Explaining Schwarz's LODEF algorithm with examples

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1 Introduction

To every linear homogeneous ordinary differential equation

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0,$$

one can associate the linear operator:

$$L(D)[y] = \left(D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n\right)[y].$$

Here, D^i is another notation for the i-th derivative of y, and the coefficients a_i belong to a differential field K. Such an operator is called a linear homogeneous differential operator either over K, or with coefficients in K. Linear homogeneous differential operators over K form a ring under the usual addition of operators and composition as multiplication.

Problem:

Decompose L as a product of operators of lower degree in D. We allow expressions algebraic over K.

Methods:

Rational solutions:

Rational solutions are functions in K. To decompose the operators of this class, Liouville [1833a,b] already gave an algorithm, but only when K is a rational function field. More general versions have been presented by Singer [1991] and Bronstein [1992]. To solve scalar equations, there are implementations by Abramov and Bronstein in Maple

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and Bernina. To solve matrix equations, there is an implementation by M. Barkatou in Isolde. Both cases are restricted to the field $\overline{\mathbb{Q}}(z)$ of rational functions in z with coefficients in the algebraic closure of $\mathbb{Q}$. In particular, there are problems with parameters.

Liouvillian solutions:

An elementary function of a variable x is a function that can be obtained from the rational functions in x by repeatedly adjoining a finite number of nested logarithms, exponentials, and algebraic numbers and functions. Since $\sqrt{-1}$ is elementary, the trigonometric functions and their inverses are also elementary, as well as all the usual functions of calculus.

A Liouvillian function of x is a function that can be obtained like a elementary function except that arbitrary antiderivatives are allowed instead of logarithms in their construction. For example,

$$\sin\left(x + \tan(x^3 - \sqrt{x^3 - x + 1})\right)$$

is elementary, while the error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int e^{-x^2} dx$$

is Liouvillian but not elementary, and the Bessel functions defined as the fundamental solutions of

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\epsilon x^2 - v^2) = 0$$

where $\epsilon = \pm 1$, are not Liouvillian (and therefore not elementary).

Liouvillian solutions are functions that can be constructed by successively substituting nested algebraic functions, integrals and exponentials of integrals of such functions. An example of such a construction is

$$x \xrightarrow{\sqrt{x}} \sqrt{x} \xrightarrow{e^{\int}} \exp\left[\int \sqrt{x}\right].$$

This class of functions includes also functions like the trigonometric functions and logarithms. In principle, it consists of nearly all closed-form functions. An important exception are the Bessel functions and other special functions.

For order two equations the method is given by Kovacic [1986], with small improvements by Weil [1994]. For order three, the methods are given by Singer and Ulmer [1993]. There are algorithms for certain steps like:

- Computing invariants: Algorithms are given by van Hoeij and Weil [1997], these algorithms are not implemented. Presently, Bronstein is working on an implementation.
- Computing symmetric powers of an order two equation: There are algorithms by Bronstein, Mulders, and Weil [1997]. An implementation of the last three authors and van Hoeij exists in Maple. In particular, Weil has simplified the code of van Hoeij. In Bernina there are implementations by Bronstein.
- Factoring an invariant: There is an algorithm, which computes one semi-invariant that factors into linear forms and computes the minimal polynomial of an algebraic solution of the Riccati equation by van Hoeij, Ragot, Ulmer, and Weil [1999].

Algebraic solutions:

Algebraic solutions are functions that lie in an algebraic extension of K; i.e., they satisfy an irreducible polynomial over K. An example for this class is $\sqrt[3]{1-\sqrt{3}}$.

Many renowned mathematicians like Pépin [1881], Fuchs [1875, 1878], Klein and Jordan [1878] were searching for an algorithm for algebraic solutions. Today, there exists an algorithm by Singer [1979] with some improvements in Singer and Ulmer [1993], but it is far from being satisfactory because it is very expensive; in this case one needs to substitute a minimal polynomial decomposed into invariants in the differential equation. Another method from Fakler [1997] combines Liouvillian solutions with the algebraic case of Risch's algorithm (see Bronstein [1997]), it has not been implemented.

Factorization:

- The eigenring method: The eigenring of a differential equation $L = 0$ is the finite dimensional C-Algebra of all the endomorphisms of the equation, where C is the subfield of constants of K. This eigenring is the set of all rational solutions of other differential equations associated to L. If this eigenring is not too trivial, then some factorizations of L can be deduced from it. The method was introduced by Singer [1996] and has been improved by van Hoeij [1997a].
- First order factors of exponential solutions: An exponential solution is a function whose logarithmic derivative lie in K. If y is a solution, then $\frac{y'}{y}$ is its logarithmic derivative. An example is $y = exp(x^3)$ since $\frac{y'}{y} = 3x^2$ lies in K. Exponential functions form the most important subclass of Liouvillian functions, without an algorithm for them, it is not possible to give an algorithm for

Liouvillian solutions. Fortunately, there are algorithms for exponential solutions. The very first one is from Beke [1894]. For the formal case, van Hoeij [1997b] has implemented classical methods. Factorizations over $\overline{\mathbb{Q}}(z)$ are based on the formal case.

- Full factorizations: Recent work of Tsarev [1996] combined the local formal factorization of the previous case with the classical work of Beke [1894].

Dividing functions into one of these classes is not always unique; e.g., for the function $y = \sqrt{x}$, it is possible to attach it either to the algebraic functions or to the exponential, since $\frac{y'}{y} = \frac{1}{2x}$, and $\frac{1}{2x} \in K$. Furthermore, it can be hard to decide for a given function whether it is Liouvillian and how one could find the simplest construction. A combination of the above methods is used in the implementation by van Hoeij in Maple.

This work is the result of the study of the first two modern algorithms for factorization of linear differential operators, namely the LODEF algorithm due to Schwarz [1989], and the Beke-Bronstein algorithm due to Bronstein [1994]. We are implementing the algorithm by Bronstein, the details of that work will be reported in the next technical report.

In this paper, we explain the LODEF algorithm with the help of examples. First, a simple example shows that additional steps, that are not explicitly mentioned in the algorithm, are necessary to get a solution. Next we give step by step solution of the example mentioned in the original paper, by the application of the algorithm. This is useful in the study of different algorithms for solving linear differential equations. We have explained the important basic notions with the help of examples, for further details see the books by Magid [1994] and van der Put and Singer [2002].

2 Preliminaries

Definition 2.1. A derivation D on the field K is an additive map $D : K \to K$ satisfying:

$$D(ab) = D(a)b + aD(b), \text{ for all } a, b \in K.$$

In this case we say that K is a differential field. The constants are the members of $C = \mathrm{Ker}(D)$, where $\mathrm{Ker}(D)$ is the kernel of map D, a subfield of K.

To every linear homogeneous ordinary differential equation,

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0 \tag{1}$$

where $a_i \in K$ for $i = 1, \ldots, n$, one can associate the linear differential operator:

$$L(D)[y] = \left(D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n\right)[y],$$

where, D^i denotes the i-th derivative of y, and the coefficients a_i belong to a differential field K. Such an operator is called a linear homogeneous differential operator over K. The set $\mathbb{D} = K[D]$ of linear homogeneous differential operators over K is a ring under the usual addition of operators and composition as multiplication.

Let us assume that K is a characteristic zero field, and C is algebraically closed.

Definition 2.2. A set of n solutions $\{y_1, \ldots, y_n\}$ of $L(y) = 0$, linearly independent over C, is a **fundamental system** of solutions of $L(y) = 0$.

Definition 2.3. Let $\{y_1, \ldots, y_n\}$ be a fundamental system of solutions of $L(y) = 0$. Let

$$\Delta = \begin{vmatrix} y & y_1 & y_2 & \dots & y_n \\ y' & y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots & \dots \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} \end{vmatrix}, \text{ and}$$

$$\omega_{S_i} = \frac{\partial \Delta}{\partial y^{(i)}}, \text{ for } i = 0, \ldots, n.$$

If we consider Δ as a function of the variables $y, y', y'', \ldots, y^{(n)}$, i.e. $\Delta = \Delta(y, y', y'', \ldots, y^{(n)})$, then the ω_{S_i} are the partial derivatives of Δ with respect to $y^{(i)}$ for $i = 0, \ldots, n$. In this case $(-1)^n \omega_{S_n}$ is the Wronskian ω of $\{y_1, \ldots, y_n\}$, and $\omega' = (-1)^{n+1}\omega_{S_{n-1}}$. By the Cramer's rule the differential equation $L(y) = 0$ is uniquely determined by:

$$L(y) = \frac{\Delta}{\omega_{S_n}} = y^{(n)} + \frac{\omega_{S_{n-1}}}{\omega_{S_n}} y^{(n-1)} + \frac{\omega_{S_{n-2}}}{\omega_{S_n}} y^{(n-2)} + \cdots + \frac{\omega_{S_0}}{\omega_{S_n}} y = 0$$

or

$$a_i = \frac{\omega_{S_{n-i}}}{\omega_{S_n}}, \quad \text{for } i = 1, \ldots, n. \tag{2}$$

This means that the logarithmic derivative of each nonzero a_i can be expressed as a quotient of two ω_{S_i}. In particular,

$$a_1 = -\frac{\omega'}{\omega_{S_n}}, \text{ or } \omega' + a_1 \omega_{S_n} = 0 \tag{3}$$

which is known as **Liouville's relation.**

Example 1. Let us consider a second order linear homogeneous differential equation

$$L_2(y) = y'' + a_1 y' + a_2 y = 0, \text{ with } a_1, a_2 \in \mathbb{Q}[x],$$

and let $\{y_1, y_2\}$ be a fundamental system of solutions of $L_2(y) = 0$. Then for this case we have

$$\Delta = \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix}.$$

We compute ω_{S_i} for $i = 0$, 1, and 2:

- for $i = 0$

$$\omega_{S_0} = \frac{\partial \Delta}{\partial y^{(0)}} = \frac{\partial}{\partial y} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix}$$

$$= \frac{\partial}{\partial y}\left(y \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} - y' \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} + y'' \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right) = \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

$$= y_1' y_2'' - y_1'' y_2',$$

  that is ,

$$\omega_{S_0} = y_1' y_2'' - y_1'' y_2'.$$

- for $i = 1$

$$\omega_{S_1} = \frac{\partial \Delta}{\partial y^{(1)}} = \frac{\partial}{\partial y'} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix}$$

$$= \frac{\partial}{\partial y'}\left(y \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} - y' \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} + y'' \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right) = - \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix}$$

$$= -(y_1 y_2'' - y_1'' y_2),$$

  that is,

$$\omega_{S_1} = y_1'' y_2 - y_1 y_2''.$$

- for $i = 2$

$$\omega_{S_2} = \frac{\partial \Delta}{\partial y^{(2)}} = \frac{\partial}{\partial y''} \begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix}$$

$$= \frac{\partial}{\partial y''}\left(y \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} - y' \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} + y'' \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= y_1 y_2' - y_1' y_2 = \omega,$$

  that is,

$$\omega_{S_2} = \omega.$$

On the other hand,

$$\omega' = \frac{d\omega}{dx} = \frac{d(y_1y_2' - y_1'y_2)}{dx} = y_1y_2'' - y_1''y_2 = -(y_1''y_2 - y_1y_2'') = -\omega_{S_1}.$$

Now, substituting y_1 and y_2 in the original equation we get the following system of two equations in the unknowns a_1 and a_2,

$$\begin{aligned} y_1'' + a_1y_1' + a_2y_1 &= 0 \\ y_2'' + a_1y_2' + a_2y_2 &= 0. \end{aligned}$$

Finally, by the Cramer's rule we get,

$$\begin{aligned} a_1 &= \frac{y_1''y_2 - y_1y_2''}{y_1y_2' - y_1'y_2} = \frac{\omega_{S_1}}{\omega_{S_2}}, \\ a_2 &= \frac{y_1'y_2'' - y_1''y_2'}{y_1y_2' - y_1'y_2} = \frac{\omega_{S_0}}{\omega_{S_2}}. \end{aligned}$$

Example 2. Let us consider a third order linear homogeneous differential equation

$$L_3(y) = y''' + a_1y'' + a_2y' + a_3y = 0, \text{ with } a_1, a_2, a_3 \in \mathbb{Q}[x], \tag{4}$$

and let $\{y_1, y_2, y_3\}$ be a fundamental system of solutions of $L_3(y) = 0$. Then for this case we have

$$\Delta = \begin{vmatrix} y & y_1 & y_2 & y_3 \\ y' & y_1' & y_2' & y_3' \\ y'' & y_1'' & y_2'' & y_3'' \\ y''' & y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

We compute ω_{S_i} for $i = 0, 1, 2$ and 3 :

- for $i = 0$

$$\omega_{S_0} = \frac{\partial \Delta}{\partial y^{(0)}} = \frac{\partial}{\partial y} \begin{vmatrix} y & y_1 & y_2 & y_3 \\ y' & y_1' & y_2' & y_3' \\ y'' & y_1'' & y_2'' & y_3'' \\ y''' & y_1''' & y_2''' & y_3''' \end{vmatrix}$$

$$= \frac{\partial}{\partial y}\left(y \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} - y' \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} + y'' \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} - y''' \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \right)$$

$$= \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix},$$

that is,

$$\omega_{S_0} = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

- for $i = 1$

$$\omega_{S_1} = \frac{\partial \Delta}{\partial y^{(1)}} = \frac{\partial}{\partial y'} \begin{vmatrix} y & y_1 & y_2 & y_3 \\ y' & y_1' & y_2' & y_3' \\ y'' & y_1'' & y_2'' & y_3'' \\ y''' & y_1''' & y_2''' & y_3''' \end{vmatrix} = -\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix},$$

that is,

$$\omega_{S_1} = -\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

- for $i = 2$

$$\omega_{S_2} = \frac{\partial \Delta}{\partial y^{(2)}} = \frac{\partial}{\partial y''} \begin{vmatrix} y & y_1 & y_2 & y_3 \\ y' & y_1' & y_2' & y_3' \\ y'' & y_1'' & y_2'' & y_3'' \\ y''' & y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix},$$

that is,

$$\omega_{S_2} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

- for $i = 3$

$$\omega_{S_3} = \frac{\partial \Delta}{\partial y^{(3)}} = \frac{\partial}{\partial y'''} \begin{vmatrix} y & y_1 & y_2 & y_3 \\ y' & y_1' & y_2' & y_3' \\ y'' & y_1'' & y_2'' & y_3'' \\ y''' & y_1''' & y_2''' & y_3''' \end{vmatrix} = -\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -\omega,$$

that is,

$$\omega_{S_3} = -\omega.$$

On the other hand,

$$\omega' = \frac{d\omega}{dx} = \frac{d}{dx} \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \omega_{S_2},$$

that is,

$$\omega' = \omega_{S_2}.$$

Now, substituting y_1, y_2 and y_3 in the original equation we get the following system of three equations in the unknowns a_1, a_2 and a_3,

$$\begin{aligned} y_1''' + a_1 y_1'' + a_2 y_1' + a_3 y_1 = 0 \\ y_2''' + a_1 y_2'' + a_2 y_2' + a_3 y_2 = 0 \\ y_3''' + a_1 y_3'' + a_2 y_3' + a_3 y_3 = 0. \end{aligned}$$

Finally, by the Cramer's rule we get,

$$
\begin{aligned}
a_1 &= \frac{y_1 y_2' y_3''' - y_1 y_2''' y_3' - y_1' y_2 y_3''' + y_1' y_2''' y_3 + y_1''' y_2 y_3' - y_1''' y_2' y_3}{-y_1 y_2' y_3'' + y_1 y_2'' y_3' + y_1' y_2 y_3'' - y_1' y_2'' y_3 - y_1'' y_2 y_3' + y_1'' y_2' y_3} &= \frac{\omega_{S_2}}{\omega_{S_3}}, \\
a_2 &= \frac{-y_1 y_2'' y_3''' + y_1 y_2''' y_3'' + y_1'' y_2 y_3''' - y_1'' y_2''' y_3 - y_1''' y_2 y_3'' + y_1''' y_2'' y_3}{-y_1 y_2' y_3'' + y_1 y_2'' y_3' + y_1' y_2 y_3'' - y_1' y_2'' y_3 - y_1'' y_2 y_3' + y_1'' y_2' y_3} &= \frac{\omega_{S_1}}{\omega_{S_3}}, \\
a_3 &= \frac{y_1' y_2'' y_3''' - y_1' y_2''' y_3'' - y_1'' y_2' y_3''' + y_1'' y_2''' y_3'' + y_1''' y_2' y_3'' - y_1''' y_2'' y_3'}{-y_1 y_2' y_3'' + y_1 y_2'' y_3' + y_1' y_2 y_3'' - y_1' y_2'' y_3 - y_1'' y_2 y_3' + y_1'' y_2' y_3} &= \frac{\omega_{S_0}}{\omega_{S_3}}.
\end{aligned}
$$

3 The LODEF Algorithm

Definition 3.1. Consider the matrix

$$
\begin{pmatrix}
y_1 & y_2 & \dots & y_n \\
y_1' & y_2' & \dots & y_n' \\
\dots & \dots & \dots & \dots \\
y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)}
\end{pmatrix} \tag{5}
$$

There are $\binom{n}{m}$ square matrices, of size $m \times m$, that may be formed out of its rows for any fixed value of $m < n$. Let the determinants, which are formed out of the $m \times m$ sub-matrices of (5), be denoted by z_ν for $\nu = 1, \dots, \binom{n}{m}$.

The set $\{z_\nu\}$ may be considered as a set of new functions, which is closed under differentiation if the original differential equation is used to substitute derivatives of order higher than $n-1$. By suitable differentiations and elimination for each of these functions, an $\binom{n}{m}$-th order linear differential equation may be obtained. These equations are called **associated equations** for the original one.

Procedure. Schwarz [1989]: The full range for the index ν is subdivided into $n - m + 1$ subintervals $I^{(k)}$ which are defined by

$$
I^{(k)} = \left\{ \nu \,\middle|\, \binom{k}{m} + 1 \le \nu \le \binom{k+1}{m},\ m - 1 \le k \le n - 1 \right\}
$$

An index ν belong to $I^{(k)}$ if in the corresponding z_ν the highest derivatives of $y_1, y_2, \dots, y_n$, which are involved are exactly of order k. Each z_ν' may be expressed as a linear homogeneous function of the z_ν, where $\nu = 1, \dots, \binom{n}{m}$, by using the original differential Equation (1) for substituting derivatives of order higher than $n-1$. As we enumerate with respect to z, these relations may be written in the form

$$
z_\nu' = \sum_{\nu \in I^{(k)} \cup I^{(k+1)}} a_{\nu\mu} z_\mu,
$$

where $a_{\nu\mu}$ is integer for $\nu \in I^{(k)}$, $m-1 \leq k \leq n-2$, and

$$z'_\nu = \sum_{\mu=1}^{\binom{n}{m}} a_{\nu\mu} z_\mu,$$

where $a_{\nu\mu}$ is linear homogeneous in $a_1, \ldots, a_n$ for $\nu \in I^{(n-1)}$. The associated equations are generalizations of Liouville's relations.

Example 3. For $n = 2$ we have the equation

$$y'' + a_1 y' + a_2 y = 0,$$

where $m = 1$ and $k = 1$, and

$$I^{(1)} = \left\{ \nu \,\middle|\, \binom{1}{1} + 1 \leq \nu \leq \binom{1+1}{1} \right\} = \{2\}.$$

There are two first-order determinants

$$z_1 = |y_1| = y_1, \;\; z_2 = |y'_1| = y'_1$$

from which we get the expressions

$$z'_1 = z_2, \;\; z''_1 + a_1 z'_1 + a_2 z_1 = 0.$$

After some operations we obtain the associated equation in this case

$$a_2 z''_2 + (a_1 a_2 - a'_2) z'_2 + (a'_1 a_2 - a_1 a'_2 + a_2^2) z_2 = 0.$$

Example 4. For $n = 3$ we have the equation

$$y''' + a_1 y'' + a_2 y' + a_3 y = 0,$$

where $m = 1, 2$.

If $m = 1$ then $k = 1$, and

$$I^{(1)} = \left\{ \nu \,\middle|\, \binom{1}{1} + 1 \leq \nu \leq \binom{1+1}{1} \right\} = \{2\}.$$

If $m = 2$ then $k = 1$ or 2, and

- for $k = 1$,

$$I^{(1)} = \left\{ \nu \,\middle|\, \binom{1}{2} + 1 \leq \nu \leq \binom{1+1}{2} \right\} = \{1\},$$

- for $k = 2$,

$$I^{(2)} = \left\{ \nu \,\middle|\, \binom{2}{2} + 1 \leq \nu \leq \binom{2+1}{2} \right\} = \{2,\ 3\}.$$

There are three second-order determinants,

$$z_1 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},\ z_2 = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix},\ z_3 = \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

From these expressions we get,

$$z_1' = z_2,\ z_2' = -q_2 z_1 - q_1 z_2 + z_3\ z_3' = -q_3 z_1 - q_1 z_3.$$

Therefore, the third order equations for z_1 and z_2 are

$$z_1''' + 2a_1 z_1'' + (a_1^2 + a_2 + a_1')z_1' + (a_1 a_2 - a_3 + a_2')z_1 = 0 \tag{6}$$

$$\begin{aligned} &a_3^2 z_2''' + 2a_3(a_1 a_3 - a_3')z_2'' + (a_1^2 a_3^2 + a_2 a_3^2 + 2a_1' a_3^2 - 3a_3' a_1 a_3 + 2a_3'^2 - a_3'' a_3)z_2 \\ &+ (a_1 a_2 a_3^2 - a_3^3 + a_1' a_1 a_3^2 - a_3' a_1^2 a_3 - 2a_1' a_3' a_3 + 2a_3'^2 - a_3'' a_1 a_3 + a_1'' a_3^2)z_2 = 0 \end{aligned} \tag{7}$$

Proposition 1. *Kreider, Kuller, Ostberg, and Perkins [1971, pg.122]*
The change of variable $u = \frac{y'}{y}$ *reduces the second order linear homogeneous differential equation:*

$$y'' + a_1 y' + a_2 y = 0, \tag{8}$$

to the Riccati equation:

$$u' + u^2 + a_1 u + a_2 = 0. \tag{9}$$

Consequently, solving Equation (8) is equivalent to solving the simultaneous system of first order equations:

$$y' = uy,\ u' = -u^2 - a_1 u - a_2$$

Equation (9) is called the Riccati equation associated to Equation (8).

Proof. By derivation and substitution we obtain the result in both cases. □

Definition 3.2. Using the change of variable $y' = uy$ for $n \geq 1$ we have

$$y'' = (u^2 + u)y,$$

$$y^{(3)} = (u^3 + 3uu' + u'')y$$

$$\cdots$$

$$y^{(n)} = (u^n + \ \cdots \ + nu^{(n-2)}u + u^{(n-1)})y$$

In particular for $n = 3$ we obtain

$$u'' + (3u + a_1)u' + u^3 + a_1u^2 + a_2u + a_3 = 0,$$

which is a nonlinear differential equation for the unknown function u. These equations may be considered as **generalized Riccati equations.**

Algorithm 1. Algorithm LODEF Schwarz [1989]:

Input: A linear differential equation

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_{n-1}y' + a_ny = 0 \tag{10}$$

with rational coefficients $a_i \in \mathbb{Z}(x)$ for all $i = 1, \ldots, n$.

Output: A right factor

$$y^{(m)} + r_1y^{(m-1)} + r_2y^{(m-2)} + \cdots + r_{m-1}y' + r_my = 0$$

of order m with $m < n$ with $r_j \in \mathbb{Z}(x)$ for all $j = 1, \ldots, m$. If no genuine factor exists, the input equation is returned unchanged.

$m := 0$, $k := 1$

1. $m := m + 1$. If $m = n$ return the input equation.
2. Determine the associated equations.
3. Determine a solution of the equation for r_1 found in step 2 with rational logarithmic derivative and determine r_1 from it. If none exists, go to step 1.
4. $k := k + 1$. If $k > m$ go to step 7.
5. Determine the equation for r_k.
6. Find rational solutions of the equation determined in step 5 and determine r_k from it. If none exists, go to step 1, else go to step 4.
7. From the coefficients r_k, construct a factor of left side of Equation (10) and return.

We can summarize this algorithm in the following steps:

Generate the associated equations for $\omega_{S_0}, \ldots, \omega_{S_m}$. (Step 2)

Look for all the nonzero solutions of the equation for ω_{S_m} whose logarithmic derivatives are in K. (Steps 3 – 6)

Substitute $\omega_{S_{m-i}}$ by $(-1)^i \omega_{S_m} r_i$ in all the other associated equations and look for all the solutions in K. (Step 7)

The above yields all the possible candidates for right-factors of order m, and trial divisions either determine an actual factor, or prove that there is no such factor.

Example 5. Let us consider the equation

$$L_1(y) = y'' + (x-1)y' - xy = 0,$$

where $a_1 = x - 1, a_2 = -x, a_1' = 1,$ and $a_2' = -1$.

In this case $n = 2$, $m = 1$, and the corresponding associated equation is

$$a_2 z_2'' + (a_1 a_2 - a_2') z_2' + (a_1' a_2 - a_1 a_2' + a_2^2) z_2 = 0.$$

After substituting the values of a_1, a_2, a_1' and a_2' in the above equation, we get

$$z_2'' + \frac{x^2 - x - 1}{x} z_2' + \frac{1 - x^2}{x} z_2 = 0.$$

However, in this case we do not have an equation for z_1, we use the original equation to find the corresponding associated Riccati equation, and it is

$$u' + u^2 + a_1 u + a_2 = 0 \text{ or } u' + u^2 + (x-1)u + (-x) = 0,$$

which has solution

$$u = 1 \Leftrightarrow \frac{y'}{y} = 1 \Leftrightarrow \frac{\omega'}{\omega_{S_2}} = 1.$$

By the **Liouville's relation**, we obtain $r_1 = -1$. Therefore,

$$y' - y = D(y) - y = (D - 1)(y).$$

This implies that $(D-1)$ is a right factor of the equation. Now, in order to find a left factor in this case we subtract the derivative of the right factor from the original equation:

$$(y'' + xy' - y' - xy) - (y' - y)' = xy' - xy = x[D(y) - y] = x(D-1)(y).$$

Then, $D + x$ is the left factor. In conclusion,

$$L_1(y) = [D^2 + (x-1)D - x](y) = (D + x)(D - 1)(y).$$

Remark 3.3. To solve the equation in Example 5 using Schwarz's algorithm, we need to find the associated Riccati equation. In order to find the associated Riccati equation, we use the original equation. We cannot get the associated Riccati equation using the method given by the algorithm. In other words, this method suffers from the problem of the actual generation of the associated equations.

Example 6. Let us consider the equation

$$L_2(y) = y''' + \frac{x-1}{x}y'' + \frac{x^2-2}{x}y' + \frac{2}{x^2}y = 0,$$

where $a_1 = \frac{x-1}{x}, a_2 = \frac{x^2-2}{x}, a_3 = \frac{2}{x^2}$. Differentiating a_1, a_2 and a_3, we get $a_1' = \frac{1}{x^2}, a_2' = \frac{x^2+2}{x^2}, a_3' = \frac{-4}{x^3}, a_1'' = \frac{-2}{x^3}, a_2'' = \frac{-4}{x^3}, a_3'' = \frac{12}{x^4}$.

In this case $n = 3$, and we can take $m = 2$. By Equation (6), the corresponding associated equations for z_1 is:

$$z_1''' + 2\left(\frac{x-1}{x}\right)z_1'' + \left(\frac{x^3+x^2-4x+2}{x^2}\right)z_1' + \left(\frac{x^3-2x+2}{x^2}\right)z_1 = 0. \quad (11)$$

By Equation (7) the corresponding associated equation for z_2 is:

$$z_2''' + \frac{2(x+1)}{x}z_2'' + \frac{(x^3+x^2+2x-1)}{x^2}z_2' + \frac{(x^4-x^3-x+1)}{x^3}z_2 = 0. \quad (12)$$

The generalized Riccati equation associated to Equation (11) is:

$$u'' + \left[3u + 2\left(\frac{x-1}{x}\right)\right]u' + u^3 + 2\left(\frac{x-1}{x}\right)u^2 + \left(\frac{x^3+x^2-4x+2}{x^2}\right)u + \left(\frac{x^3-2x+2}{x^2}\right) = 0,$$

which has solution

$$u = 1 \Leftrightarrow \frac{y'}{y} = 1 \Leftrightarrow \frac{\omega'}{\omega_{S_3}} = 1.$$

By the **Liouville's relation**, we obtain $r_1 = -1$. To obtain the equation for r_2, substitute

$$z_2 = ve^{r_1} = ve^{-1}$$

in the associated Equation (12). We get

$$v''' - \frac{(x-2)}{x}v'' + \frac{(x^3-x-1)}{x^2}v' - \frac{(x^3-1)}{x^3}v = 0,$$

and its generalized Riccati associated equation is:

$$u'' + \left[3u - \frac{(x-2)}{x}\right]u' + u^3 - \frac{(x-2)}{x}u^2 + \frac{(x^3-x-1)}{x^2}u - \frac{(x^3-1)}{x^3} = 0.$$

The solution of the generalized Riccati equation is

$$v = \frac{x^2-1}{x}, \text{ and then } r_2 = \frac{x^2-1}{x}.$$

Therefore,

$$y'' + y' + \frac{(x^2-1)}{x}y = D^2(y) + D(y) + \frac{(x^2-1)}{x}y = \left[D^2 + D + \frac{(x^2-1)}{x}\right](y)$$

is a right factor of the equation. Now, in order to find a left factor in this case we subtract the derivative of the right factor from the original equation and rearrange the terms to obtain:

$$-\frac{1}{x}\left(y'' + y' + \frac{(x^2-1)}{x}y\right).$$

Then, $D - \frac{1}{x}$ is the left factor. We can factor $L_2(y)$, as:

$$L_2(y) = \left(D - \frac{1}{x}\right)\left[D^2 + D + \frac{(x^2-1)}{x}\right](y).$$

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