$OG \leq OI \leq OH$

- Textbook formulation:

Let O be the circumcenter, I the incenter, G the centroid and H the orthocenter of a triangle. Then $OG \leq OI \leq OH$.

- Input to the system:

```
Proposition["Cad_points", any[II, A, B, C, P, H, M, N, O, G],
 point[II, A, B] ∧
   incircle[II, A, B, C, P] ∧
   inter[H, tline[A, B, C], tline[B, A, C]] ∧
   median[C, M, A, B, C] ∧
   median[B, N, A, B, C] ∧
   inter[O, tline[M, A, B], tline[N, A, C]] ∧ circle[O, A] ∧
   inter[G, line[C, M], line[B, N]]
  ⇒ inequation[seglengthsq[O, H] - seglengthsq[O, II] >= 0] ∧
   inequation[seglengthsq[O, II] - seglengthsq[O, G] >= 0]]
```

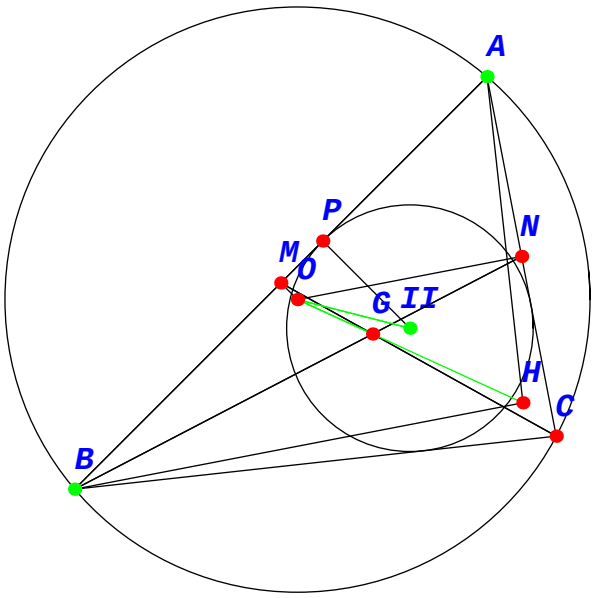
- Input to the system:

```
KnowledgeBase["Cad_points_g", any[A, B, II],
 {{A, {0, 0}}, {B, {-123, -123}}, {II, {-23, -75}}}]
```

- Input to the system:

```
Simplify[Proposition["Cad_points"],
 by -> GraphicSimplifier, using → KnowledgeBase["Cad_points_g"]]
```

- Output generated automatically by the system



$\overline{HO}^2 \geq \overline{II\,O}^2$ for this configuration of the points

$\overline{II\,O}^2 \geq \overline{GO}^2$ for this configuration of the points

- End of output

- Input to the system:

```
Prove[Proposition["Cad_points"],
 by → GeometryProver, ProverOptions → {Method → "AreaCAD"}]

- ProofObject -
```

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Cad_points))

$$\forall_{II,A,B,C,P,H,M,N,O,G}\ (\text{point}[II, A, B] \wedge \text{incircle}[II, A, B, C, P] \wedge \text{inter}[H, \text{tline}[A, B, C], \text{tline}[B, A, C]] \wedge \text{median}[C, M, A, B, C] \wedge \text{median}[B, N, A, B, C] \wedge \text{inter}[O, \text{tline}[M, A, B], \text{tline}[N, A, C]] \wedge \text{circle}[O, A] \wedge \text{inter}[G, \text{line}[C, M], \text{line}[B, N]] \Rightarrow \text{inequation}[(\text{seglengthsq}[O, H] - \text{seglengthsq}[O, II]) \geq 0] \wedge \text{inequation}[(\text{seglengthsq}[O, II] - \text{seglengthsq}[O, G]) \geq 0])$$

with no assumptions.

As the proposition contains inequalities, we have to use the CAD method. We shall use the area method first to obtain a simpler input for CAD.

First we have to transform the problem in internal form for the AreaCAD. We have to prove, that constructions

$\{II, A, B\}$ free points
$\alpha_A\ B \perp A\ II$ and $\alpha_A \in A\ II$ with ndg. $\{A \neq II\}$
$\alpha 1_A\ B \parallel B\ \alpha_A$ and $\frac{B\ \alpha 1_A}{B\ \alpha_A} = 2$ with ndg. $\{B \neq \alpha_A\}$
$\alpha_B\ A \perp B\ II$ and $\alpha_B \in B\ II$ with ndg. $\{B \neq II\}$
$\alpha 1_B\ A \parallel A\ \alpha_B$ and $\frac{A\ \alpha 1_B}{A\ \alpha_B} = 2$ with ndg. $\{A \neq \alpha_B\}$
$C = A\ \alpha 1_A \cap B\ \alpha 1_B$ with ndg. $\{A \neq \alpha 1_A, B \neq \alpha 1_B, A\ \alpha 1_A \not\parallel B\ \alpha 1_B\}$
$P\ II \perp A\ B$ and $P \in A\ B$ with ndg. $\{A \neq B\}$
$\alpha_C\ II \perp A\ C$ and $\alpha_C \in A\ C$ with ndg. $\{A \neq C\}$
$\gamma_H\ B \perp A\ C$ and $\gamma_H \in A\ C$ with ndg. $\{A \neq C\}$
$\beta_H\ A \perp B\ C$ and $\beta_H \in B\ C$ with ndg. $\{B \neq C\}$
$H = A\ \beta_H \cap B\ \gamma_H$ with ndg. $\{A \neq \beta_H, B \neq \gamma_H, \beta_H\ A \not\parallel \gamma_H\ B\}$
$M\ A \parallel A\ B$ and $\frac{A\ M}{A\ B} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$N\ A \parallel A\ C$ and $\frac{A\ N}{A\ C} = \frac{1}{2}$ with ndg. $\{A \neq C\}$
$N\ \gamma_O \perp N\ A$ and $\frac{N\ \gamma_O}{N\ A} = r_{23}$ with ndg. $\{N \neq A\}$
$M\ \beta_O \perp M\ A$ and $\frac{M\ \beta_O}{M\ A} = r_{22}$ with ndg. $\{M \neq A\}$
$O = M\ \beta_O \cap N\ \gamma_O$ with ndg. $\{M \neq \beta_O, N \neq \gamma_O, \beta_O\ M \not\parallel \gamma_O\ N\}$
$G = C\ M \cap B\ N$ with ndg. $\{C \neq M, B \neq N, C\ M \not\parallel B\ N\}$

with additional constraints

- $R_{\{A,P,P,B\}} > 0$
- $R_{\{A,\alpha_C,\alpha_C,C\}} > 0$

imply that $\overline{HO}^2 + -\overline{IIO}^2 \geq 0$
$-\overline{GO}^2 + \overline{IIO}^2 \geq 0$

Step 1

Now we try to obtain a simpler form of the constraints and of the conclusion using the area method.

Explanations

Expression 1

Using constructions

$\{II, A, B\}$ free points
$\alpha_A\, B \perp A\, II$ and $\alpha_A \in A\, II$ with ndg. $\{A \neq II\}$
$\alpha 1_A\, B \parallel B\, \alpha_A$ and $\frac{\overline{B\, \alpha 1_A}}{\overline{B\, \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$\alpha_B\, A \perp B\, II$ and $\alpha_B \in B\, II$ with ndg. $\{B \neq II\}$
$\alpha 1_B\, A \parallel A\, \alpha_B$ and $\frac{\overline{A\, \alpha 1_B}}{\overline{A\, \alpha_B}} = 2$ with ndg. $\{A \neq \alpha_B\}$
$C = A\, \alpha 1_A \cap B\, \alpha 1_B$ with ndg. $\{A \neq \alpha 1_A,\ B \neq \alpha 1_B,\ A\, \alpha 1_A \nparallel B\, \alpha 1_B\}$
$P\, II \perp A\, B$ and $P \in A\, B$ with ndg. $\{A \neq B\}$
$\alpha_C\, II \perp A\, C$ and $\alpha_C \in A\, C$ with ndg. $\{A \neq C\}$
$\gamma_H\, B \perp A\, C$ and $\gamma_H \in A\, C$ with ndg. $\{A \neq C\}$
$\beta_H\, A \perp B\, C$ and $\beta_H \in B\, C$ with ndg. $\{B \neq C\}$
$H = A\, \beta_H \cap B\, \gamma_H$ with ndg. $\{A \neq \beta_H,\ B \neq \gamma_H,\ \beta_H\, A \nparallel \gamma_H\, B\}$
$M\, A \parallel A\, B$ and $\frac{\overline{A\, M}}{\overline{A\, B}} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$N\, A \parallel A\, C$ and $\frac{\overline{A\, N}}{\overline{A\, C}} = \frac{1}{2}$ with ndg. $\{A \neq C\}$
$N\, \gamma_O \perp N\, A$ and $\frac{\overline{N\, \gamma_O}}{\overline{N\, A}} = r_{23}$ with ndg. $\{N \neq A\}$
$M\, \beta_O \perp M\, A$ and $\frac{\overline{M\, \beta_O}}{\overline{M\, A}} = r_{22}$ with ndg. $\{M \neq A\}$
$O = M\, \beta_O \cap N\, \gamma_O$ with ndg. $\{M \neq \beta_O,\ N \neq \gamma_O,\ \beta_O\, M \nparallel \gamma_O\, N\}$
$G = C\, M \cap B\, N$ with ndg. $\{C \neq M,\ B \neq N,\ C\, M \nparallel B\, N\}$

we have to compute $\bullet R_{\{A,P,P,B\}}$

We should verify the non–degenerateconditions, but you asked to skip this phase, so let's proceed with the computation

Eliminating point P:

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$$\bullet R_{\{A,P,P,B\}} = \frac{\bullet L_{\{A,B\}}{}^2 + \bullet L_{\{II,A\}}{}^2 + - \bullet L_{\{II,B\}}{}^2}{\bullet L_{\{A,B\}}{}^2 + - \bullet L_{\{II,A\}}{}^2 + \bullet L_{\{II,B\}}{}^2}$$

Expression 2

Expression 3

Expression 4

Using constructions

$\{II, A, B\}$ free points
$\alpha_A\, B \perp A\, II$ and $\alpha_A \in A\, II$ with ndg. $\{A \neq II\}$
$\alpha 1_A\, B \parallel B\, \alpha_A$ and $\frac{\overline{B\, \alpha 1_A}}{\overline{B\, \alpha_A}} = 2$ with ndg. $\{B \neq \alpha_A\}$
$\alpha_B\, A \perp B\, II$ and $\alpha_B \in B\, II$ with ndg. $\{B \neq II\}$
$\alpha 1_B\, A \parallel A\, \alpha_B$ and $\frac{\overline{A\, \alpha 1_B}}{\overline{A\, \alpha_B}} = 2$ with ndg. $\{A \neq \alpha_B\}$
$C = A\, \alpha 1_A \cap B\, \alpha 1_B$ with ndg. $\{A \neq \alpha 1_A,\ B \neq \alpha 1_B,\ A\, \alpha 1_A \nparallel B\, \alpha 1_B\}$

$P\ II \perp A\ B$ and $P \in A\ B$ with ndg. $\{A \neq B\}$
$\alpha_C\ II \perp A\ C$ and $\alpha_C \in A\ C$ with ndg. $\{A \neq C\}$
$\gamma_H\ B \perp A\ C$ and $\gamma_H \in A\ C$ with ndg. $\{A \neq C\}$
$\beta_H\ A \perp B\ C$ and $\beta_H \in B\ C$ with ndg. $\{B \neq C\}$
$H = A\ \beta_H \bigcap B\ \gamma_H$ with ndg. $\{A \neq \beta_H, B \neq \gamma_H, \beta_H\ A \nparallel \gamma_H\ B\}$
$M\ A \parallel A\ B$ and $\frac{A\ M}{A\ B} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$N\ A \parallel A\ C$ and $\frac{A\ N}{A\ C} = \frac{1}{2}$ with ndg. $\{A \neq C\}$
$N\ \gamma_O \perp N\ A$ and $\frac{N\ \gamma_O}{N\ A} = r_{23}$ with ndg. $\{N \neq A\}$
$M\ \beta_O \perp M\ A$ and $\frac{M\ \beta_O}{M\ A} = r_{22}$ with ndg. $\{M \neq A\}$
$O = M\ \beta_O \bigcap N\ \gamma_O$ with ndg. $\{M \neq \beta_O, N \neq \gamma_O, \beta_O\ M \nparallel \gamma_O\ N\}$
$G = C\ M \bigcap B\ N$ with ndg. $\{C \neq M, B \neq N, C\ M \nparallel B\ N\}$

we have to compute $-\overline{G\ O}^2 + \overline{II\ O}^2$

We should verify the non−degenerateconditions, but you asked to skip this phase, so let's proceed with the computation

Eliminating point G:

Eliminating point O:

Eliminating point β_O:

Eliminating point γ_O:

Eliminating point N:

Eliminating point M:

Eliminating point C:

Eliminating point $\alpha 1_B$:

Eliminating point α_B:

Eliminating point $\alpha 1_A$:

Eliminating point α_A:

All the constructed points have been eliminated, so we have to rewrite the expression in area coordinates of the free points:

We try to simplify further this expression using the possible relations between areas and lengths of segments

Thus

$$
\begin{aligned}
&\bullet P_{\{II,O,II\}} + -\bullet P_{\{O,G,O\}} = \\
&\Big(2\ \Big(\bullet L_{\{A,B\}}{}^8 + -2\ \bullet L_{\{A,B\}}{}^6\ (\bullet L_{\{II,A\}}{}^2 + \bullet L_{\{II,B\}}{}^2) + \\
&\qquad (\bullet L_{\{II,A\}}{}^2 + -\bullet L_{\{II,B\}}{}^2)^2\ (\bullet L_{\{II,A\}}{}^4 + \bullet L_{\{II,B\}}{}^4) + \\
&\qquad \bullet L_{\{A,B\}}{}^4\ (2\ \bullet L_{\{II,A\}}{}^4 + -7\ \bullet L_{\{II,A\}}{}^2\ \bullet L_{\{II,B\}}{}^2 + 2\ \bullet L_{\{II,B\}}{}^4) + \\
&\qquad \bullet L_{\{A,B\}}{}^2\ (-2\ \bullet L_{\{II,A\}}{}^6 + 11\ \bullet L_{\{II,A\}}{}^4\ \bullet L_{\{II,B\}}{}^2 + \\
&\qquad\quad 11\ \bullet L_{\{II,A\}}{}^2\ \bullet L_{\{II,B\}}{}^4 + -2\ \bullet L_{\{II,B\}}{}^6)\Big)\Big) \Big/ \\
&\Big(9\ \bullet L_{\{A,B\}}{}^2\ (-\bullet L_{\{A,B\}}{}^2 + \bullet L_{\{II,A\}}{}^2 + \bullet L_{\{II,B\}}{}^2)^2\Big)
\end{aligned}
$$

Step 2

Our initial problem becomes

(1)

$$\forall_{A,B,II} \left(\frac{\bullet L_{\{A,B\}}^2 + \bullet L_{\{II,A\}}^2 + -\bullet L_{\{II,B\}}^2}{\bullet L_{\{A,B\}}^2 + -\bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2} > 0 \bigwedge \right.$$

$$\frac{-\bullet L_{\{A,B\}}^4 + \bullet L_{\{II,A\}}^4 + 2\, \bullet L_{\{A,B\}}^2\, \bullet L_{\{II,B\}}^2 + -\bullet L_{\{II,B\}}^4}{\bullet L_{\{A,B\}}^4 + (\bullet L_{\{II,A\}}^2 + -\bullet L_{\{II,B\}}^2)^2 + -2\, \bullet L_{\{A,B\}}^2\, (\bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2)} > 0 \Rightarrow$$

$$\Big(-2\, \Big(\bullet L_{\{A,B\}}^{12} + -4\, \bullet L_{\{A,B\}}^{10}\, (\bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2) +$$
$$(\bullet L_{\{II,A\}}^2 + -\bullet L_{\{II,B\}}^2)^4\, (\bullet L_{\{II,A\}}^4 + \bullet L_{\{II,B\}}^4) +$$
$$7\, \bullet L_{\{A,B\}}^8\, (\bullet L_{\{II,A\}}^4 + \bullet L_{\{II,A\}}^2\, \bullet L_{\{II,B\}}^2 + \bullet L_{\{II,B\}}^4) +$$
$$-\bullet L_{\{A,B\}}^2\, (\bullet L_{\{II,A\}}^2 + -\bullet L_{\{II,B\}}^2)^2$$
$$(4\, \bullet L_{\{II,A\}}^6 + -\bullet L_{\{II,A\}}^4\, \bullet L_{\{II,B\}}^2 +$$
$$-\bullet L_{\{II,A\}}^2\, \bullet L_{\{II,B\}}^4 + 4\, \bullet L_{\{II,B\}}^6) +$$
$$-\bullet L_{\{A,B\}}^6\, (8\, \bullet L_{\{II,A\}}^6 + \bullet L_{\{II,A\}}^4\, \bullet L_{\{II,B\}}^2 +$$
$$\bullet L_{\{II,A\}}^2\, \bullet L_{\{II,B\}}^4 + 8\, \bullet L_{\{II,B\}}^6) +$$
$$7\, \bullet L_{\{A,B\}}^4\, (\bullet L_{\{II,A\}}^8 + -\bullet L_{\{II,A\}}^6\, \bullet L_{\{II,B\}}^2 +$$
$$-\bullet L_{\{II,A\}}^2\, \bullet L_{\{II,B\}}^6 + \bullet L_{\{II,B\}}^8)\Big)\Big) /$$
$$\Big(\bullet L_{\{A,B\}}^2\, (-\bullet L_{\{A,B\}}^2 + \bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2)^2$$
$$\Big(\bullet L_{\{A,B\}}^4 + (\bullet L_{\{II,A\}}^2 + -\bullet L_{\{II,B\}}^2)^2 +$$
$$-2\, \bullet L_{\{A,B\}}^2\, (\bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2)\Big)\Big) \geq 0 \bigwedge$$
$$\Big(2\, \Big(\bullet L_{\{A,B\}}^8 + -2\, \bullet L_{\{A,B\}}^6\, (\bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2) +$$
$$(\bullet L_{\{II,A\}}^2 + -\bullet L_{\{II,B\}}^2)^2$$
$$(\bullet L_{\{II,A\}}^4 + \bullet L_{\{II,B\}}^4) + \bullet L_{\{A,B\}}^4$$
$$(2\, \bullet L_{\{II,A\}}^4 + -7\, \bullet L_{\{II,A\}}^2\, \bullet L_{\{II,B\}}^2 + 2\, \bullet L_{\{II,B\}}^4) +$$
$$\bullet L_{\{A,B\}}^2\, (-2\, \bullet L_{\{II,A\}}^6 + 11\, \bullet L_{\{II,A\}}^4\, \bullet L_{\{II,B\}}^2 +$$
$$11\, \bullet L_{\{II,A\}}^2\, \bullet L_{\{II,B\}}^4 + -2\, \bullet L_{\{II,B\}}^6)\Big)\Big) /$$
$$\Big(9\, \bullet L_{\{A,B\}}^2\, (-\bullet L_{\{A,B\}}^2 + \bullet L_{\{II,A\}}^2 + \bullet L_{\{II,B\}}^2)^2\Big) \geq 0\Big)$$

Choosing a cartesian coordinate system with the x–axis $\{II, B\}$
and performing substitution: $\{x_{II} \to 0,\ y_{II} \to 0,\ x_B \to 1,\ y_B \to 0\}$
the problem becomes:

(2)

$$\forall_{x_A, y_A} \left(\frac{-(-x_A + x_A^2 + y_A^2)}{(-1) + x_A} > 0 \bigwedge \frac{-x_A\, (-x_A + x_A^2 + y_A^2)}{y_A^2} > 0 \Rightarrow \right.$$

$$\Big(6\, x_A^6 + -4\, x_A^7 + x_A^8 + -2\, x_A^2\, y_A^2 + y_A^4\, ((-1) + y_A^2)^2 +$$
$$-x_A^5\, (4 + y_A^2) + x_A^3\, y_A^2\, (3 + 2\, y_A^2) + x_A^4\, (1 + -2\, y_A^4) +$$
$$-x_A\, (y_A^4 + y_A^6)\Big) / (x_A^2\, y_A^2\, (1 + -2\, x_A + x_A^2 + y_A^2)) \geq 0 \bigwedge$$
$$(-3\, x_A^5 + 4\, x_A^6 + 4\, y_A^4 + x_A^3\, ((-3) + 2\, y_A^2) +$$
$$x_A^4\, ((-2) + 8\, y_A^2) + 5\, x_A\, (y_A^2 + y_A^4) +$$
$$x_A^2\, (4 + -6\, y_A^2 + 4\, y_A^4)) /$$
$$(9\, x_A^2\, (1 + -2\, x_A + x_A^2 + y_A^2)) \geq 0\Big)$$

If there are square roots of squares in this expression we eliminate them adding the necessary conditions. After performing substitution $\{x_A \to u_1,\ y_A \to u_2\}$ the problem becomes:

(3)

$$\forall_{u_1,u_2} \left(\frac{-(-u_1+u_1^2+u_2^2)}{(-1)+u_1} > 0 \bigwedge \frac{-u_1\,(-u_1+u_1^2+u_2^2)}{u_2^2} > 0 \bigwedge u_1 \neq 0 \bigwedge u_2 \neq 0 \Rightarrow \right.$$
$$\left(6\,u_1^6 + -4\,u_1^7 + u_1^8 + -2\,u_1^2\,u_2^2 + u_2^4\,((-1)+u_2^2)^2 + \right.$$
$$-u_1^5\,(4+u_2^2) + u_1^3\,u_2^2\,(3+2\,u_2^2) + u_1^4\,(1+-2\,u_2^4) +$$
$$\left.-u_1\,(u_2^4+u_2^6)\right) \Big/ (u_1^2\,u_2^2\,(1+-2\,u_1+u_1^2+u_2^2)) \geq 0 \bigwedge$$
$$(-3\,u_1^5 + 4\,u_1^6 + 4\,u_2^4 + u_1^3\,((-3)+2\,u_2^2) + u_1^4\,((-2)+$$
$$8\,u_2^2) + 5\,u_1\,(u_2^2+u_2^4) + u_1^2\,(4+-6\,u_2^2+4\,u_2^4)) /$$
$$\left.(9\,u_1^2\,(1+-2\,u_1+u_1^2+u_2^2)) \geq 0\right)$$

Applying the CAD algorithm to this expression we obtain that the proposition is true.

□

- End of inserted notebook