Incircle and Circumcircle

- Textbook formulation:

Let ABC a triangle, r and R the radius of the incircle resp. circumcircle. Show that $2\,r \le R$ and $9\,R^2 \le a^2 + b^2 + c^2$.

- Input to the system:

```
Proposition["Cad_incircle", any[O, A, B, C, P, X, Y, O1],
 point[O, A, B] ∧
   incircle[O, A, B, C, P] ∧
   midpoint[X, A, B] ∧
   midpoint[Y, A, C] ∧ inter[O1, tline[X, A, B], tline[Y, A, C]] ∧
   circle[O1, A]
  ⇒ inequation[4 * seglengthsq[O, P] <= seglengthsq[O1, A]] ∧
   inequation[9 * seglengthsq[O1, A] >=
     seglengthsq[A, B] + seglengthsq[A, C] + seglengthsq[B, C]]]
```

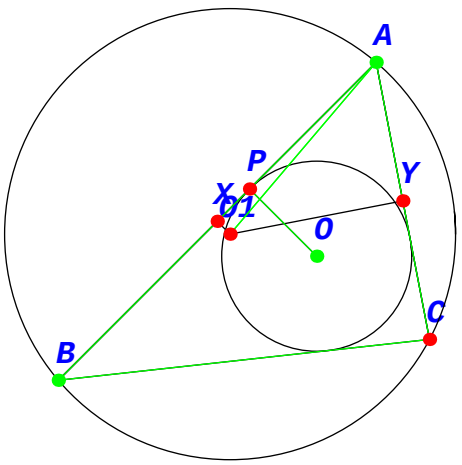
- Input to the system:

```
KnowledgeBase["Cad_incircle_g", any[A, B, O],
 {{A, {0, 0}}, {B, {-123, -123}}, {O, {-23, -75}}}]
```

- Input to the system:

```
Simplify[Proposition["Cad_incircle"],
 by -> GraphicSimplifier, using → KnowledgeBase["Cad_incircle_g"]]
```

- Output generated automatically by the system



$4\,\overline{PO}^2 \le \overline{AO1}^2$ for this configuration of the points

$9\,\overline{AO1}^2 \ge \overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2$ for this configuration of the points

- End of output

- Input to the system:

```
Prove[Proposition["Cad_incircle"],
 by → GeometryProver, ProverOptions → {Method → "AreaCAD"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Cad_incircle))

$$\forall_{O,A,B,C,P,X,Y,O1} (\text{point}[O, A, B] \wedge \text{incircle}[O, A, B, C, P] \wedge \text{midpoint}[X, A, B] \wedge \text{midpoint}[Y, A, C] \wedge \text{inter}[O1, \text{tline}[X, A, B], \text{tline}[Y, A, C]] \wedge \text{circle}[O1, A] \Rightarrow \text{inequation}[4\ \text{seglengthsq}[O, P] \leq \text{seglengthsq}[O1, A]] \wedge \text{inequation}[9\ \text{seglengthsq}[O1, A] \geq \text{seglengthsq}[A, B] + \text{seglengthsq}[A, C] + \text{seglengthsq}[B, C]])$$

with no assumptions.

As the proposition contains inequalities, we have to use the CAD method. We shall use the area method first to obtain a simpler input for CAD.

First we have to transform the problem in internal form for the AreaCAD. We have to prove, that constructions

$\{O, A, B\}$ free points
$\alpha_A\ B \perp A\ O$ and $\alpha_A \in A\ O$ with ndg. $\{A \neq O\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{\overline{B\ \alpha 1_A}}{\overline{B\ \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$\alpha_B\ A \perp B\ O$ and $\alpha_B \in B\ O$ with ndg. $\{B \neq O\}$
$\alpha 1_B\ A \parallel A\ \alpha_B$ and $\frac{\overline{A\ \alpha 1_B}}{\overline{A\ \alpha_B}} = 2$ with ndg. $\{A \neq \alpha_B\}$
$C = A\ \alpha 1_A \cap B\ \alpha 1_B$ with ndg. $\{A \neq \alpha 1_A, B \neq \alpha 1_B, A\ \alpha 1_A \not\parallel B\ \alpha 1_B\}$
$P\ O \perp A\ B$ and $P \in A\ B$ with ndg. $\{A \neq B\}$
$\alpha_C\ O \perp A\ C$ and $\alpha_C \in A\ C$ with ndg. $\{A \neq C\}$
$X\ A \parallel A\ B$ and $\frac{\overline{A\ X}}{\overline{A\ B}} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$Y\ A \parallel A\ C$ and $\frac{\overline{A\ Y}}{\overline{A\ C}} = \frac{1}{2}$ with ndg. $\{A \neq C\}$
$Y\ \gamma_{O1} \perp Y\ A$ and $\frac{\overline{Y\ \gamma_{O1}}}{\overline{Y\ A}} = r_{21}$ with ndg. $\{Y \neq A\}$
$X\ \beta_{O1} \perp X\ A$ and $\frac{\overline{X\ \beta_{O1}}}{\overline{X\ A}} = r_{20}$ with ndg. $\{X \neq A\}$
$O1 = X\ \beta_{O1} \cap Y\ \gamma_{O1}$ with ndg. $\{X \neq \beta_{O1}, Y \neq \gamma_{O1}, \beta_{O1}\ X \not\parallel \gamma_{O1}\ Y\}$

with additional constraints

- $R_{\{A,P,P,B\}} > 0$
- $R_{\{A,\alpha_C,\alpha_C,C\}} > 0$

imply that $-\overline{A\ O1}^2 + 4\ \overline{P\ O}^2 \leq 0$
$-\overline{A\ B}^2 + -\overline{A\ C}^2 + -\overline{B\ C}^2 + 9\ \overline{A\ O1}^2 \geq 0$

Step 1

Now we try to obtain a simpler form of the constraints and of the conclusion using the area method.

Explanations

Expression 1

Using constructions

$\{O, A, B\}$ free points
$\alpha_A\ B \perp A\ O$ and $\alpha_A \in A\ O$ with ndg. $\{A \neq O\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{\overline{B\ \alpha 1_A}}{\overline{B\ \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$\alpha_B\ A \perp B\ O$ and $\alpha_B \in B\ O$ with ndg. $\{B \neq O\}$

$\alpha 1_B\ A \parallel A\ \alpha_B$ and $\frac{A\ \alpha 1_B}{A\ \alpha_B} = 2$ with ndg. $\{A \neq \alpha_B\}$
$C = A\ \alpha 1_A \bigcap B\ \alpha 1_B$ with ndg. $\{A \neq \alpha 1_A, B \neq \alpha 1_B, A\ \alpha 1_A \not\parallel B\ \alpha 1_B\}$
$P\ O \perp A\ B$ and $P \in A\ B$ with ndg. $\{A \neq B\}$
$\alpha_C\ O \perp A\ C$ and $\alpha_C \in A\ C$ with ndg. $\{A \neq C\}$
$X\ A \parallel A\ B$ and $\frac{A\ X}{A\ B} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$Y\ A \parallel A\ C$ and $\frac{A\ Y}{A\ C} = \frac{1}{2}$ with ndg. $\{A \neq C\}$
$Y\ \gamma_{01} \perp Y\ A$ and $\frac{Y\ \gamma_{01}}{Y\ A} = r_{21}$ with ndg. $\{Y \neq A\}$
$X\ \beta_{01} \perp X\ A$ and $\frac{X\ \beta_{01}}{X\ A} = r_{20}$ with ndg. $\{X \neq A\}$
$O1 = X\ \beta_{01} \bigcap Y\ \gamma_{01}$ with ndg. $\{X \neq \beta_{01}, Y \neq \gamma_{01}, \beta_{01}\ X \not\parallel \gamma_{01}\ Y\}$

we have to compute $\bullet R_{\{A,P,P,B\}}$

We should verify the non−degenerateconditions, but you asked to skip this phase, so let's proceed with the computation

Eliminating point P:

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$$\bullet R_{\{A,P,P,B\}} = \frac{\bullet L_{\{A,B\}}^{\ 2} + \bullet L_{\{O,A\}}^{\ 2} + -\bullet L_{\{O,B\}}^{\ 2}}{\bullet L_{\{A,B\}}^{\ 2} + -\bullet L_{\{O,A\}}^{\ 2} + \bullet L_{\{O,B\}}^{\ 2}}$$

Expression 2

Expression 3

Expression 4

Using constructions

$\{O, A, B\}$ free points
$\alpha_A\ B \perp A\ O$ and $\alpha_A \in A\ O$ with ndg. $\{A \neq O\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{B\ \alpha 1_A}{B\ \alpha_A} = 2$ with ndg. $\{B \neq \alpha_A\}$
$\alpha_B\ A \perp B\ O$ and $\alpha_B \in B\ O$ with ndg. $\{B \neq O\}$
$\alpha 1_B\ A \parallel A\ \alpha_B$ and $\frac{A\ \alpha 1_B}{A\ \alpha_B} = 2$ with ndg. $\{A \neq \alpha_B\}$
$C = A\ \alpha 1_A \bigcap B\ \alpha 1_B$ with ndg. $\{A \neq \alpha 1_A, B \neq \alpha 1_B, A\ \alpha 1_A \not\parallel B\ \alpha 1_B\}$
$P\ O \perp A\ B$ and $P \in A\ B$ with ndg. $\{A \neq B\}$
$\alpha_C\ O \perp A\ C$ and $\alpha_C \in A\ C$ with ndg. $\{A \neq C\}$
$X\ A \parallel A\ B$ and $\frac{A\ X}{A\ B} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$Y\ A \parallel A\ C$ and $\frac{A\ Y}{A\ C} = \frac{1}{2}$ with ndg. $\{A \neq C\}$
$Y\ \gamma_{01} \perp Y\ A$ and $\frac{Y\ \gamma_{01}}{Y\ A} = r_{21}$ with ndg. $\{Y \neq A\}$
$X\ \beta_{01} \perp X\ A$ and $\frac{X\ \beta_{01}}{X\ A} = r_{20}$ with ndg. $\{X \neq A\}$
$O1 = X\ \beta_{01} \bigcap Y\ \gamma_{01}$ with ndg. $\{X \neq \beta_{01}, Y \neq \gamma_{01}, \beta_{01}\ X \not\parallel \gamma_{01}\ Y\}$

we have to compute $-A\ B^2 + -A\ C^2 + -B\ C^2 + 9\ A\ O1^2$

We should verify the non−degenerateconditions, but you asked to skip this phase, so let's proceed with the computation

Eliminating point $O1$:

Eliminating point β_{01}:

Eliminating point γ_{01}:

We eliminate point γ_{01} using construction:
$Y\ \gamma_{01} \perp Y\ A$ and $\frac{Y\ \gamma_{01}}{Y\ A} = r_{21}$ with ndg. $\{Y \neq A\}$

We obtain:

Eliminating point Y:

We eliminate point Y using construction:
$Y\ A \parallel A\ C$ and $\frac{A\ Y}{A\ C} = \frac{1}{2}$ with ndg. $\{A \neq C\}$

We obtain:

$$\frac{1}{64 \cdot S_{\{A,C,X\}}^{\ 2}} (9 \cdot P_{\{A,C,A\}}^{\ 2} \cdot P_{\{A,X,A\}} + 4 \cdot P_{\{A,C,A\}} (9 \cdot P_{\{A,X,A\}} (\cdot P_{\{A,X,C\}} + -\cdot P_{\{C,X,C\}}) + -16 \cdot S_{\{A,C,X\}}^{\ 2}) + 4 (-16 (\cdot P_{\{A,B,A\}} + \cdot P_{\{B,C,B\}}) \cdot S_{\{A,C,X\}}^{\ 2} + 9 \cdot P_{\{A,X,A\}} (\cdot P_{\{A,X,C\}}^{\ 2} + -2 \cdot P_{\{A,X,C\}} \cdot P_{\{C,X,C\}} + \cdot P_{\{C,X,C\}}^{\ 2} + 16 \cdot S_{\{A,C,X\}}^{\ 2}))))$$

Eliminating point X:

Eliminating point C:

Eliminating point $\alpha 1_B$:

Eliminating point α_B:

Eliminating point $\alpha 1_A$:

Eliminating point α_A:

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$$-\cdot P_{\{A,B,A\}} + -\cdot P_{\{A,C,A\}} + 9 \cdot P_{\{A,01,A\}} + -\cdot P_{\{B,C,B\}} = \left(9 \cdot L_{\{A,B\}}^{\ 12} + -36 \cdot L_{\{A,B\}}^{\ 10} (\cdot L_{\{O,A\}}^{\ 2} + \cdot L_{\{O,B\}}^{\ 2}) + 128 \cdot L_{\{A,B\}}^{\ 2} (\cdot L_{\{O,A\}}^{\ 2} + -\cdot L_{\{O,B\}}^{\ 2})^2 (\cdot L_{\{O,A\}}^{\ 2} + \cdot L_{\{O,B\}}^{\ 2}) \cdot S_{\{O,A,B\}}^{\ 2} + -64 (\cdot L_{\{O,A\}}^{\ 2} + -\cdot L_{\{O,B\}}^{\ 2})^2 (\cdot L_{\{O,A\}}^{\ 4} + \cdot L_{\{O,B\}}^{\ 4}) \cdot S_{\{O,A,B\}}^{\ 2} + -4 \cdot L_{\{A,B\}}^{\ 6} (\cdot L_{\{O,A\}}^{\ 2} + \cdot L_{\{O,B\}}^{\ 2}) (9 \cdot L_{\{O,A\}}^{\ 4} + 9 \cdot L_{\{O,B\}}^{\ 4} + 40 \cdot S_{\{O,A,B\}}^{\ 2}) + \cdot L_{\{A,B\}}^{\ 8} (54 \cdot L_{\{O,A\}}^{\ 4} + 72 \cdot L_{\{O,A\}}^{\ 2} \cdot L_{\{O,B\}}^{\ 2} + 54 \cdot L_{\{O,B\}}^{\ 4} + 80 \cdot S_{\{O,A,B\}}^{\ 2}) + \cdot L_{\{A,B\}}^{\ 4} (9 \cdot L_{\{O,A\}}^{\ 8} + 9 \cdot L_{\{O,B\}}^{\ 8} + 160 \cdot L_{\{O,A\}}^{\ 2} \cdot L_{\{O,B\}}^{\ 2} \cdot S_{\{O,A,B\}}^{\ 2} + 16 \cdot L_{\{O,B\}}^{\ 4} \cdot S_{\{O,A,B\}}^{\ 2} + 2 \cdot L_{\{O,A\}}^{\ 4} (9 \cdot L_{\{O,B\}}^{\ 4} + 8 \cdot S_{\{O,A,B\}}^{\ 2}))) \Big/ \left(32 \cdot L_{\{A,B\}}^{\ 2} (-\cdot L_{\{A,B\}}^{\ 2} + \cdot L_{\{O,A\}}^{\ 2} + \cdot L_{\{O,B\}}^{\ 2})^2 \cdot S_{\{O,A,B\}}^{\ 2}\right)$$

Step 2

Our initial problem becomes

(1)

$$\forall_{A,B,O} \left(\frac{\bullet L_{\{A,B\}}^2 + \bullet L_{\{O,A\}}^2 + -\bullet L_{\{O,B\}}^2}{\bullet L_{\{A,B\}}^2 + -\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2} > 0 \bigwedge \right.$$

$$\frac{-\bullet L_{\{A,B\}}^4 + \bullet L_{\{O,A\}}^4 + 2\,\bullet L_{\{A,B\}}^2\,\bullet L_{\{O,B\}}^2 + -\bullet L_{\{O,B\}}^4}{\bullet L_{\{A,B\}}^4 + (\bullet L_{\{O,A\}}^2 + -\bullet L_{\{O,B\}}^2)^2 + -2\,\bullet L_{\{A,B\}}^2\,(\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2)} > 0 \Rightarrow$$

$$-\left(\bullet L_{\{A,B\}}^{12} + -4\,\bullet L_{\{A,B\}}^{10}\,(\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2) + \right.$$
$$64\,(\bullet L_{\{O,A\}}^4 + -\bullet L_{\{O,B\}}^4)^2\,\bullet S_{\{O,A,B\}}^2 +$$
$$-256\,\bullet L_{\{A,B\}}^2\,(\bullet L_{\{O,A\}}^6 + \bullet L_{\{O,A\}}^4\,\bullet L_{\{O,B\}}^2 +$$
$$\bullet L_{\{O,A\}}^2\,\bullet L_{\{O,B\}}^4 + \bullet L_{\{O,B\}}^6)\,\bullet S_{\{O,A,B\}}^2 +$$
$$-4\,\bullet L_{\{A,B\}}^6\,(\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2)\,(\bullet L_{\{O,A\}}^4 +$$
$$\bullet L_{\{O,B\}}^4 + 72\,\bullet S_{\{O,A,B\}}^2) + \bullet L_{\{A,B\}}^8\,(6\,\bullet L_{\{O,A\}}^4 +$$
$$8\,\bullet L_{\{O,A\}}^2\,\bullet L_{\{O,B\}}^2 + 6\,\bullet L_{\{O,B\}}^4 + 80\,\bullet S_{\{O,A,B\}}^2) +$$
$$\bullet L_{\{A,B\}}^4\,(\bullet L_{\{O,A\}}^8 + \bullet L_{\{O,B\}}^8 + 544\,\bullet L_{\{O,A\}}^2$$
$$\bullet L_{\{O,B\}}^2\,\bullet S_{\{O,A,B\}}^2 + 400\,\bullet L_{\{O,B\}}^4\,\bullet S_{\{O,A,B\}}^2 +$$
$$\left.2\,\bullet L_{\{O,A\}}^4\,(\bullet L_{\{O,B\}}^4 + 200\,\bullet S_{\{O,A,B\}}^2))\right) /$$
$$\left(32\,\bullet L_{\{A,B\}}^2\,(-\bullet L_{\{A,B\}}^2 + \bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2)^2\right.$$
$$\left.\bullet S_{\{O,A,B\}}^2\right) \le 0 \bigwedge$$
$$\left(9\,\bullet L_{\{A,B\}}^{12} + -36\,\bullet L_{\{A,B\}}^{10}\,(\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2) + \right.$$
$$128\,\bullet L_{\{A,B\}}^2\,(\bullet L_{\{O,A\}}^2 + -\bullet L_{\{O,B\}}^2)^2$$
$$(\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2)\,\bullet S_{\{O,A,B\}}^2 +$$
$$-64\,(\bullet L_{\{O,A\}}^2 + -\bullet L_{\{O,B\}}^2)^2\,(\bullet L_{\{O,A\}}^4 + \bullet L_{\{O,B\}}^4)$$
$$\bullet S_{\{O,A,B\}}^2 + -4\,\bullet L_{\{A,B\}}^6\,(\bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2)$$
$$(9\,\bullet L_{\{O,A\}}^4 + 9\,\bullet L_{\{O,B\}}^4 + 40\,\bullet S_{\{O,A,B\}}^2) +$$
$$\bullet L_{\{A,B\}}^8\,(54\,\bullet L_{\{O,A\}}^4 + 72\,\bullet L_{\{O,A\}}^2\,\bullet L_{\{O,B\}}^2 +$$
$$54\,\bullet L_{\{O,B\}}^4 + 80\,\bullet S_{\{O,A,B\}}^2) +$$
$$\bullet L_{\{A,B\}}^4\,(9\,\bullet L_{\{O,A\}}^8 + 9\,\bullet L_{\{O,B\}}^8 + 160\,\bullet L_{\{O,A\}}^2$$
$$\bullet L_{\{O,B\}}^2\,\bullet S_{\{O,A,B\}}^2 + 16\,\bullet L_{\{O,B\}}^4\,\bullet S_{\{O,A,B\}}^2 +$$
$$\left.2\,\bullet L_{\{O,A\}}^4\,(9\,\bullet L_{\{O,B\}}^4 + 8\,\bullet S_{\{O,A,B\}}^2))\right) /$$
$$\left(32\,\bullet L_{\{A,B\}}^2\,(-\bullet L_{\{A,B\}}^2 + \bullet L_{\{O,A\}}^2 + \bullet L_{\{O,B\}}^2)^2\right.$$
$$\left.\bullet S_{\{O,A,B\}}^2\right) \ge 0\Big)$$

Choosing a cartesian coordinate system with the x–axis $\{O, B\}$
and performing substitution: $\{x_O \to 0, y_O \to 0, x_B \to 1, y_B \to 0\}$
the problem becomes:

(2)

$$\forall_{x_A, y_A} \left(\frac{-(-x_A + x_A^2 + y_A^2)}{(-1) + x_A} > 0 \wedge \frac{-x_A\,(-x_A + x_A^2 + y_A^2)}{y_A^2} > 0 \Rightarrow \right.$$
$$-\left(-4\,x_A^7 + x_A^8 + y_A^4\,(1 + y_A^2)^2 + -4\,x_A^5\,(1 + 3\,y_A^2) + \right.$$
$$-4\,x_A^3\,y_A^2\,(2 + 3\,y_A^2) + x_A^6\,(6 + 4\,y_A^2) +$$
$$x_A^4\,(1 + 14\,y_A^2 + 6\,y_A^4) + -4\,x_A\,(y_A^4 + y_A^6) +$$
$$\left. 2\,x_A^2\,(y_A^2 + -27\,y_A^4 + 2\,y_A^6)\right) /$$
$$(8\,x_A^2\,y_A^2\,(1 + -2\,x_A + x_A^2 + y_A^2)) \le 0 \wedge$$
$$(-36\,x_A^7 + 9\,x_A^8 + -12\,x_A^5\,(3 + y_A^2) + x_A^6\,(54 + 4\,y_A^2) +$$
$$4\,x_A^3\,y_A^2\,(6 + 5\,y_A^2) + x_A^4\,(9 + -2\,y_A^2 + -10\,y_A^4) +$$
$$2\,x_A^2\,y_A^2\,((-7) + -3\,y_A^2 + 2\,y_A^4) +$$
$$y_A^4\,(9 + -14\,y_A^2 + 9\,y_A^4) + -4\,x_A\,(y_A^4 + y_A^6)) /$$
$$\left.(8\,x_A^2\,y_A^2\,(1 + -2\,x_A + x_A^2 + y_A^2)) \ge 0\right)$$

If there are square roots of squares in this expression we eliminate them adding the necessary conditions. After performing substitution $\{x_A \to u_1,\ y_A \to u_2\}$ the problem becomes:

(3)

$$\forall_{u_1, u_2} \left(\frac{-(-u_1 + u_1^2 + u_2^2)}{(-1) + u_1} > 0 \wedge \frac{-u_1\,(-u_1 + u_1^2 + u_2^2)}{u_2^2} > 0 \wedge u_1 \neq 0 \wedge u_2 \neq 0 \Rightarrow \right.$$
$$-\left(-4\,u_1^7 + u_1^8 + u_2^4\,(1 + u_2^2)^2 + -4\,u_1^5\,(1 + 3\,u_2^2) + -4\,u_1^3\,u_2^2\right.$$
$$(2 + 3\,u_2^2) + u_1^6\,(6 + 4\,u_2^2) + u_1^4\,(1 + 14\,u_2^2 + 6\,u_2^4) +$$
$$\left. -4\,u_1\,(u_2^4 + u_2^6) + 2\,u_1^2\,(u_2^2 + -27\,u_2^4 + 2\,u_2^6)\right) /$$
$$(8\,u_1^2\,u_2^2\,(1 + -2\,u_1 + u_1^2 + u_2^2)) \le 0 \wedge$$
$$(-36\,u_1^7 + 9\,u_1^8 + -12\,u_1^5\,(3 + u_2^2) + u_1^6\,(54 + 4\,u_2^2) +$$
$$4\,u_1^3\,u_2^2\,(6 + 5\,u_2^2) + u_1^4\,(9 + -2\,u_2^2 + -10\,u_2^4) +$$
$$2\,u_1^2\,u_2^2\,((-7) + -3\,u_2^2 + 2\,u_2^4) +$$
$$u_2^4\,(9 + -14\,u_2^2 + 9\,u_2^4) + -4\,u_1\,(u_2^4 + u_2^6)) /$$
$$\left.(8\,u_1^2\,u_2^2\,(1 + -2\,u_1 + u_1^2 + u_2^2)) \ge 0\right)$$

Applying the CAD algorithm to this expression we obtain that the proposition is true.

□

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