A NOTE ON THE NUMBER OF RHOMBUS TILINGS OF A SYMMETRIC HEXAGON AND SYMBOLIC SUMMATION

CARSTEN SCHNEIDER

ABSTRACT. In order to count the number of certain rhombus tilings of a symmetric hexagon, in [FK00] a closed form evaluation of a definite multisum expression has been derived with the help of highly involved transformations of special functions. In this article we will illustrate, how this identity can be found algorithmically in a straightforward manner by using the summation package *Sigma* whose algebraic platform is based on difference field theory.

1. INTRODUCTION

In [FK00] M. Fulmek and C. Krattenthaler count the number of rhombus tilings of a symmetric hexagon with side lengths N, M, N, N, M, N, with N and M having the same parity, which contain a particular rhombus next to the center of the hexagon. For further explanations and graphical illustrations we refer to [FK00].

In [FK98] this has been achieved for an arbitrary but fixed rhombus on the symmetry axis which cuts through the sides of length M. Moreover, in [FK00] the authors try to count the number of rhombus tilings where a fixed rhombus is on the "other" symmetry axis which runs in parallel to the sides of length M. But in contrast to [FK98], they failed to count the total number for such an arbitrary rhombus in [FK00], but only for rhombi which are close to the center. More precisely, they are able to solve this problem for a rhombus which is by one (Theorems 1 and 2), two (Theorems 3 and 4) or three (Theorems 5 and 6) "half units" off the center. Note that the computations of the proofs are increasingly more involved, when the rhombus is shifted farther away from the center.

The basic idea to count these numbers of rhombus tilings is based on an approach of Helfgott and Gessel [HG99]. More precisely, the proposed technique enables to write the number of rhombus tilings of a hexagon in form of a determinant where a particular rhombus may lie arbitrarily on the symmetry axis. In other words the number of rhombus tilings with a specific rhombus on the symmetry axis (Theorems 1 to 6) has been obtained by evaluating the determinant for various special cases. In particular in the proofs of Theorem 5 and 6 this task is getting more and more involved. To be more precise, one has to evaluate a certain Hankel matrix of Bernoulli numbers. In order to compute this evaluation, the authors combine results from continuous Hahn polynomials and certain orthogonal polynomials. Moreover within this evaluation they have to find a closed form evaluation $S_n^{(1)} + \frac{n!}{(n+3)!} S_n^{(2)}$ for the two sums

$$S_n^{(1)} := \sum_{k=0}^{n-1} \frac{(-1)^{n+k}(n+k+4)! \sum_{j=0}^{k} \frac{1}{j+1}}{(k+2)!(k+3)!(n-k-1)!}, \quad S_n^{(2)} := \sum_{k=1}^{n-1} \frac{(-1)^k (n+k+4)!(1-(-1)^n(n+2))}{(k+1)(k+2)!^2(n-k-1)!}.$$

In order to achieve this, the authors derive for these sums closed form evaluations, namely,

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$$S_n^{(1)} = -\frac{1+(-1)^n(n^2+3n+3)}{n+2} - (1-(-1)^n(n+2))\sum_{k=0}^{n}\frac{2}{k+1} \quad \text{and} \tag{1}$$

$$S_n^{(2)} = (1-(-1)^n(n+2))\left(\sum_{k=0}^{n}\frac{2}{k+1} - \frac{(-1)^n+2n^3+11n^2+19n+11}{(n+1)(n+2)(n+3)}\right) \tag{2}$$

which finally delivers the identity

$$S_n^{(1)} + \frac{n!}{(n+3)!}S_n^{(2)} = (-1)^n(n+2)-2 \tag{3}$$

for all $n \geq 0$. We want to emphasize that the proof of this identity, given in [FK00, Lemma 26], requires about four pages involving highly complicated transformations of special functions.

It is worthwhile to mention that the authors indicate another way of proving the identity (3) purely algorithmically. The basic strategy in this alternative proof consists of first computing two linear recurrence relations for the two sum expressions $S_n^{(1)}$ and $S_n^{(2)}$ by using the summation package *Sigma* [Sch00] and the Zeilberger Algorithm [Zei90, PWZ96], and afterwards adding the two recurrences with the aid of [SZ94] or [Mal96] to one recurrence of order 10 which contains $S_n^{(1)} + \frac{n!}{(n+3)!}S_n^{(2)}$ as a solution. Then it is a simple task to check that the right hand side in identity (3) is also a solution of this combined recurrence. The fact that both sides in the equation (3) agree with the first 10 initial values shows the correctness of (3). To this approach the authors remark:

> However, in the present implementation, these algorithms are not able to find the explicit evaluations, in terms of harmonic numbers, of the sums...

In this work we will show how these explicit evaluations of the sums in (1) and (2) can be obtained with purely computer algebra methods based on the summation package *Sigma* [Sch00]. In particular we will illustrate how the proof of [FK00, Lemma 26] in combination with *Sigma* becomes straightforward and hence feasible without advanced knowledge of hypergeometric functions and its transformations.

2. Symbolic Summation in Difference Fields

Sigma [Sch00, Sch01] is a summation package implemented in the computer algebra system Mathematica that enables one to find closed form evaluations of a very general class of nested multisum identities. Nested sum expressions, including summations involving harmonic numbers and hypergeometric terms, are translated in a natural way into the corresponding difference field setting of $\Pi\Sigma$-fields [Kar81, Kar85, Sch01], and, by using a very general algebraic machinery [Bro00, Sch02c, Sch02b, Sch02a], the problem is solved there. In the *Sigma* package a streamlined version of Karr's indefinite summation algorithm [Kar81] is implemented which is in a sense the summation counterpart to Risch's algorithm [Ris69, Ris70] for indefinite integration. Moreover the summation package *Sigma* contains several generalizations that enables one to deal also with definite multisum identities.

In the sequel we will find the closed form evaluation in (1) and (2) with the help of *Sigma*. We start a Mathematica session by loading the package

In[1]:= << **Sigma**`

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Sigma - A summation package by Carsten Schneider © RISC-Linz
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and setting up the summation problem $S_n^{(1)} =$ `mySum` as follows:

In[2]:= **mySum = SigmaSum[−SigmaHNumber[k]SigmaPower[−1, k]SigmaPower[−1, n] SigmaFactorial[n + k + 3]/(SigmaFactorial[k + 1]SigmaFactorial[k + 2] SigmaFactorial[n − k]), {k, 1, n}]**

Out[2]= $\sum_{k=1}^{n} -\frac{H_k\ (3+k+n)!\ (-1)^{k}\ (-1)^{n}}{(1+k)!\ (2+k)!\ (-k+n)!}$

Remark. The functions `SigmaSum` and `SigmaProduct` are used to define nested sums and products that can be formulated in the difference field setting. For this purpose there are also several other functions available, like `SigmaHNumber`, `SigmaBinomial` or `SigmaPower` to define harmonic numbers, binomials or powers in terms of sums and products which itself can be converted into the difference field setting. For instance, `SigmaHNumber[k]` produces the kth harmonic number H_k which alternatively could be described by `SigmaSum[1/i,{i,1,k}]`. □

Given this sum expression, we are able to compute a recurrence relation of order three by the *Sigma*-function call

In[3]:= **rec = GenerateRecurrence[mySum, RecOrder → 3]**

Out[3]= $\{(2+n)\ (4+n)\ (9+2\ n)\ \mathrm{SUM}[n] - (9+2\ n)\ (13+8\ n+n^2)\ \mathrm{SUM}[1+n] - (5+2\ n)\ (6+6\ n+n^2)\ \mathrm{SUM}[2+n] + (3+n)\ (5+n)\ (5+2\ n)\ \mathrm{SUM}[3+n] == -2\ (5+2\ n)\ (7+2\ n)\ (9+2\ n)\ (-1)^{n}\}$

This means that SUM[n] $= S_n^{(1)}$ (=mySum) satisfies the output recurrence Out[3].

Remark. In order to compute such recurrences, Zeilberger's creative telescoping [Zei90] has been extended from hypergeometric expressions to terms in $\Pi\Sigma$-fields; for more information see [Sch01]. □

In the next step we try to find solutions of that recurrence. For this task, *Sigma* provides the following function call.

In[4]:= **SolveRecurrence[rec[[1]], SUM[n]]**

Out[4]= $\{\{0, 1\}, \{0, (2+n)\ (-1)^{n}\}\}$

Internally *Sigma* constructs the underlying difference ring $\mathbb{A} = \mathbb{Q}(n)[(-1)^n]$ given by the objects in the recurrence and afterwards tries to solve the recurrence formulated in this algebraic setting $\mathbb{A}$. In this case *Sigma* finds two linearly independent solutions of the homogeneous version of the recurrence, namely 1 and $(n+2)(-1)^n$.

Obviously, those solutions are not sufficient to describe the whole set of solutions of the given recurrence. Hence we try to extend the underlying difference ring, in order to find further solutions. In our concrete situation, *Sigma* is able to find two additional solutions, namely

$$A_n := \sum_{k=0}^{n}(2k+3)(-1)^k \sum_{i=1}^{k} \frac{(-1)^i}{i(i+2)} \quad \text{and} \quad B_n := 2\sum_{k=0}^{n}(2k+3)(-1)^k \sum_{i=1}^{k} \frac{i+1}{i(i+2)}$$

where A_n is one additional solution of the homogeneous version of the recurrence which is linearly independent from the other two solutions 1 and $(n+2)(-1)^n$, and where B_n is one particular solution of the inhomogeneous recurrence itself.

Remark. Such nested sum solutions are included in the class of d'Alembertian solutions which have been considered in [AP94]; further results can be found in [HS99, Sch01]. □

In this example *Sigma* succeeded completely since it was able to compute three linearly independent solutions of the homogeneous version of the recurrence and one particular solution of the inhomogeneous recurrence itself. Hence we could find a "closed form" of $S_n^{(1)}$ by taking a particular linear combination of the homogeneous solutions plus the inhomogeneous solution B_n that together match the initial values of $S_n^{(1)}$. But in this case the "closed form" is not simpler, more precisely not less nested, since the inhomogeneous solution B_n, is also a two-nested sum expression. The important observation is that the two-nested sum solutions A_n and B_n —and in general most of the d'Alembertian solutions of a given recurrence— fit exactly into the input class of *Sigma*'s indefinite summation algorithm. So we can simplify for instance B_n with *Sigma* as follows:

In[5]:= $\mathbf{B} = \sum_{\iota_1=0}^{\mathbf{n}} (\mathbf{3}+\mathbf{2}\ \iota_1)\ (-\mathbf{1})^{\iota_1}\ \sum_{\iota_2=1}^{\iota_1} \frac{\mathbf{1}+\iota_2}{\iota_2\ (\mathbf{2}+\iota_2)};$

In[6]:= $\mathbf{SigmaReduce}[\mathbf{B}]$

Out[6]= $\frac{-3\ (1+\mathtt{n})\ (2+\mathtt{n})+2\ \left(3+3\ \mathtt{n}+\mathtt{n}^2\right)\ (-1)^{\mathtt{n}}+4\ (1+\mathtt{n})\ (2+\mathtt{n})^2\ (-1)^{\mathtt{n}}\ \sum_{\iota_1=1}^{\mathtt{n}}\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}}{4\ (1+\mathtt{n})\ (2+\mathtt{n})}$

Similarly, we can simplify A_n to a single-nested sum expression, namely

$$A_n = \frac{2-(n+1)(n+2)(-1)^n+4(n+1)\sum_{i=1}^{k}\frac{i+1}{i(i+2)}}{4(n+1)},$$

by applying our indefinite summation algorithm. All the above steps, namely (a) constructing the two additional sum solutions for the recurrence `rec` and (b) the simplification of that nested sum solutions, are carried out by setting the option **NestedSumExt** $\to\infty$ in the following function call.

In[7]:= $\mathbf{recSol} = \mathbf{SolveRecurrence}[\mathbf{rec}[[\mathbf{1}]],\mathbf{SUM}[\mathbf{n}],\mathbf{NestedSumExt}\to\infty]$

Out[7]= $\Big\{\{0,1\},\{0,(2+\mathtt{n})\ (-1)^{\mathtt{n}}\},\Big\{0,\frac{\mathtt{n}-2\ (1+\mathtt{n})\ \sum_{\iota_1=1}^{\mathtt{n}}\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}}{2\ (1+\mathtt{n})}\Big\},$

$\Big\{1,\frac{\left(3+3\ \mathtt{n}+\mathtt{n}^2\right)\ (-1)^{\mathtt{n}}+2\ (1+\mathtt{n})\ (2+\mathtt{n})^2\ (-1)^{\mathtt{n}}\ \sum_{\iota_1=1}^{\mathtt{n}}\frac{1+\iota_1}{\iota_1\ (2+\iota_1)}}{(1+\mathtt{n})\ (2+\mathtt{n})}\Big\}\Big\}$

Note that these two additional sum solutions are equivalent to A_n and B_n up to specific linear combinations of the homogeneous solutions. Now looking closer at this result, from the partial fraction decomposition on the summand

$$\sum_{i=1}^{n}\frac{i+1}{i(i+2)} = \frac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{i}+\sum_{i=1}^{n}\frac{1}{i+2}\right)$$

one sees immediately that this sum can be expressed in terms of the harmonic numbers H_n.

This motivates to simplify Out[7] further by asking *Sigma* for a representation of the expression `recSol` in terms of H_n. This is carried out by the following command.

In[8]:= $\mathbf{recSol} = \mathbf{SigmaReduce}\big[\mathbf{recSol},\mathbf{n},\mathbf{Tower}\to\{\mathbf{H_k}\}\big]$

Out[8]= $\Big\{\{0,1\},\{0,(2+\mathtt{n})(-1)^{\mathtt{n}}\},\Big\{0,\frac{\mathtt{n}(9+5\mathtt{n})-4(1+\mathtt{n})(2+\mathtt{n})\mathtt{H_n}}{4(1+\mathtt{n})(2+\mathtt{n})}\Big\},$

$\Big\{1,\frac{(6-4\mathtt{n}-9\mathtt{n}^2-3\mathtt{n}^3+4(1+\mathtt{n})(2+\mathtt{n})^2\mathtt{H_n})(-1)^{\mathtt{n}}}{2(1+\mathtt{n})(2+\mathtt{n})}\Big\}\Big\}$

Finally, the closed form of $S_n^{(1)}$ is the particular linear combination of the homogeneous solutions plus the inhomogeneous solution that matches the first three initial values of $S_n^{(1)}$.

In[9]:= $\mathbf{solution} = \mathbf{FindLinearCombination}[\mathbf{recSol},\mathbf{mySum},\mathbf{3}]$

Out[9]= $\frac{-5-3\ \mathtt{n}-2\ (1+\mathtt{n})\ (2+\mathtt{n})\ \mathtt{H_n}+\left(5+2\ \mathtt{n}-2\ \mathtt{n}^2-\mathtt{n}^3+2\ (1+\mathtt{n})\ (2+\mathtt{n})^2\ \mathtt{H_n}\right)\ (-1)^{\mathtt{n}}}{(1+\mathtt{n})\ (2+\mathtt{n})}$

In the same spirit we are able to find a closed form evaluation for the hypergeometric sum

$$T_n := \sum_{k=1}^{n-1} \frac{(-1)^k (n+k+4)!}{(k+1)(k+2)!^2 (n-k-1)!}$$

where $S_n^{(2)} = (1-(-1)^n(n+2))T_n$. More precisely, we first compute a recurrence either with *Sigma* or any other implementation that can deal with creative telescoping for definite hypergeometric sums [Zei90, PWZ96], like for instance [PS95]:

$$(n+2)(2n+5)T_{n+2} - 6(n+3)T_{n+1} - (n+4)(2n+7)T_n = 2(n+2)(n+4)(2n+5)(2n+7).$$

Afterwards we solve this recurrence by finding its d'Alembertian solutions [AP94, Sch01]. This leads to the two linearly independent solutions $(-1)^n$ and $(n+1)(n+2)(n+3)$ of the homogeneous version of the recurrence and the particular solution

$$C_n := \frac{1}{6}(12(n+1)(n+2)(n+3)H_n - n(13n^2 + 60n + 65)$$

of the inhomogeneous recurrence itself. Finally we look at the first two initial values of T_n and find the right linear combination of the homogeneous solutions plus the particular solution C_n which give us the closed form evaluation

$$T_n = 2(n+1)(n+2)(n+3)H_n - (-1)^n - 2n^3 - 9n^2 - 9n + 1,$$

or the identity

$$S_n^{(2)} = -(n+3)\left[2n^2+3n-1-(-1)^n(2n^3+7n^2+6n-1)+2H_n(n^2+3n+2)((-1)^n(n+2)-1)\right].$$

3. Conclusions

In this work we illustrated how the two closed form evaluations of $S_n^{(1)}$ and $S_n^{(2)}$ can be derived purely algorithmically with the help of the summation package *Sigma*. In particular the user is completely dispensed from working explicitly in difference fields or rings, the algebraic setting of this package; instead one can work conveniently in terms of usual sum and product expressions.

Finally we want to suggest that the *Sigma* package might be helpful for further development following the ideas in [FK00]. First recall that the identity (3) is needed in order to prove Theorems 5 and 6 in [FK00] which give the exact number of rhombus tilings where a fixed rhombus is at most three "half units" off the center. In the "Concluding comments and open problems" section of [FK00] the authors raise the question wether one could compute the number of rhombus tilings where the fixed rhombus tiling is even farther away from the center. Here they refer to the already quite involved proof of identity (3) and estimate that even more complicated identities might occur, if the fixed rhombus is shifted further. Similarly to identity (3) the summation package *Sigma* might assist in finding such closed form evaluations.

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(C. Schneider) Johann Radon Institute for Computational and Applied Mathematics (RICAM), Johannes Kepler University Linz, A–4040 Linz, Austria

E-mail address: Carsten.Schneider@oeaw.ac.at