Interior Bisector

- Textbook formulation:

Let ABC be a triangle with $\overline{AB} \leq \overline{AC}$ and AD the interior bisector of $\angle BAC$ ($D \in BC$). Prove that $\overline{BD} \leq \overline{DC}$.

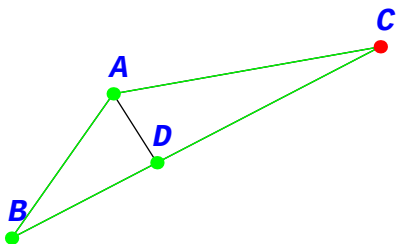
- Input to the system:

```
Proposition["Cad_bisect", any[A, B, C, D],
 point[A, D, B] ∧
   bisectint[A, D, A, B, C] ∧
   hypothesis[seglengthsq[A, B] <= seglengthsq[A, C], D]
  ⇒ inequation[seglengthsq[B, D] <= seglengthsq[D, C]]]
```

- Input to the system:

```
Simplify[Proposition["Cad_bisect"], by -> GraphicSimplifier]
```

- Output generated automatically by the system



$\overline{BD}^2 \leq \overline{CD}^2$ for this configuration of the points

- End of output

```
Prove[Proposition["Cad_bisect"],
 by → GeometryProver, ProverOptions → {Method → "AreaCAD"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Cad_bisect))

$$\forall_{A,B,C,D} \left(\text{point}[A, D, B] \wedge \text{bisectint}[A, D, A, B, C] \wedge \text{hypothesis}[\text{seglengthsq}[A, B] \leq \text{seglengthsq}[A, C], D] \Rightarrow \text{inequation}[\text{seglengthsq}[B, D] \leq \text{seglengthsq}[D, C]]\right)$$

with no assumptions.

As the proposition contains inequalities, we have to use the CAD method. We shall use the area method first to obtain a simpler input for CAD.

First we have to transform the problem in internal form for the AreaCAD. We have to prove, that constructions

$\{A, D, B\}$ free points
$\alpha_A\ B \perp A\ D$ and $\alpha_A \in A\ D$ with ndg. $\{A \neq D\}$

$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{\overline{B\ \alpha 1_A}}{\overline{B\ \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$C = A\ \alpha 1_A \cap B\ D$ with ndg. $\{A \neq \alpha 1_A,\ B \neq D,\ A\ \alpha 1_A \not\parallel B\ D\}$

with additional constraints

$\bullet R_{\{D,B,D,C\}} < 0$
$\overline{A\ B}^2 + -\overline{A\ C}^2 \leq 0$

imply that $\overline{B\ D}^2 + -\overline{C\ D}^2 \leq 0$

Step 1

Now we try to obtain a simpler form of the constraints and of the conclusion using the area method.

Explanations

Expression 1

Using constructions

$\{A,\ D,\ B\}$ free points
$\alpha_A\ B \perp A\ D$ and $\alpha_A \in A\ D$ with ndg. $\{A \neq D\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{\overline{B\ \alpha 1_A}}{\overline{B\ \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$C = A\ \alpha 1_A \cap B\ D$ with ndg. $\{A \neq \alpha 1_A,\ B \neq D,\ A\ \alpha 1_A \not\parallel B\ D\}$

we have to compute $\bullet R_{\{D,B,D,C\}}$

First we have to verify the non−degenerateconditions for each construction, then to prove our statement.

Eliminating point C:

Eliminating point $\alpha 1_A$:

Eliminating point α_A:

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$\bullet R_{\{D,B,D,C\}} = \frac{-\bullet L_{\{A,B\}}^2 + \bullet L_{\{D,B\}}^2}{\bullet L_{\{A,D\}}^2}$

Expression 2

Using constructions

$\{A,\ D,\ B\}$ free points
$\alpha_A\ B \perp A\ D$ and $\alpha_A \in A\ D$ with ndg. $\{A \neq D\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{\overline{B\ \alpha 1_A}}{\overline{B\ \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$C = A\ \alpha 1_A \cap B\ D$ with ndg. $\{A \neq \alpha 1_A,\ B \neq D,\ A\ \alpha 1_A \not\parallel B\ D\}$

we have to compute $\overline{A\ B}^2 + -\overline{A\ C}^2$

First we have to verify the non−degenerateconditions for each construction, then to prove our statement.

Eliminating point C:

Eliminating point $\alpha 1_A$:

Eliminating point α_A :

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$$\bullet \mathrm{P}_{\{A,B,A\}} + - \bullet \mathrm{P}_{\{A,C,A\}} = \frac{2 \bullet \mathrm{L}_{\{A,B\}}^{2} \left(\bullet \mathrm{L}_{\{A,B\}}^{4} + - \bullet \mathrm{L}_{\{A,D\}}^{4} + -2 \bullet \mathrm{L}_{\{A,B\}}^{2} \bullet \mathrm{L}_{\{D,B\}}^{2} + \bullet \mathrm{L}_{\{D,B\}}^{4} \right)}{\left(\bullet \mathrm{L}_{\{A,B\}}^{2} + - \bullet \mathrm{L}_{\{D,B\}}^{2} \right)^{2}}$$

Expression 3

Using constructions

$\{A, D, B\}$ free points
$\alpha_A\ B \perp A\ D$ and $\alpha_A \in A\ D$ with ndg. $\{A \neq D\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{B\ \alpha 1_A}{B\ \alpha_A} = 2$ with ndg. $\{B \neq \alpha_A\}$
$C = A\ \alpha 1_A \bigcap B\ D$ with ndg. $\{A \neq \alpha 1_A, B \neq D, A\ \alpha 1_A \not\parallel B\ D\}$

we have to compute $\overline{B\ D}^{2} + -\overline{C\ D}^{2}$

First we have to verify the non−degenerateconditions for each construction, then to prove our statement.

Eliminating point C :

Eliminating point $\alpha 1_A$:

Eliminating point α_A :

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$$\bullet \mathrm{P}_{\{D,B,D\}} + - \bullet \mathrm{P}_{\{D,C,D\}} = \frac{2 \bullet \mathrm{L}_{\{D,B\}}^{2} \left(\bullet \mathrm{L}_{\{A,B\}}^{4} + - \bullet \mathrm{L}_{\{A,D\}}^{4} + -2 \bullet \mathrm{L}_{\{A,B\}}^{2} \bullet \mathrm{L}_{\{D,B\}}^{2} + \bullet \mathrm{L}_{\{D,B\}}^{4} \right)}{\left(\bullet \mathrm{L}_{\{A,B\}}^{2} + - \bullet \mathrm{L}_{\{D,B\}}^{2} \right)^{2}}$$

Step 2

Our initial problem becomes

(1)

$$\forall_{A,B,D} \left(\frac{- \bullet \mathrm{L}_{\{A,B\}}^{2} + \bullet \mathrm{L}_{\{D,B\}}^{2}}{\bullet \mathrm{L}_{\{A,D\}}^{2}} < 0 \bigwedge \frac{2 \bullet \mathrm{L}_{\{A,B\}}^{2} \left(\bullet \mathrm{L}_{\{A,B\}}^{4} + - \bullet \mathrm{L}_{\{A,D\}}^{4} + -2 \bullet \mathrm{L}_{\{A,B\}}^{2} \bullet \mathrm{L}_{\{D,B\}}^{2} + \bullet \mathrm{L}_{\{D,B\}}^{4} \right)}{\left(\bullet \mathrm{L}_{\{A,B\}}^{2} + - \bullet \mathrm{L}_{\{D,B\}}^{2} \right)^{2}} \leq 0 \Rightarrow \frac{2 \bullet \mathrm{L}_{\{D,B\}}^{2} \left(\bullet \mathrm{L}_{\{A,B\}}^{4} + - \bullet \mathrm{L}_{\{A,D\}}^{4} + -2 \bullet \mathrm{L}_{\{A,B\}}^{2} \bullet \mathrm{L}_{\{D,B\}}^{2} + \bullet \mathrm{L}_{\{D,B\}}^{4} \right)}{\left(\bullet \mathrm{L}_{\{A,B\}}^{2} + - \bullet \mathrm{L}_{\{D,B\}}^{2} \right)^{2}} \leq 0 \right)$$

Choosing a cartesian coordinate system with the x−axis $\{B, A\}$
and performing substitution: $\{x_B \to 0, y_B \to 0, x_A \to 1, y_A \to 0\}$
the problem becomes:

(2)

$$\forall_{x_D, y_D} \left(\frac{(-1)+x_D^2+y_D^2}{((-1)+x_D)^2+y_D^2} < 0 \wedge \frac{8((-1)+x_D)(-x_D+x_D^2+y_D^2)}{((-1)+x_D^2+y_D^2)^2} \leq 0 \Rightarrow \frac{8((-1)+x_D)(x_D^2+y_D^2)(-x_D+x_D^2+y_D^2)}{((-1)+x_D^2+y_D^2)^2} \leq 0 \right)$$

If there are square roots of squares in this expression we eliminate them adding the necessary conditions. After performing substitution $\{x_D \to u_1, y_D \to u_2\}$ the problem becomes:

(3)

$$\forall_{u_1, u_2} \left(\frac{(-1)+u_1^2+u_2^2}{((-1)+u_1)^2+u_2^2} < 0 \wedge \frac{8((-1)+u_1)(-u_1+u_1^2+u_2^2)}{((-1)+u_1^2+u_2^2)^2} \leq 0 \wedge u_1 \neq 0 \wedge u_2 \neq 0 \Rightarrow \frac{8((-1)+u_1)(u_1^2+u_2^2)(-u_1+u_1^2+u_2^2)}{((-1)+u_1^2+u_2^2)^2} \leq 0 \right)$$

Applying the CAD algorithm to this expression we obtain that the proposition is true.

□

- End of inserted notebook