Line Intersections

- Textbook formulation:

Let *O* and *U* be two points in the plane of triangle *ABC*. Let *AO*, *BO*, *CO* intersect the opposite sides *BC*, *CA*, *AB* in *P, Q, R*. Let *PU, QU, RU* intersect *QR, RP, PQ* respectively in *X,Y,Z*. Show that *AX, BY, CZ* are concurrent.

- Input to the system:

```
Proposition["Chou26", any[A, B, C, O, U, P, Q, R, X, Y, Z, II],
 triangle[A, B, C] ∧
  point[O, U] ∧
  inter[P, line[A, O], line[B, C]] ∧
  inter[Q, line[B, O], line[C, A]] ∧
  inter[R, line[C, O], line[A, B]] ∧
  inter[X, line[P, U], line[Q, R]] ∧
  inter[Y, line[Q, U], line[R, P]] ∧
  inter[Z, line[R, U], line[P, Q]] ∧
  inter[II, line[A, X], line[B, Y]]
 ⇒ collinear[II, C, Z]]
```

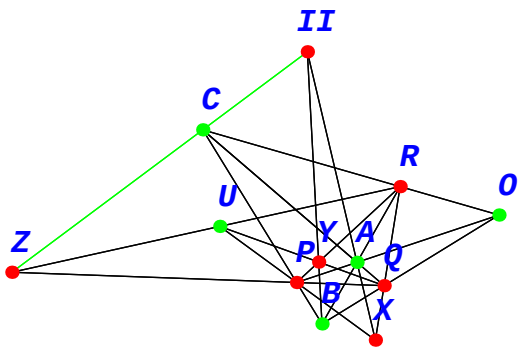
- Input to the system:

```
KnowledgeBase["Chou26_g", any[A, B, C, O, U],
 {{A, {1, 3}}, {B, {-36, -62}},
  {C, {-162, 143}}, {O, {151, 53}}, {U, {-144, 41}}}]
```

- Input to the system:

```
Simplify[Proposition["Chou26"],
 by -> GraphicSimplifier, using → KnowledgeBase["Chou26_g"]]
```

- Output generated automatically by the system



Points ***II C*** and ***Z*** are collinear for this configuration of the points

- End of output

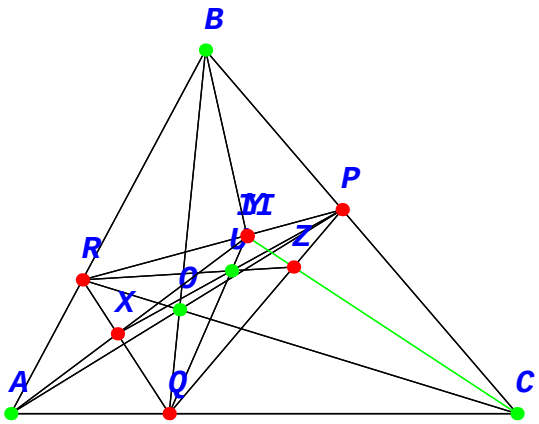
- Input to the system:

```
KnowledgeBase["Chou26_g1", any[A, B, C, O, U],
 {{A, {10, 20}}, {B, {160, 300}},
  {C, {400, 20}}, {O, {140, 100}}, {U, {180, 130}}}]
```

- Input to the system:

```
Simplify[Proposition["Chou26"],
 by -> GraphicSimplifier, using → KnowledgeBase["Chou26_g1"]]
```

- Output generated automatically by the system



Points **II** **C** and **Z** are collinear for this configuration of the points

- End of output

- Input to the system:

```
Prove[Proposition["Chou26"], by → GeometryProver,
 ProverOptions → {Method → "Wu", Iterate → False}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Chou26))

$$\forall_{A,B,C,O,U,P,Q,R,X,Y,Z,II} \Big(\text{triangle}[A, B, C] \wedge \text{point}[O, U] \wedge \text{inter}[P, \text{line}[A, O], \text{line}[B, C]] \wedge \text{inter}[Q, \text{line}[B, O], \text{line}[C, A]] \wedge \text{inter}[R, \text{line}[C, O], \text{line}[A, B]] \wedge \text{inter}[X, \text{line}[P, U], \text{line}[Q, R]] \wedge \text{inter}[Y, \text{line}[Q, U], \text{line}[R, P]] \wedge \text{inter}[Z, \text{line}[R, U], \text{line}[P, Q]] \wedge \text{inter}[II, \text{line}[A, X], \text{line}[B, Y]] \Rightarrow \text{collinear}[II, C, Z]\Big)$$

with no assumptions.

To prove the above statement we shall use Wu's method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point **P**, and points { } and {**B**, **C**} on the two axes.

Using this coordinate system we have the following points:

```
{{P, 0, 0}, {A, u1, u2}, {B, 0, u3}, {C, 0, u4},
 {U, u5, u6}, {O, u7, x1}, {Q, x2, x3}, {R, x4, x5},
 {X, x6, x7}, {Y, x8, x9}, {Z, x10, x11}, {II, x12, x13}}
```

The algebraic form of the given construction is:

(1)

$$\forall_{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13}} (u_2 u_7 + -u_1 x_1 == 0 \wedge$$
$$u_3 u_7 + -u_3 x_2 + x_1 x_2 + -u_7 x_3 == 0 \wedge$$
$$u_1 u_4 + u_2 x_2 + -u_4 x_2 + -u_1 x_3 == 0 \wedge$$
$$u_4 u_7 + -u_4 x_4 + x_1 x_4 + -u_7 x_5 == 0 \wedge$$
$$-u_1 u_3 + -u_2 x_4 + u_3 x_4 + u_1 x_5 == 0 \wedge u_6 x_6 + -u_5 x_7 == 0 \wedge$$
$$x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 == 0 \wedge$$
$$-u_6 x_2 + u_5 x_3 + u_6 x_8 + -x_3 x_8 + -u_5 x_9 + x_2 x_9 == 0 \wedge$$
$$-x_5 x_8 + x_4 x_9 == 0 \wedge$$
$$-u_6 x_4 + u_5 x_5 + u_6 x_{10} + -x_5 x_{10} + -u_5 x_{11} + x_4 x_{11} == 0 \wedge$$
$$x_3 x_{10} + -x_2 x_{11} == 0 \wedge$$
$$u_2 x_6 + -u_1 x_7 + -u_2 x_{12} + x_7 x_{12} + u_1 x_{13} + -x_6 x_{13} == 0 \wedge$$
$$u_3 x_8 + -u_3 x_{12} + x_9 x_{12} + -x_8 x_{13} == 0 \Rightarrow$$
$$u_4 x_{10} + -u_4 x_{12} + x_{11} x_{12} + -x_{10} x_{13} == 0)$$

First we compute the characteristic set of the hypothesis polynomials.

Characteristics of the characteristic set:

Time needed to compute the characteristic set: `422.9 Seconds`.

During the computation the following factors were removed:

The maximum number of monomials in a polynomial during the computation was: `1769`.

The set of initials of the polynomials in the characteristic set is:

The index set of the polynomials in the characteristic set is:

```
{{2, 1, 1}, {4, 2, 1}, {6, 3, 1}, {4, 4, 1}, {6, 5, 1},
 {7, 6, 1}, {7, 7, 1}, {26, 8, 1}, {29, 9, 1},
 {26, 10, 1}, {29, 11, 1}, {46, 12, 1}, {64, 13, 1}}
```

The characteristic set is:

Next we compute the pseudoremainder of the conclusion polynomial $u_4 x_{10} + -u_4 x_{12} + x_{11} x_{12} + -x_{10} x_{13}$ with respect to the obtained characteristic set.

The remainder is: `0`

Time needed to compute the remainder: `2.7 Seconds`.

The maximum number of monomials in a polynomial during the computation was: `162`.

As the obtained remainder is 0 the statement is generically true.

□

- End of inserted notebook
- Input to the system:

```
Prove[Proposition["Chou26"],
 by → GeometryProver, ProverOptions → {Method → "Wu"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Chou26))

$$\forall_{A,B,C,O,U,P,Q,R,X,Y,Z,II} (\text{triangle}[A, B, C] \wedge \text{point}[O, U] \wedge \text{inter}[P, \text{line}[A, O], \text{line}[B, C]] \wedge \text{inter}[Q, \text{line}[B, O], \text{line}[C, A]] \wedge \text{inter}[R, \text{line}[C, O], \text{line}[A, B]] \wedge \text{inter}[X, \text{line}[P, U], \text{line}[Q, R]] \wedge \text{inter}[Y, \text{line}[Q, U], \text{line}[R, P]] \wedge \text{inter}[Z, \text{line}[R, U], \text{line}[P, Q]] \wedge \text{inter}[II, \text{line}[A, X], \text{line}[B, Y]] \Rightarrow \text{collinear}[II, C, Z])$$

with no assumptions.

To prove the above statement we shall use Wu's method. First we have to transform the problem into algebraic form.

Algebraic Form:

First we compute the characteristic set of the hypothesis polynomials.We use successive steps, introducing polynomials for each point in a separate step.

Characteristics of the characteristic set:

Time needed to compute the characteristic set: 63.42 Seconds.

During the computation the following factors were removed:

The maximum number of monomials in a polynomial during the computation was: 503.

The set of initials of the polynomials in the characteristic set is:

The index set of the polynomials in the characteristic set is:

```
{{2, 1, 1}, {4, 2, 1}, {6, 3, 1}, {4, 4, 1}, {6, 5, 1},
 {7, 6, 1}, {7, 7, 1}, {26, 8, 1}, {29, 9, 1},
 {26, 10, 1}, {29, 11, 1}, {46, 12, 1}, {64, 13, 1}}
```

The characteristic set is:

Next we compute the pseudoremainder of the conclusion polynomial $u_4 x_{10} + -u_4 x_{12} + x_{11} x_{12} + -x_{10} x_{13}$ with respect to the obtained characteristic set.

The remainder is: 0.

Time needed to compute the remainder: 2.7 Seconds.

The maximum number of monomials in a polynomial during the computation was: 162.

As the obtained remainder is 0 the statement is generically true.

□

- End of inserted notebook
- Input to the system:

```
Prove[Proposition["Chou26"], by → GeometryProver,
 ProverOptions → {Method → "GroebnerProver"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Chou26))

$$\forall_{A,B,C,O,U,P,Q,R,X,Y,Z,II} (\text{triangle}[A, B, C] \wedge \text{point}[O, U] \wedge \text{inter}[P, \text{line}[A, O], \text{line}[B, C]] \wedge \text{inter}[Q, \text{line}[B, O], \text{line}[C, A]] \wedge \text{inter}[R, \text{line}[C, O], \text{line}[A, B]] \wedge \text{inter}[X, \text{line}[P, U], \text{line}[Q, R]] \wedge \text{inter}[Y, \text{line}[Q, U], \text{line}[R, P]] \wedge \text{inter}[Z, \text{line}[R, U], \text{line}[P, Q]] \wedge \text{inter}[II, \text{line}[A, X], \text{line}[B, Y]] \Rightarrow \text{collinear}[II, C, Z])$$

with no assumptions.

To prove the above statement we shall use the Gröbner basis method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point P, and points $\{\}$ and $\{B, C\}$ on the two axes.

Using this coordinate system we have the following points:

$$\{\{P, 0, 0\}, \{A, u_1, u_2\}, \{B, 0, u_3\}, \{C, 0, u_4\}, \{U, u_5, u_6\}, \{O, u_7, x_1\}, \{Q, x_2, x_3\}, \{R, x_4, x_5\}, \{X, x_6, x_7\}, \{Y, x_8, x_9\}, \{Z, x_{10}, x_{11}\}, \{II, x_{12}, x_{13}\}\}$$

The algebraic form of the given construction is:

(1)

$$\forall_{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13}} (u_2 u_7 + -u_1 x_1 == 0 \wedge u_3 u_7 + -u_3 x_2 + x_1 x_2 + -u_7 x_3 == 0 \wedge u_1 u_4 + u_2 x_2 + -u_4 x_2 + -u_1 x_3 == 0 \wedge u_4 u_7 + -u_4 x_4 + x_1 x_4 + -u_7 x_5 == 0 \wedge -u_1 u_3 + -u_2 x_4 + u_3 x_4 + u_1 x_5 == 0 \wedge u_6 x_6 + -u_5 x_7 == 0 \wedge x_3 x_4 + -x_2 x_5 + -x_3 x_6 + x_5 x_6 + x_2 x_7 + -x_4 x_7 == 0 \wedge -u_6 x_2 + u_5 x_3 + u_6 x_8 + -x_3 x_8 + -u_5 x_9 + x_2 x_9 == 0 \wedge -x_5 x_8 + x_4 x_9 == 0 \wedge -u_6 x_4 + u_5 x_5 + u_6 x_{10} + -x_5 x_{10} + -u_5 x_{11} + x_4 x_{11} == 0 \wedge x_3 x_{10} + -x_2 x_{11} == 0 \wedge u_2 x_6 + -u_1 x_7 + -u_2 x_{12} + x_7 x_{12} + u_1 x_{13} + -x_6 x_{13} == 0 \wedge u_3 x_8 + -u_3 x_{12} + x_9 x_{12} + -x_8 x_{13} == 0 \Rightarrow u_4 x_{10} + -u_4 x_{12} + x_{11} x_{12} + -x_{10} x_{13} == 0)$$

We have to compute the Gröbner bases of the hypothesis polynomials (GB).

We compute the normal form of the conclusion polynomial modulo GB.

The normal form is: 0

As the obtained normal form equals to 0 the statement is generically true.

Statistics:

The length of the Gröbner bases is 13.

Time needed to compute the Gröbner bases: 1.65 Seconds.

Time needed to compute the normal form of the conclusion polynomial modulo GB: 4.35 Seconds.

□

- End of inserted notebook

```
Prove[Proposition["Chou26"], by → GeometryProver,
 ProverOptions → {Method → "GroebnerProver", MyAxis → False}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Chou26))

$$\forall_{A,B,C,O,U,P,Q,R,X,Y,Z,II} (\text{triangle}[A, B, C] \wedge \text{point}[O, U] \wedge \text{inter}[P, \text{line}[A, O], \text{line}[B, C]] \wedge \text{inter}[Q, \text{line}[B, O], \text{line}[C, A]] \wedge \text{inter}[R, \text{line}[C, O], \text{line}[A, B]] \wedge \text{inter}[X, \text{line}[P, U], \text{line}[Q, R]] \wedge \text{inter}[Y, \text{line}[Q, U], \text{line}[R, P]] \wedge \text{inter}[Z, \text{line}[R, U], \text{line}[P, Q]] \wedge \text{inter}[II, \text{line}[A, X], \text{line}[B, Y]] \Rightarrow \text{collinear}[II, C, Z])$$

with no assumptions.

To prove the above statement we shall use the Gröbner basis method. First we have to transform the problem into algebraic form.

Algebraic Form:

To transform the geometric problem into algebraic form we have to chose first an orthogonal coordinate system.

Let's have the origin in point A, and points $\{\}$ and $\{B, R\}$ on the two axes.

Using this coordinate system we have the following points:

$$\{\{A, 0, 0\}, \{B, 0, u_1\}, \{C, u_2, u_3\}, \{O, u_4, u_5\}, \{U, u_6, u_7\}, \{P, x_1, x_2\}, \{Q, x_3, x_4\}, \{R, 0, x_5\}, \{X, x_6, x_7\}, \{Y, x_8, x_9\}, \{Z, x_{10}, x_{11}\}, \{II, x_{12}, x_{13}\}\}$$

The algebraic form of the given construction is:

(1)

$$\forall_{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13}} (u_5 x_1 + -u_4 x_2 == 0 \wedge u_1 u_2 + -u_1 x_1 + u_3 x_1 + -u_2 x_2 == 0 \wedge u_1 u_4 + -u_1 x_3 + u_5 x_3 + -u_4 x_4 == 0 \wedge -u_3 x_3 + u_2 x_4 == 0 \wedge u_3 u_4 + -u_2 u_5 + u_2 x_5 + -u_4 x_5 == 0 \wedge -u_7 x_1 + u_6 x_2 + u_7 x_6 + -x_2 x_6 + -u_6 x_7 + x_1 x_7 == 0 \wedge -x_3 x_5 + -x_4 x_6 + x_5 x_6 + x_3 x_7 == 0 \wedge -u_7 x_3 + u_6 x_4 + u_7 x_8 + -x_4 x_8 + -u_6 x_9 + x_3 x_9 == 0 \wedge x_1 x_5 + x_2 x_8 + -x_5 x_8 + -x_1 x_9 == 0 \wedge u_6 x_5 + u_7 x_{10} + -x_5 x_{10} + -u_6 x_{11} == 0 \wedge x_2 x_3 + -x_1 x_4 + -x_2 x_{10} + x_4 x_{10} + x_1 x_{11} + -x_3 x_{11} == 0 \wedge x_7 x_{12} + -x_6 x_{13} == 0 \wedge u_1 x_8 + -u_1 x_{12} + x_9 x_{12} + -x_8 x_{13} == 0 \Rightarrow u_3 x_{10} + -u_2 x_{11} + -u_3 x_{12} + x_{11} x_{12} + u_2 x_{13} + -x_{10} x_{13} == 0)$$

We have to compute the Gröbner bases of the hypothesis polynomials (GB).

We compute the normal form of the conclusion polynomial modulo GB.

The normal form is: `0`

As the obtained normal form equals to 0 the statement is generically true.

Statistics:

The length of the Gröbner bases is `13`

Time needed to compute the Gröbner bases: `3.16 Seconds`.

Time needed to compute the normal form of the conclusion polynomial modulo GB: `17.63 Seconds`.

□

- End of inserted notebook

- Input to the system:

```
Prove[Proposition["Chou26"], by → GeometryProver,
 ProverOptions → {Method → "Area", OrtCoordSyst → False}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Chou26))

$$\forall_{A,B,C,O,U,P,Q,R,X,Y,Z,II} (\text{triangle}[A, B, C] \wedge \text{point}[O, U] \wedge \text{inter}[P, \text{line}[A, O], \text{line}[B, C]] \wedge \text{inter}[Q, \text{line}[B, O], \text{line}[C, A]] \wedge \text{inter}[R, \text{line}[C, O], \text{line}[A, B]] \wedge \text{inter}[X, \text{line}[P, U], \text{line}[Q, R]] \wedge \text{inter}[Y, \text{line}[Q, U], \text{line}[R, P]] \wedge \text{inter}[Z, \text{line}[R, U], \text{line}[P, Q]] \wedge \text{inter}[II, \text{line}[A, X], \text{line}[B, Y]] \Rightarrow \text{collinear}[II, C, Z])$$

with no assumptions.

We shall use the area method.

First we have to transform the problem into the internal form for the area method. We have to prove, that constructions

{A, B, C, O, U} free points
P = A O ⋂ B C with ndg. {A ≠ O, B ≠ C, A O ∦ B C}
Q = B O ⋂ C A with ndg. {B ≠ O, C ≠ A, B O ∦ A C}
R = C O ⋂ A B with ndg. {C ≠ O, A ≠ B, C O ∦ A B}
X = P U ⋂ Q R with ndg. {P ≠ U, Q ≠ R, P U ∦ Q R}
Y = Q U ⋂ R P with ndg. {Q ≠ U, R ≠ P, Q U ∦ P R}
Z = R U ⋂ P Q with ndg. {R ≠ U, P ≠ Q, R U ∦ P Q}
II = A X ⋂ B Y with ndg. {A ≠ X, B ≠ Y, A X ∦ B Y}

imply that points {II, C, Z} are collinear

Explanations

First we have to verify the non−degenerateconditions for each construction, then to prove our statement.

Step 1

Step 2

Step 3

Step 4

We have to verify the non−degenerateconditions for

X = P U ⋂ Q R with ndg. {P ≠ U, Q ≠ R, P U ∦ Q R}

That is, constructions

{A, B, C, O, U} free points
P = A O ⋂ B C with ndg. {A ≠ O, B ≠ C, A O ∦ B C}
Q = B O ⋂ C A with ndg. {B ≠ O, C ≠ A, B O ∦ A C}
R = C O ⋂ A B with ndg. {C ≠ O, A ≠ B, C O ∦ A B}

do not imply

P U ∥ Q R

We use the elimination method described in the above explanations.

Eliminating point R:

Eliminating point Q:

Eliminating point P:

As the last expression contains no constructed points we have to rewrite the expression in area coordinates of the free points:

As the expression is still not 0, we try to use the possible relations between areas and lengths of segments

Non−degeneratecondition is verified.

Step 5

Step 6

Step 7

Step 8

As all the non−degenerateconditions are verified now we have to prove

{A, B, C, O, U} free points
P = A O ⋂ B C with ndg. {A ≠ O, B ≠ C, A O ∦ B C}
Q = B O ⋂ C A with ndg. {B ≠ O, C ≠ A, B O ∦ A C}
R = C O ⋂ A B with ndg. {C ≠ O, A ≠ B, C O ∦ A B}
X = P U ⋂ Q R with ndg. {P ≠ U, Q ≠ R, P U ∦ Q R}
Y = Q U ⋂ R P with ndg. {Q ≠ U, R ≠ P, Q U ∦ P R}
Z = R U ⋂ P Q with ndg. {R ≠ U, P ≠ Q, R U ∦ P Q}
II = A X ⋂ B Y with ndg. {A ≠ X, B ≠ Y, A X ∦ B Y}
⇒points {II, C, Z} are collinear

Eliminating point II:

Eliminating point Z:

We eliminate point Z using construction:
Z = R U ⋂ P Q with ndg. {R ≠ U, P ≠ Q, R U ∦ P Q}

After ordering the points we obtain:

The simplified expression is:

$$(\bullet S_{\{A,C,U\}}\bullet S_{\{B,X,Y\}}\bullet S_{\{P,Q,R\}}+-\bullet S_{\{A,C,R\}}\bullet S_{\{B,X,Y\}}\bullet S_{\{U,P,Q\}}+\bullet S_{\{A,B,Y\}}(\bullet S_{\{C,U,X\}}\bullet S_{\{P,Q,R\}}+-\bullet S_{\{C,R,X\}}\bullet S_{\{U,P,Q\}}))/((\bullet S_{\{A,B,X\}}+\bullet S_{\{A,X,Y\}})(\bullet S_{\{P,Q,R\}}+-\bullet S_{\{U,P,Q\}}))$$

Eliminating point Y:

We eliminate point Y using construction:
Y = Q U ⋂ R P with ndg. {Q ≠ U, R ≠ P, Q U ∦ P R}

After ordering the points we obtain:

The simplified expression is:

$$\begin{aligned}&(-\bullet S_{\{A,C,R\}}\bullet S_{\{B,U,X\}}\bullet S_{\{P,Q,R\}}\bullet S_{\{U,P,Q\}}+\bullet S_{\{A,B,U\}}\bullet S_{\{P,Q,R\}}(-\bullet S_{\{C,U,X\}}\bullet S_{\{P,Q,R\}}+\bullet S_{\{C,R,X\}}\bullet S_{\{U,P,Q\}})+\\&-\bullet S_{\{A,B,Q\}}\bullet S_{\{C,U,X\}}\bullet S_{\{P,Q,R\}}\bullet S_{\{U,P,R\}}+\\&-\bullet S_{\{A,C,R\}}\bullet S_{\{B,Q,X\}}\bullet S_{\{U,P,Q\}}\bullet S_{\{U,P,R\}}+\\&\bullet S_{\{A,B,Q\}}\bullet S_{\{C,R,X\}}\bullet S_{\{U,P,Q\}}\bullet S_{\{U,P,R\}}+\bullet S_{\{A,C,U\}}\bullet S_{\{P,Q,R\}}(\bullet S_{\{B,U,X\}}\bullet S_{\{P,Q,R\}}+\bullet S_{\{B,Q,X\}}\bullet S_{\{U,P,R\}}))/\\&((\bullet S_{\{P,Q,R\}}+-\bullet S_{\{U,P,Q\}})(\bullet S_{\{A,U,X\}}\bullet S_{\{P,Q,R\}}+\\&\bullet S_{\{A,Q,X\}}\bullet S_{\{U,P,R\}}+-\bullet S_{\{A,B,X\}}(\bullet S_{\{U,P,Q\}}+\bullet S_{\{U,Q,R\}})))\end{aligned}$$

Eliminating point X:

We eliminate point X using construction:
X = P U ⋂ Q R with ndg. {P ≠ U, Q ≠ R, P U ∦ Q R}

After ordering the points we obtain:

The simplified expression is:

$$\begin{aligned}&-(-\bullet S_{\{A,C,R\}}\bullet S_{\{B,Q,R\}}\bullet S_{\{U,P,Q\}}^2\bullet S_{\{U,P,R\}}+\\&\bullet S_{\{A,B,Q\}}\bullet S_{\{C,Q,R\}}\bullet S_{\{U,P,Q\}}\bullet S_{\{U,P,R\}}^2+\\&\bullet S_{\{A,C,R\}}\bullet S_{\{B,U,P\}}\bullet S_{\{P,Q,R\}}\bullet S_{\{U,P,Q\}}\bullet S_{\{U,Q,R\}}+\\&\bullet S_{\{A,B,Q\}}\bullet S_{\{C,U,P\}}\bullet S_{\{P,Q,R\}}\bullet S_{\{U,P,R\}}\bullet S_{\{U,Q,R\}}+\\&\bullet S_{\{A,C,U\}}\bullet S_{\{P,Q,R\}}(\bullet S_{\{B,Q,R\}}\bullet S_{\{U,P,Q\}}\bullet S_{\{U,P,R\}}+\\&-\bullet S_{\{B,U,P\}}\bullet S_{\{P,Q,R\}}\bullet S_{\{U,Q,R\}})+\\&\bullet S_{\{A,B,U\}}\bullet S_{\{P,Q,R\}}(\bullet S_{\{C,Q,R\}}\bullet S_{\{U,P,Q\}}\bullet S_{\{U,P,R\}}+\end{aligned}$$

$$\bullet S_{\{C,U,P\}}\ \bullet S_{\{P,Q,R\}}\ \bullet S_{\{U,Q,R\}}\big)\big)\ /$$
$$\big(\big(\bullet S_{\{P,Q,R\}} + -\bullet S_{\{U,P,Q\}}\big)\ \big(-\bullet S_{\{A,Q,R\}}\ \bullet S_{\{U,P,Q\}}\ \bullet S_{\{U,P,R\}} +$$
$$\bullet S_{\{A,B,U\}}\ \bullet S_{\{P,Q,R\}}\ \big(\bullet S_{\{U,P,Q\}} + \bullet S_{\{U,Q,R\}}\big) + -\bullet S_{\{U,Q,R\}}$$
$$\big(-\bullet S_{\{A,U,P\}}\ \bullet S_{\{P,Q,R\}} + \bullet S_{\{A,B,P\}}\ \big(\bullet S_{\{U,P,Q\}} + \bullet S_{\{U,Q,R\}}\big)\big)\big)\big)$$

Eliminating point R:

Eliminating point Q:

Eliminating point P:

We eliminate point P using construction:
$P = A\,O \bigcap B\,C$ with ndg. $\{A \neq O,\ B \neq C,\ A\,O \nparallel B\,C\}$

After ordering the points we obtain:

The simplified expression is:

As the last expression contains no constructed points we have to rewrite the expression in area coordinates of the free points:

Rewriting

Ordering

Simplifying

The initial problem was proved, as we obtained 0 == 0.

□

- End of inserted notebook