$$0.4 < \frac{r}{h} < 0.5.$$

- Textbook formulation:

Let ABC be a rectangular triangle ($\angle BAC = 90°$), r the radius of the inscribed circle and h the altitude attached to A. Prove that

$$0.4 < \frac{r}{h} < 0.5.$$

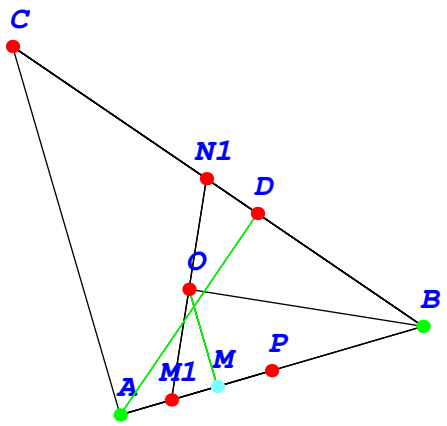
- Input to the system:

```
Proposition["Cad_61_32", any[A, B, P, M, O, M1, N1, C, D],
 point[A, B] ∧ midpoint[P, A, B] ∧ pon[M, segment[A, P]] ∧
   pon[O, tline[M, M, A]] ∧ eqdist[O, M, M, A] ∧
   inter[M1, line[A, B], tline[O, O, B]] ∧ divpoint[N1, M1, O, 2] ∧ inter[C,
    line[B, N1], tline[A, A, B]] ∧ inter[D, line[B, C], tline[A, B, C]]
  ⇒ inequation[4 / 25 <= seglengthsq[O, M] / seglengthsq[A, D]] ∧
   inequation[seglengthsq[O, M] / seglengthsq[A, D] ≤ 1 / 4]]
```

- End of output

```
Simplify[Proposition["Cad_61_32"], by -> GraphicSimplifier]
```

- Output generated automatically by the system



$\frac{4}{25} \leq \frac{\overline{MO}^2}{\overline{AD}^2}$ for this configuration of the points

$\frac{\overline{MO}^2}{\overline{AD}^2} \leq \frac{1}{4}$ for this configuration of the points

- End of output

- Input to the system:

```
Prove[Proposition["Cad_61_32"],
 by → GeometryProver, ProverOptions → {Method -> "AreaCAD"}]
```

- ProofObject -

- Inserted notebook generated automatically by the system

Prove:

(Proposition (Cad_61_32))

$$\forall_{A,B,P,M,O,M1,N1,C,D} \; (\text{point}[A, B] \wedge \text{midpoint}[P, A, B] \wedge \text{pon}[M, \text{segment}[A, P]] \wedge \text{pon}[O, \text{tline}[M, M, A]] \wedge \text{eqdist}[O, M, M, A] \wedge \text{inter}[M1, \text{line}[A, B], \text{tline}[O, O, B]] \wedge \text{divpoint}[N1, M1, O, 2] \wedge \text{inter}[C, \text{line}[B, N1], \text{tline}[A, A, B]] \wedge \text{inter}[D, \text{line}[B, C], \text{tline}[A, B, C]] \Rightarrow \text{inequation}[\tfrac{4}{25} \leq \tfrac{\text{seglengthsq}[O,M]}{\text{seglengthsq}[A,D]}] \bigwedge \text{inequation}[\tfrac{\text{seglengthsq}[O,M]}{\text{seglengthsq}[A,D]} \leq \tfrac{1}{4}])$$

with no assumptions.

As the proposition contains inequalities, we have to use the CAD method. We shall use the area method first to obtain a simpler input for CAD.

First we have to transform the problem in internal form for the AreaCAD. We have to prove, that constructions

$\{A, B\}$ free points
$P\,A \parallel A\,B$ and $\frac{\overline{A\,P}}{\overline{A\,B}} = \frac{1}{2}$ with ndg. $\{A \neq B\}$
$M\,A \parallel A\,P$ and $\frac{\overline{A\,M}}{\overline{A\,P}} = r_M$ with ndg. $\{A \neq P\}$
$M\,O \perp M\,A$ and $\frac{\overline{M\,O}}{\overline{M\,A}} = 1$ with ndg. $\{M \neq A\}$
$O\,\beta_{M1} \perp O\,B$ and $\frac{\overline{O\,\beta_{M1}}}{\overline{O\,B}} = r_8$ with ndg. $\{O \neq B\}$
$M1 = A\,B \bigcap O\,\beta_{M1}$ with ndg. $\{A \neq B, O \neq \beta_{M1}, A\,B \not\parallel \beta_{M1}\,O\}$
$N1\,M1 \parallel M1\,O$ and $\frac{\overline{M1\,N1}}{\overline{M1\,O}} = 2$ with ndg. $\{M1 \neq O\}$
$A\,\beta_C \perp A\,B$ and $\frac{\overline{A\,\beta_C}}{\overline{A\,B}} = r_{11}$ with ndg. $\{A \neq B\}$
$C = B\,N1 \bigcap A\,\beta_C$ with ndg. $\{B \neq N1, A \neq \beta_C, B\,N1 \not\parallel \beta_C\,A\}$
$D\,A \perp B\,C$ and $D \in B\,C$ with ndg. $\{B \neq C\}$

with additional constraints

$r_M > 0$
$(-1) + r_M < 0$

imply that $\frac{4}{25} + \frac{-\overline{M\,O}^2}{\overline{A\,D}^2} \leq 0$
$(\frac{-1}{4}) + \frac{\overline{M\,O}^2}{\overline{A\,D}^2} \leq 0$

Step 1

Now we try to obtain a simpler form of the constraints and of the conclusion using the area method.

Explanations

Expression 1

Expression r_M cannot be simplified by the area method

Expression 2

Expression $(-1) + r_M$ cannot be simplified by the area method

Expression 3

Expression 4

Using constructions

{**A**, **B**} free points
P A ∥ **A B** and $\frac{\overline{A P}}{\overline{A B}} = \frac{1}{2}$ with ndg. {**A** ≠ **B**}
M A ∥ **A P** and $\frac{\overline{A M}}{\overline{A P}} = r_M$ with ndg. {**A** ≠ **P**}
M O ⊥ **M A** and $\frac{\overline{M O}}{\overline{M A}} = 1$ with ndg. {**M** ≠ **A**}
O β_{M1} ⊥ **O B** and $\frac{\overline{O \beta_{M1}}}{\overline{O B}} = r_8$ with ndg. {**O** ≠ **B**}
M1 = **A B** ⋂ **O** β_{M1} with ndg. {**A** ≠ **B**, **O** ≠ β_{M1}, **A B** ⋔ β_{M1} **O**}
N1 M1 ∥ **M1 O** and $\frac{\overline{M1\ N1}}{\overline{M1\ O}} = 2$ with ndg. {**M1** ≠ **O**}
A β_C ⊥ **A B** and $\frac{\overline{A \beta_C}}{\overline{A B}} = r_{11}$ with ndg. {**A** ≠ **B**}
C = **B N1** ⋂ **A** β_C with ndg. {**B** ≠ **N1**, **A** ≠ β_C, **B N1** ⋔ β_C **A**}
D A ⊥ **B C** and **D**∈**B C** with ndg. {**B** ≠ **C**}

we have to compute $\left(\frac{-1}{4}\right) + \frac{\overline{A M}^2}{\overline{A D}^2}$

We should verify the non−degenerateconditions, but you asked to skip this phase, so let's proceed with the computation

Eliminating point **D**:

We eliminate point **D** using construction:
D A ⊥ **B C** and **D**∈**B C** with ndg. {**B** ≠ **C**}

We obtain:

$$\left(\frac{-1}{4}\right) + \frac{\bullet P_{\{A,M,A\}}\ \bullet P_{\{B,C,B\}}}{\bullet P_{\{A,B,C\}}\ (\bullet P_{\{A,C,A\}} + -\bullet P_{\{A,C,B\}}) + \bullet P_{\{A,B,A\}}\ \bullet P_{\{A,C,B\}}}$$

Eliminating point **C**:

We eliminate point **C** using construction:
C = **B N1** ⋂ **A** β_C with ndg. {**B** ≠ **N1**, **A** ≠ β_C, **B N1** ⋔ β_C **A**}

We obtain:

$$\begin{aligned}&(\bullet P_{\{A,B,N1\}}\ (-\bullet P_{\{A,N1,A\}}\ \bullet S_{\{A,B,\beta_C\}} + \\ &\quad \bullet P_{\{A,N1,B\}}\ \bullet S_{\{A,B,\beta_C\}} + \bullet P_{\{A,B,A\}}\ \bullet S_{\{A,N1,\beta_C\}}) + \\ &\quad -\bullet P_{\{A,B,A\}}\ (\bullet P_{\{B,N1,B\}}\ \bullet S_{\{A,N1,\beta_C\}} + \\ &\quad \bullet P_{\{A,N1,B\}}\ (\bullet S_{\{A,B,N1\}} + -\bullet S_{\{B,N1,\beta_C\}})) + 4\ \bullet P_{\{A,M,A\}} \\ &\quad \bullet P_{\{B,N1,B\}}\ (\bullet S_{\{A,B,N1\}} + \bullet S_{\{A,N1,\beta_C\}} + -\bullet S_{\{B,N1,\beta_C\}}))\ / \\ &(4\ (\bullet P_{\{A,B,N1\}}\ (\bullet P_{\{A,N1,A\}}\ \bullet S_{\{A,B,\beta_C\}} + -\bullet P_{\{A,N1,B\}}\ \bullet S_{\{A,B,\beta_C\}} + \\ &\quad -\bullet P_{\{A,B,A\}}\ \bullet S_{\{A,N1,\beta_C\}}) + \bullet P_{\{A,B,A\}}\ (\bullet P_{\{B,N1,B\}} \\ &\quad \bullet S_{\{A,N1,\beta_C\}} + \bullet P_{\{A,N1,B\}}\ (\bullet S_{\{A,B,N1\}} + -\bullet S_{\{B,N1,\beta_C\}}))))\end{aligned}$$

Eliminating point β_C:

Eliminating point **N1**:

Eliminating point **M1**:

Eliminating point β_{M1}:

Eliminating point **O**:

Eliminating point **M**:

Eliminating point **P**:

We eliminate point P using construction:
$P\,A \parallel A\,B$ and $\frac{\overline{A\,P}}{\overline{A\,B}} = \frac{1}{2}$ with ndg. $\{A \neq B\}$

We obtain:

$$\frac{r_M\,((-4) + 7\,r_M + -4\,r_M^{\,2} + r_M^{\,3})}{4\,((-2) + r_M)^2}$$

Thus

$$\left(\frac{-1}{4}\right) + \frac{\bullet P_{\{A,M,A\}}}{\bullet P_{\{A,D,A\}}} = \frac{r_M\,((-4)+7\,r_M+-4\,r_M^{\,2}+r_M^{\,3})}{4\,((-2)+r_M)^2}$$

Step 2

Our initial problem becomes

(1)

$$\forall_M \left(r_M > 0 \wedge (-1) + r_M < 0 \Rightarrow \frac{(-36)+136\,r_M+-184\,r_M^{\,2}+100\,r_M^{\,3}+-25\,r_M^{\,4}}{100\,((-2)+r_M)^2} \leq 0 \bigwedge \frac{r_M\,((-4)+7\,r_M+-4\,r_M^{\,2}+r_M^{\,3})}{4\,((-2)+r_M)^2} \leq 0\right)$$

If there are square roots of squares in this expression we eliminate them adding the necessary conditions. After performing substitution $\{r_M \to u_1\}$ the problem becomes:

(2)

$$\forall_{u_1} \left(u_1 > 0 \wedge (-1) + u_1 < 0 \wedge u_1 \neq 0 \Rightarrow \frac{(-36)+136\,u_1+-184\,u_1^{\,2}+100\,u_1^{\,3}+-25\,u_1^{\,4}}{100\,((-2)+u_1)^2} \leq 0 \bigwedge \frac{u_1\,((-4)+7\,u_1+-4\,u_1^{\,2}+u_1^{\,3})}{4\,((-2)+u_1)^2} \leq 0\right)$$

Applying the CAD algorithm to this expression we obtain that the proposition is true.

□

- End of inserted notebook