mathrm{Sin}[\bullet A_{\alpha V}+-\bullet A_D]^2\,d_1^{\,3}$$
$$((-\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_{\alpha U}]\,\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_{\alpha V}]^3\,\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_{\alpha V}]^2$$
$$\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_D]\,d_1^{\,7}+\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_{\alpha V}]$$
$$\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_D]^2\,\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_D]^2\,d_1^{\,5}$$
$$(\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_{\alpha U}]\,\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_D]\,d_1^{\,2}+$$
$$\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_{\alpha V}]\,\mathrm{Sin}[\bullet A_{\alpha V}+-\bullet A_D]\,d_1^{\,2})+$$
$$\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_{\alpha V}]^2\,\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_{\alpha V}]\,d_1^{\,3}$$
$$(-\mathrm{Sin}[\bullet A_{\alpha O}+-\bullet A_{\alpha U}]^2\,\mathrm{Sin}[\bullet A_{\alpha V}+-\bullet A_D]$$
$$d_1^{\,3}\,(-3\,\mathrm{Sin}[\bullet A_{\alpha U}+-\bullet A_{\alpha V}]\,d_1+$$
$$-2\,\mathrm{Cos}[\bullet A_{\alpha U}+-\bullet A_D]\,\mathrm{Sin}[\bullet A_{\alpha V}+-\bullet A_D]\,d_1)+$$

$$
\begin{aligned}
&-\sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^3 \\
&\quad (-\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]\, d_1 + 2\cos[\bullet A_{\alpha U} + -\bullet A_{D}] \sin[\\
&\qquad \bullet A_{\alpha V} + -\bullet A_{D}]\, d_1) + \sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}] \\
&\quad \sin[\bullet A_{\alpha O} + -\bullet A_{D}]\, d_1^2\, (\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 d_1^2 + \\
&\qquad 2\sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 d_1^2)) + \\
&\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha V}] \sin[\bullet A_{\alpha U} + -\bullet A_{D}]\, d_1^2 \\
&\quad (-3\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}]^2 \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \\
&\qquad \sin[\bullet A_{\alpha O} + -\bullet A_{D}] \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^5 + \\
&\qquad -\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \sin[\bullet A_{\alpha O} + -\bullet A_{D}] \\
&\qquad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^3 \\
&\qquad (\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 d_1^2 + \sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 d_1^2 + \\
&\qquad\quad -2\cos[\bullet A_{\alpha U} + -\bullet A_{D}] \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \sin[\\
&\qquad\qquad \bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^2) + -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}]\, d_1 \\
&\qquad (4\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 \sin[\bullet A_{\alpha V} + -\bullet A_{D}]^2 d_1^4 + \\
&\qquad\quad \sin[\bullet A_{\alpha O} + -\bullet A_{D}]^2 d_1^2\, (2\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 \\
&\qquad\qquad d_1^2 + \sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 d_1^2)))) \Big/ \\
&(-\sin[\bullet A_{\alpha O} + -\bullet A_{D}] \sin[\bullet A_{\alpha U} + -\bullet A_{D}]\, d_1^2 + \\
&\quad -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha V}]\, d_1\, (-\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]\, d_1 + \\
&\qquad -2\cos[\bullet A_{\alpha U} + -\bullet A_{D}] \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1))^2 + \\
&(\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \sin[\bullet A_{\alpha O} + -\bullet A_{D}]^2 d_1^3 \\
&\quad (-\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha V}]^4 \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \\
&\qquad \sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^8 + \\
&\qquad -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}] \sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 d_1^3 \\
&\qquad (\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 \sin[\bullet A_{\alpha O} + -\bullet A_{D}]^3 d_1^5 + \\
&\qquad\quad -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}]^2 \sin[\bullet A_{\alpha O} + -\bullet A_{D}] \\
&\qquad\quad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]^2 d_1^5 + \\
&\qquad\quad -2\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}] \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \\
&\qquad\quad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^3\, (-\sin[\bullet A_{\alpha O} + -\bullet A_{D}]^2 d_1^2 + \\
&\qquad\qquad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]^2 d_1^2)) + \\
&\qquad -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha V}] \sin[\bullet A_{\alpha U} + -\bullet A_{D}]\, d_1^2 \\
&\qquad (3\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}]^3 \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \\
&\qquad\quad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]^2 d_1^6 + -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}]^2 \\
&\qquad\quad \sin[\bullet A_{\alpha O} + -\bullet A_{D}] \sin[\bullet A_{\alpha V} + -\bullet A_{D}] \\
&\qquad\quad d_1^4\, (2\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 d_1^2 + \\
&\qquad\qquad -\sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 d_1^2 + 2\cos[\bullet A_{\alpha U} + -\bullet A_{D}] \\
&\qquad\qquad\quad \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^2) + \\
&\qquad\quad -2\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 \sin[\bullet A_{\alpha O} + -\bullet A_{D}] \\
&\qquad\quad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]\, d_1^4\, (\sin[\bullet A_{\alpha O} + -\bullet A_{D}]^2 d_1^2 + \\
&\qquad\qquad \sin[\bullet A_{\alpha V} + -\bullet A_{D}]^2 d_1^2) + -\sin[\bullet A_{\alpha O} + -\bullet A_{\alpha U}] \\
&\qquad\quad \sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}] \sin[\bullet A_{\alpha O} + -\bullet A_{D}]^2 d_1^4 \\
&\qquad\quad (2\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 d_1^2 + \sin[\bullet A_{\alpha U} + -\bullet A_{D}]^2 \\
&\qquad\qquad d_1^2 + 5\sin[\bullet A_{\alpha V} + -\bullet A_{D}]^2 d_1^2)) + \\
&\quad \sin[\bullet A_{\alpha O} + -\bullet A_{\alpha V}]^3 \sin[\bullet A_{\alpha U} + -\bullet A_{D}]\, d_1^4 \\
&\qquad (\sin[\bullet A_{\alpha O} + -\bullet A_{D}] \sin[\bullet A_{\alpha V} + -\bullet A_{D}] \\
&\qquad\quad d_1^2\, (3\sin[\bullet A_{\alpha U} + -\bullet A_{\alpha V}]^2 d_1^2 +
\end{aligned}
$$

```
                    Sin[•Aαu + -•AD]^2 d1^2 + -2 Cos[•Aαu + -•AD]
                     Sin[•Aαu + -•Aαv] Sin[•Aαv + -•AD] d1^2) +
                -Sin[•Aαo + -•Aαu] d1 (-Sin[•Aαu + -•Aαv]^3
                    d1^3 + -3 Sin[•Aαu + -•Aαv]
                    Sin[•Aαv + -•AD]^2 d1^3)) +
              Sin[•Aαo + -•Aαv]^2 d1^2 (Sin[•Aαo + -•Aαu]^2
                   Sin[•Aαu + -•Aαv] Sin[•Aαu + -•AD]^2
                   Sin[•Aαv + -•AD] d1^6 + -Sin[•Aαo + -•Aαu]
                   Sin[•Aαo + -•AD] d1^2 (Sin[•Aαu + -•Aαv]^4 d1^4 +
                     2 Sin[•Aαu + -•Aαv]^2 Sin[•Aαu + -•AD]^2 d1^4 +
                     Sin[•Aαv + -•AD]^2 d1^2 (3 Sin[•Aαu + -•Aαv]^2
                         d1^2 + Sin[•Aαu + -•AD]^2 d1^2)) +
                -Sin[•Aαu + -•Aαv] Sin[•Aαv + -•AD] d1^2
                 (2 Sin[•Aαu + -•AD]^2 Sin[•Aαv + -•AD]^2 d1^4 +
                   Sin[•Aαo + -•AD]^2 d1^2 (2 Sin[•Aαu + -•Aαv]^2
                       d1^2 + 3 Sin[•Aαu + -•AD]^2 d1^2 +
                      -2 Cos[•Aαu + -•AD] Sin[•Aαu + -•Aαv]
                       Sin[•Aαv + -•AD] d1^2))))) /
       (Sin[•Aαo + -•Aαv]^2 Sin[•Aαu + -•Aαv]
            Sin[•Aαu + -•AD] d1^4 +
           Sin[•Aαu + -•AD] Sin[•Aαv + -•AD] d1^2
            (Sin[•Aαo + -•Aαu] Sin[•Aαo + -•AD] d1^2 +
              2 Sin[•Aαu + -•Aαv] Sin[•Aαv + -•AD] d1^2) +
           Sin[•Aαo + -•Aαv] d1 (-3 Sin[•Aαo + -•Aαu]
                Sin[•Aαu + -•Aαv] Sin[•Aαv + -•AD] d1^3 +
               -Sin[•Aαo + -•AD] d1 (Sin[•Aαu + -•AD]^2 d1^2 +
                  -2 Cos[•Aαu + -•AD] Sin[•Aαu + -•Aαv]
                   Sin[•Aαv + -•AD] d1^2)))^2)) /
  (Sin[•Aαo + -•Aαv] Sin[•Aαu + -•Aαv]
       d1^2 +
     -Sin[•Aαo + -•AD]
      Sin[
       •Aαu +
        -•AD] d1^2)^2
```

Simplifying

```
0
```

The initial problem was proved, as we obtained 0 == 0.

□

- End of inserted notebook