Solving Equations Involving Sequence Variables and Sequence Functions

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Abstract. Term equations involving individual and sequence variables, and individual and sequence function symbols are studied. Function symbols can have fixed or flexible arity. A new unification procedure for solving such equations is presented. Decidability of unification is proved. Completeness and almost minimality of the procedure is shown.

1 Introduction

We study term equations with sequence variables and sequence function symbols. A sequence variable can be instantiated by any finite sequence of terms, including the empty sequence. A sequence function abbreviates a finite sequence of functions all having the same argument lists[1]. An instance of such a function is $\mathsf{IntegerDivision(x,y)}$ that abbreviates the sequence $\mathsf{Quotient}(\mathsf{x}, \mathsf{y}), \mathsf{Remainder}(\mathsf{x}, \mathsf{y})$.

Bringing sequence functions in the language naturally allows Skolemization over sequence variables: Let x, y be individual variables, $\overline{x}$ be a sequence variable, and p be a flexible arity predicate symbol. Then $\forall x \forall y \exists \overline{x}.p(x, y, \overline{x})$ Skolemizes to $\forall x \forall y.p(x, y, \overline{f}(x, y))$, where $\overline{f}$ is a binary Skolem sequence function symbol. Another example, $\forall \overline{y} \exists \overline{x}.p(\overline{y}, \overline{x})$, where $\overline{y}$ is a sequence variable, after Skolemization introduces a flexible arity sequence function symbol $\overline{g}$: $\forall \overline{y}.p(\overline{y}, \overline{g}(\overline{y}))$.

Equation solving with sequence variables play important role in several applications in automated reasoning, artificial intelligence, and programming. At the end of the paper we briefly review some of the works related to this topic.

We contribute to this area by introducing a new unification procedure for solving equations in the free theory with individual and sequence variables, and individual and sequence function symbols. Function symbols can have fixed or flexible arity. We prove that solvability of an equation is decidable in such a theory, and provide a unification procedure that enumerates an almost minimal complete set of solutions. The procedure terminates if the set is finite. This work is an extension and refinement of our previous results on unification with sequence variables [Kut02].

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1 Semantically, sequence functions can be interpreted as multi-valued functions.

The paper is organized as follows: Section 2 introduces basic notions. In Section 3 decidability is proved. Section 4 is about the unification type. In Section 5 unification procedure is defined and its properties are studied. In Section 6 two cases on which the procedure terminates are identified. Section 7 overviews some implementation details. Section 8 reviews some of the related work. Section 9 is Conclusion.

2 Preliminaries

We assume that the reader is familiar with the standard notions of unification theory [BS01].

We consider an alphabet consisting of the following pairwise disjoint sets of symbols:

1. A countably infinite set of individual variables $\mathcal{V}_{\mathrm{Ind}}$.
2. A countably infinite set of sequence variables $\mathcal{V}_{\mathrm{Seq}}$.
3. Punctuation: '(', ')' and ','.
4. A finite or countably infinite set of fixed arity individual function symbols $\mathcal{F}^{\mathrm{Fix}}_{\mathrm{Ind}}$.
5. A finite or countably infinite set of flexible arity individual function symbols $\mathcal{F}^{\mathrm{Flex}}_{\mathrm{Ind}}$.
6. A finite or countably infinite set of fixed arity sequence function symbols $\mathcal{F}^{\mathrm{Fix}}_{\mathrm{Seq}}$.
7. A finite or countably infinite set of flexible arity sequence function symbols $\mathcal{F}^{\mathrm{Flex}}_{\mathrm{Seq}}$.

We denote

- by $\mathcal{V}$, the set of variables $\mathcal{V}_{\mathrm{Ind}} \cup \mathcal{V}_{\mathrm{Seq}}$,
- by $\mathcal{F}_{\mathrm{Ind}}$, the set of individual function symbols $\mathcal{F}^{\mathrm{Fix}}_{\mathrm{Ind}} \cup \mathcal{F}^{\mathrm{Flex}}_{\mathrm{Ind}}$,
- by $\mathcal{F}_{\mathrm{Seq}}$, the set of sequence function symbols $\mathcal{F}^{\mathrm{Fix}}_{\mathrm{Seq}} \cup \mathcal{F}^{\mathrm{Flex}}_{\mathrm{Seq}}$,
- by $\mathcal{F}^{\mathrm{Fix}}$, the set of fixed arity function symbols $\mathcal{F}^{\mathrm{Fix}}_{\mathrm{Ind}} \cup \mathcal{F}^{\mathrm{Fix}}_{\mathrm{Seq}}$,
- by $\mathcal{F}^{\mathrm{Flex}}$, the set of flexible arity function symbols $\mathcal{F}^{\mathrm{Flex}}_{\mathrm{Ind}} \cup \mathcal{F}^{\mathrm{Flex}}_{\mathrm{Seq}}$,
- by $\mathcal{F}$, the set of all function symbols $\mathcal{F}_{\mathrm{Ind}} \cup \mathcal{F}_{\mathrm{Seq}} = \mathcal{F}^{\mathrm{Fix}} \cup \mathcal{F}^{\mathrm{Flex}}$.

The *arity* of $f \in \mathcal{F}^{\mathrm{Fix}}$ is denoted by $\mathcal{A}rity(f)$. A function symbol $c \in \mathcal{F}^{\mathrm{Fix}}$ is called a *constant* if $\mathcal{A}rity(c) = 0$.

Definition 1. *We define the notion of* term *over $\mathcal{F}$ and $\mathcal{V}$:*

1. *If $t \in \mathcal{V}_{\mathrm{Ind}}$, then t is an individual term .*
2. *If $t \in \mathcal{V}_{\mathrm{Seq}}$, then t is a sequence term.*
3. *If $f \in \mathcal{F}^{\mathrm{Fix}}_{\mathrm{Ind}}$, $\mathcal{A}rity(f) = n$, $n \geq 0$, and $t_1, \ldots, t_n$ are individual terms, then $f(t_1, \ldots, t_n)$ is an individual term.*
4. *If $f \in \mathcal{F}^{\mathrm{Fix}}_{\mathrm{Seq}}$, $\mathcal{A}rity(f) = n$, $n \geq 0$, and $t_1, \ldots, t_n$ are individual terms, then $f(t_1, \ldots, t_n)$ is a sequence term.*

5. *If* $f \in \mathcal{F}_{\mathrm{Ind}}^{\mathrm{Flex}}$ *and* $t_1, \ldots, t_n$ $(n \geq 0)$ *are (individual or sequence) terms, then* $f(t_1, \ldots, t_n)$ *is an individual term.*
6. *If* $\overline{f} \in \mathcal{F}_{\mathrm{Seq}}^{\mathrm{Flex}}$ *and* $t_1, \ldots, t_n$ $(n \geq 0)$ *are (individual or sequence) terms, then* $\overline{f}(t_1, \ldots, t_n)$ *is a sequence term.*

A term *is either an individual or a sequence term.*

The function symbol f is called the *head* of $f(t_1, \ldots, t_n)$. The head of a term t is denoted by $\mathcal{H}ead(t)$. The set of all individual terms (resp. all sequence terms) over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}_{\mathrm{Ind}}(\mathcal{F}, \mathcal{V})$ (resp. $\mathcal{T}_{\mathrm{Seq}}(\mathcal{F}, \mathcal{V})$) . The set of all terms over $\mathcal{F}$ and $\mathcal{V}$ is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$.

Definition 2. *The* depth *of a term* t*, denoted* $\mathcal{D}epth(t)$*, is defined recursively as follows:*

- *If* $t \in \mathcal{V}$*, then* $\mathcal{D}epth(t) = 1$.
- *If* t *has a form* $f(s_1, \ldots, s_n)$*, where* $f \in \mathcal{F}$ *and* $n \geq 0$*, then* $\mathcal{D}epth(t) = max(\mathcal{D}epth(s_1), \ldots, \mathcal{D}epth(s_n)) + 1$.

Definition 3. *An* equation *over* $\mathcal{F}$ *and* $\mathcal{V}$ *is a pair* $\langle s, t \rangle$*, denoted by* $s \simeq t$*, where* $s, t \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F}, \mathcal{V})$.

The terms s and t are called *sides* of the equation $s \simeq t$: s is the *left hand side* and t is the *right hand side.*

Example 1. Let $x, y \in \mathcal{V}_{\mathrm{Ind}}$, $\overline{x} \in \mathcal{V}_{\mathrm{Seq}}$, $f \in \mathcal{F}_{\mathrm{Ind}}^{\mathrm{Flex}}$, $g \in \mathcal{F}_{\mathrm{Ind}}^{\mathrm{Fix}}$, $\overline{f} \in \mathcal{F}_{\mathrm{Seq}}^{\mathrm{Flex}}$ and $\overline{g} \in \mathcal{F}_{\mathrm{Seq}}^{\mathrm{Fix}}$, $\mathcal{A}rity(g) = 2$ and $\mathcal{A}rity(\overline{g}) = 1$.

1. $f(\overline{x}, g(x, y))$ is an individual term.
2. $f(\overline{x}, \overline{f}(x, \overline{x}, y))$ is an individual term.
3. $\overline{f}(\overline{x}, \overline{f}(x, \overline{x}, y))$ is a sequence term.
4. $\overline{g}(f(x, \overline{x}, y))$ is a sequence term.
5. $f(\overline{x}, g(x, y)) \simeq g(x, y)$ is an equation.
6. $f(\overline{x}, g(\overline{x}, y))$ is not a term, because $\overline{x}$ occurs as an argument of g.
7. $f(\overline{x}, \overline{g}(\overline{x}))$ is not a term, because $\overline{x}$ occurs as an argument of $\overline{g}$.
8. $f(\overline{x}, \overline{g}(x, y))$ is not a term, because $\overline{g}$ is unary.
9. $f(\overline{x}, g(\overline{x}, y)) \simeq g(x, y)$ is not an equation, because $f(\overline{x}, g(\overline{x}, y))$ is not a term.
10. $\overline{x} \simeq f(\overline{x})$ is not an equation, because $\overline{x}$ is not an individual term.
11. $\overline{g}(x) \simeq f(\overline{x})$ is not an equation, because $\overline{g}(x)$ is not an individual term.

If not otherwise stated, the following symbols, with or without indices, are used as metavariables: x and y – over individual variables; $\overline{x}$, $\overline{y}$ and $\overline{z}$ – over sequence variables; v – over (individual or sequence) variables; f, g and h – over individual function symbols; $\overline{f}$, $\overline{g}$ and $\overline{h}$ – over sequence function symbols; a, b and c – over individual constants; $\overline{a}$, $\overline{b}$ and $\overline{c}$ – over sequence constants; s, t, r, q – over terms. We use some other denotations as well: Let T be a term, a sequence of terms, or a set of terms. Then we denote by

$\mathcal{IV}ar(T)$ – the set of all individual variables occurring in T,

$\mathcal{SV}ar(T)$ – the set of all sequence variables occurring in T,
$\mathcal{V}ar(T)$ – the set $\mathcal{IV}ar(T) \cup \mathcal{SV}ar(T)$,
$\mathcal{IF}un(T)$ – the set of all individual function symbols occurring in T,
$\mathcal{SF}un(T)$ – the set of all sequence function symbols occurring in T,
$\mathcal{F}ix(T)$ – the set of all fixed arity function symbols occurring in T,
$\mathcal{F}lex(T)$ – the set of all flexible arity function symbols occurring in T.

Definition 4. *A* variable binding *is either a pair $x \mapsto t$ where $t \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F}, \mathcal{V})$ and $t \neq x$, or an expression $\overline{x} \mapsto \ulcorner t_1, \ldots, t_n \urcorner$*[2] *where $n \geq 0$, for all $1 \leq i \leq n$ we have $t_i \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, and if $n = 1$ then $t_1 \neq \overline{x}$.*

Definition 5. *A* sequence function symbol binding *is an expression of the form $\overline{f} \mapsto \ulcorner \overline{g_1}, \ldots, \overline{g_m} \urcorner$, where $m \geq 1$, if $m = 1$ then $\overline{f} \neq \overline{g_1}$, and either $\overline{f}, \overline{g_1}, \ldots, \overline{g_m} \in \mathcal{F}^{\mathrm{Fix}}_{\mathrm{Seq}}$, with $\mathcal{A}rity(\overline{f}) = \mathcal{A}rity(\overline{g_1}) = \cdots = \mathcal{A}rity(\overline{g_m})$, or $\overline{f}, \overline{g_1}, \ldots, \overline{g_m} \in \mathcal{F}^{\mathrm{Flex}}_{\mathrm{Seq}}$*[3]*.*

Definition 6. *A* generalized substitution*, or, simply* a substitution*, is a finite set of variable and sequence function symbol bindings*

$$\{\, x_1 \mapsto t_1, \ldots, x_n \mapsto t_n, \overline{x}_1 \mapsto \ulcorner s^1_1, \ldots, s^1_{k_1} \urcorner, \ldots, \overline{x}_m \mapsto \ulcorner s^m_1, \ldots, s^m_{k_m} \urcorner,$$
$$\overline{f_1} \mapsto \ulcorner \overline{g^1_1}, \ldots, \overline{g^1_{l_1}} \urcorner, \ldots, \overline{f_r} \mapsto \ulcorner \overline{g^r_1}, \ldots, \overline{g^r_{l_r}} \urcorner \}$$

where $n, m, r \geq 0$, $x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_m$ are distinct variables and $\overline{f_1}, \ldots, \overline{f_r}$ are distinct sequence function symbols.

Lower case Greek letters are used to denote substitutions. The empty substitution is denoted by ε.

Definition 7. *The* instance *of a term t with respect to a substitution σ, denoted $t\sigma$, is defined recursively as follows:*

1. $x\sigma = \begin{cases} t, & \text{if } x \mapsto t \in \sigma, \\ x, & \text{otherwise.} \end{cases}$
2. $\overline{x}\sigma = \begin{cases} t_1, \ldots, t_n, & \text{if } \overline{x} \mapsto \ulcorner t_1, \ldots, t_n \urcorner \in \sigma,\ n \geq 0, \\ \overline{x}, & \text{otherwise.} \end{cases}$
3. $f(t_1, \ldots, t_n)\sigma = f(t_1\sigma, \ldots, t_n\sigma)$.
4. $\overline{f}(t_1, \ldots, t_n)\sigma =$
$\begin{cases} \overline{g_1}(t_1\sigma, \ldots, t_n\sigma), \ldots, \overline{g_m}(t_1\sigma, \ldots, t_n\sigma), & \text{if } \overline{f} \mapsto \ulcorner \overline{g_1}, \ldots, \overline{g_m} \urcorner \in \sigma, \\ \overline{f}(t_1\sigma, \ldots, t_n\sigma), & \text{otherwise.} \end{cases}$

[2] To improve the readability, we write sequences that bind sequence variables between $\ulcorner$ and $\urcorner$.

[3] We could define the notion of sequence function symbol binding alternatively using a slightly extended lambda-notation, allowing abstraction with a sequence of expressions: like, for instance $\overline{f} \mapsto \lambda x_1 \ldots x_n.\ulcorner \overline{g_1}(x_1 \ldots x_n), \ldots, \overline{g_m}(x_1 \ldots x_n) \urcorner$ for an n-ary sequence function symbol $\overline{f}$, or $\overline{f} \mapsto \lambda \overline{x}.\ulcorner \overline{g_1}(\overline{x}), \ldots, \overline{g_m}(\overline{x}) \urcorner$ for a flexible arity sequence function symbol $\overline{f}$. Such a definition would be probably more intuitive, but we chose the one above because it leads to shorter notation.

Example 2. Let $\sigma = \{x \mapsto a, y \mapsto f(\overline{x}), \overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner a, f(\overline{x}), b\urcorner, \overline{g} \mapsto \ulcorner \overline{g_1}, \overline{g_2}\urcorner\}$. Then $f(x, \overline{x}, \overline{g}(y, \overline{g}), \overline{y}))\sigma = f(a, \overline{g_1}(f(\overline{x}), \overline{g_1}, \overline{g_2}), \overline{g_2}(f(\overline{x}), \overline{g_1}, \overline{g_2}), a, f(\overline{x}), b)$.

Definition 8. *An* application *of a substitution σ on a sequence function symbol $\overline{f}$, denoted $\overline{f}\sigma$, is defined as follows:*

$$\overline{f}\sigma = \begin{cases} \overline{g_1}, \dots, \overline{g_m}, & \text{if } \overline{f} \mapsto \ulcorner \overline{g_1}, \dots, \overline{g_m} \urcorner \in \sigma, \\ \overline{f}, & \text{otherwise.} \end{cases}$$

Applying a substitution θ on a sequence of terms $\ulcorner t_1, \dots, t_n \urcorner$ gives a sequence of terms $\ulcorner t_1\theta, \dots, t_n\theta \urcorner$.

Definition 9. *Let σ be a substitution.*

1. *The* domain *of σ is the set of variables and sequence function symbols*

$$\mathcal{D}om(\sigma) = \{l \mid l\sigma \neq l\}.$$

2. *The* codomain *of σ is the set of terms and sequence function symbols*

$$\mathcal{C}od(\sigma) = \{l\sigma \mid l \in \mathcal{D}om(\sigma)\}.$$

3. *The* range *of σ is the set of variables*

$$\mathcal{R}an(\sigma) = \mathcal{V}ar(\mathcal{C}od(\sigma)).$$

Note that the codomain of a substitution is a set of terms and sequence function symbols, not a set consisting of terms, sequences of terms, sequence function symbols and sequences of sequence function symbols. For instance, $\mathcal{C}od(\{x \mapsto f(a), \overline{x} \mapsto \ulcorner a, a, b\urcorner, \overline{c} \mapsto \ulcorner \overline{c_1}, \overline{c_2}\urcorner\}) = \{f(a), a, b, \overline{c_1}, \overline{c_2}\}$.

A substitution σ is called *ground* iff $\mathcal{R}an(\sigma) = \emptyset$.

We also use the following denotations:

1. $\mathcal{VD}om(\sigma)$, for the set $\mathcal{D}om(\sigma) \cap \mathcal{V}$;
2. $\mathcal{VC}od(\sigma)$, for the set $\{v\sigma \mid v \in \mathcal{VD}om(\sigma)\}$;
3. $\mathcal{FD}om(\sigma)$, for the set $\mathcal{D}om(\sigma) \cap \mathcal{F}$;
4. $\mathcal{FC}od(\sigma)$ for the set $\{\overline{f}\sigma \mid \overline{f} \in \mathcal{FD}om(\sigma)\}$.

Definition 10. *The* restriction *of a substitution σ to a set of variables and sequence function symbols S, denoted $\sigma|_S$, is the substitution defined by $l\sigma|_S = l\sigma$ if $l \in S$, and $l\sigma|_S = l$ otherwise.*

Definition 11. *Let σ and ϑ be two substitutions:*

$$\begin{aligned}\sigma = \{ & x_1 \mapsto t_1, \ldots, x_n \mapsto t_n, \\ & \overline{x_1} \mapsto \ulcorner s_1^1, \ldots, s_{k_1}^1 \urcorner, \ldots, \overline{x_m} \mapsto \ulcorner s_1^m, \ldots, s_{k_m}^m \urcorner, \\ & \overline{f_1} \mapsto \ulcorner \overline{f_1^1}, \ldots, \overline{f_{l_1}^1} \urcorner, \ldots, \overline{f_r} \mapsto \ulcorner \overline{f_1^r}, \ldots, \overline{f_{l_r}^r} \urcorner \}, \\ \vartheta = \{ & y_1 \mapsto r_1, \ldots, y_{n'} \mapsto r_{n'}, \\ & \overline{y_1} \mapsto \ulcorner q_1^1, \ldots, q_{k'_1}^1 \urcorner, \ldots, \overline{y_{m'}} \mapsto \ulcorner q_1^{m'}, \ldots, q_{k'_{m'}}^{m'} \urcorner, \\ & \overline{g_1} \mapsto \ulcorner \overline{g_1^1}, \ldots, \overline{g_{l'_1}^1} \urcorner, \ldots, \overline{g_{r'}} \mapsto \ulcorner \overline{g_1^{r'}}, \ldots, \overline{g_{l'_{r'}}^{r'}} \urcorner \}.\end{aligned}$$

Then the composition *of σ and ϑ is the substitution, denoted by $\sigma\vartheta$, obtained from the set*

$$\begin{aligned}\{ & x_1 \mapsto t_1\vartheta, \ldots, x_n \mapsto t_n\vartheta, \\ & \overline{x_1} \mapsto \ulcorner s_1^1\vartheta, \ldots, s_{k_1}^1\vartheta \urcorner, \ldots, \overline{x_m} \mapsto \ulcorner s_1^m\vartheta, \ldots, s_{k_m}^m\vartheta \urcorner, \\ & \overline{f_1} \mapsto \ulcorner \overline{f_1^1}\theta, \ldots, \overline{f_{l_1}^1}\theta \urcorner, \ldots, \overline{f_r} \mapsto \ulcorner \overline{f_1^r}\theta, \ldots, \overline{f_{l_r}^r}\theta \urcorner, \\ & y_1 \mapsto r_1, \ldots, y_{n'} \mapsto r_{n'}, \\ & \overline{y_1} \mapsto \ulcorner q_1^1, \ldots, q_{k'_1}^1 \urcorner, \ldots, \overline{y_{m'}} \mapsto \ulcorner q_1^{m'}, \ldots, q_{k'_{m'}}^{m'} \urcorner, \\ & \overline{g_1} \mapsto \ulcorner \overline{g_1^1}, \ldots, \overline{g_{l'_1}^1} \urcorner, \ldots, \overline{g_{r'}} \mapsto \ulcorner \overline{g_1^{r'}}, \ldots, \overline{g_{l'_{r'}}^{r'}} \urcorner \}\end{aligned}$$

by deleting

1. *all the bindings $x_i \mapsto t_i\vartheta$ $(1 \le i \le n)$ for which $x_i = t_i\vartheta$,*
2. *all the bindings $\overline{x_i} \mapsto \ulcorner s_1^i\vartheta, \ldots, s_{k_i}^i\vartheta \urcorner$ $(1 \le i \le m)$ for which the sequence $s_1^i\vartheta, \ldots, s_{k_i}^i\vartheta$ consists of a single term $\overline{x_i}$,*
3. *all the sequence function symbol bindings $\overline{f_i} \mapsto \ulcorner \overline{f_1^i}\vartheta, \ldots, \overline{f_{l_i}^i}\vartheta \urcorner$ $(1 \le i \le r)$ such that the sequence $\overline{f_1^i}\vartheta, \ldots, \overline{f_{l_r}^i}\vartheta$ consists of a single function symbol $\overline{f_i}$,*
4. *all the bindings $y_i \mapsto r_i$ $(1 \le i \le n')$ such that $y_i \in \{x_1, \ldots, x_n\}$,*
5. *all the bindings $\overline{y_i} \mapsto \ulcorner q_1^i, \ldots, q_{k'_i}^i \urcorner$ $(1 \le i \le m')$ with $\overline{y_i} \in \{\overline{x_1}, \ldots, \overline{x_m}\}$,*
6. *all the sequence function symbol bindings $\overline{g_i} \mapsto \ulcorner \overline{g_1^i}, \ldots, \overline{g_{l'_i}^i} \urcorner$ $(1 \le i \le r')$ such that $\overline{g_i} \in \{\overline{f_1}, \ldots, \overline{f_r}\}$.*

Example 3. Let σ and ϑ be two substitutions:

$$\begin{aligned}\sigma &= \{x \mapsto y,\ \overline{x} \mapsto \ulcorner \overline{y}, \overline{x} \urcorner, \overline{y} \mapsto \ulcorner f(a,b), y, \overline{g}(\overline{x}) \urcorner, \overline{f} \mapsto \ulcorner \overline{g}, \overline{h} \urcorner\}. \\ \vartheta &= \{y \mapsto x,\ \overline{y} \mapsto \overline{x},\ \overline{x} \mapsto \ulcorner\urcorner, \overline{g} \mapsto \ulcorner \overline{g_1}, \overline{g_2} \urcorner\}.\end{aligned}$$

Then

$$\sigma\vartheta = \{y \mapsto x,\ \overline{y} \mapsto \ulcorner f(a,b), x, \overline{g_1}(), \overline{g_2}() \urcorner, \overline{f} \mapsto \ulcorner \overline{g_1}, \overline{g_2}, \overline{h} \urcorner, \overline{g} \mapsto \ulcorner \overline{g_1}, \overline{g_2} \urcorner\}.$$

Proposition 1. *Let t be a term, $\overline{f}$ be a sequence function symbol, and σ, ϑ and φ be substitutions. Then*

1. $(\overline{f}\sigma)\vartheta = \overline{f}(\sigma\vartheta)$.

2. $(t\sigma)\vartheta = t(\sigma\vartheta)$.
3. $(\sigma\vartheta)\varphi = \sigma(\vartheta\varphi)$.

Proof. Let σ and ϑ be two substitutions:

$$\begin{aligned}\sigma = \{\ & x_1 \mapsto t_1, \dots, x_n \mapsto t_n,\\ & \overline{x_1} \mapsto \ulcorner s_1^1, \dots, s_{k_1}^1 \urcorner, \dots, \overline{x_m} \mapsto \ulcorner s_1^m, \dots, s_{k_m}^m \urcorner,\\ & \overline{f_1} \mapsto \ulcorner \overline{f_1^1}, \dots, \overline{f_{l_1}^1} \urcorner, \dots, \overline{f_r} \mapsto \ulcorner \overline{f_1^r}, \dots, \overline{f_{l_r}^r} \urcorner\}.\\ \vartheta = \{\ & y_1 \mapsto r_1, \dots, y_{n'} \mapsto r_{n'},\\ & \overline{y_1} \mapsto \ulcorner q_1^1, \dots, q_{k'_1}^1 \urcorner, \dots, \overline{y_{m'}} \mapsto \ulcorner q_1^{m'}, \dots, q_{k'_{m'}}^{m'} \urcorner,\\ & \overline{g_1} \mapsto \ulcorner \overline{g_1^1}, \dots, \overline{g_{l'_1}^1} \urcorner, \dots, \overline{g_{r'}} \mapsto \ulcorner \overline{g_1^{r'}}, \dots, \overline{g_{l'_{r'}}^{r'}} \urcorner\}.\end{aligned}$$

1. We distinguish the following cases:
   (a) $\overline{f} = \overline{f_i}$ for $1 \le i \le r$ and $\{\overline{f_1^i}, \dots, \overline{f_{l_i}^i}\} \cap \mathcal{D}om(\vartheta) = \emptyset$. Then $(\overline{f}\sigma)\vartheta = \ulcorner \overline{f_1^i}, \dots, \overline{f_{l_i}^i} \urcorner \vartheta = \ulcorner \overline{f_1^i}\vartheta, \dots, \overline{f_{l_i}^i}\vartheta \urcorner = \ulcorner \overline{f_1^i}, \dots, \overline{f_{l_i}^i} \urcorner = \overline{f}(\sigma\vartheta)$.
   (b) $\overline{f} = \overline{f_i}$ for $1 \le i \le r$ and $\{\overline{f_1^i}, \dots, \overline{f_{l_i}^i}\} \cap \mathcal{D}om(\vartheta) \notin \emptyset$. Assume without loss of generality that $\overline{f_1^i} = \overline{g_j}$ for some $1 \le j \le r'$. Then we have $\overline{f} \mapsto \ulcorner \overline{g_1^j}, \dots, \overline{g_{l'_j}^j}, \overline{f_2^i}, \dots, \overline{f_{l_i}^i} \urcorner \in \sigma\vartheta$ and $(\overline{f}\sigma)\vartheta = \ulcorner \overline{f_1^i}, \dots, \overline{f_{l_i}^i} \urcorner \vartheta = \ulcorner \overline{f_1^i}\vartheta, \dots, \overline{f_{l_i}^i}\vartheta \urcorner = \ulcorner \overline{g_1^j}, \dots, \overline{g_{l'_j}^j}, \overline{f_2^i}, \dots, \overline{f_{l_i}^i} \urcorner = \overline{f}(\sigma\vartheta)$.
   (c) $\overline{f} = \overline{g_j}$ for $1 \le j \le r'$ with $\overline{g_j} \notin \mathcal{D}om(\sigma)$. Then $(\overline{f}\sigma)\vartheta = \overline{f}\vartheta = \ulcorner \overline{g_1^j}, \dots, \overline{g_{l'_j}^j} \urcorner = \overline{f}(\sigma\vartheta)$.
   (d) $\overline{f} \notin \mathcal{D}om(\sigma) \cup \mathcal{D}om(\vartheta)$. Then $(\overline{f}\sigma)\vartheta = \overline{f} = f(\sigma\vartheta)$.
2. We use induction on depth of t.
   Let first t be a variable v. We distinguish the following cases:
   (a) $v = x_i$ for $1 \le i \le n$. Then $(v\sigma)\vartheta = t_i\vartheta = v(\sigma\vartheta)$.
   (b) $v = \overline{x_i}$ for $1 \le i \le m$. Then $(v\sigma)\vartheta = \ulcorner s_1^i, \dots, s_{k_i}^i \urcorner \vartheta = \ulcorner s_1^i\vartheta, \dots, s_{k_i}^i\vartheta \urcorner = v(\sigma\vartheta)$.
   (c) $v = y_j$ for $1 \le j \le n'$ with $y_j \notin \mathcal{D}om(\sigma)$. Then $(v\sigma)\vartheta = v\vartheta = r_j = v(\sigma\vartheta)$.
   (d) $v = \overline{y_j}$ for $1 \le j \le m'$ with $\overline{y_j} \notin \mathcal{D}om(\sigma)$. Then $(v\sigma)\vartheta = v\vartheta = \ulcorner q_1^j, \dots, q_{k'_j}^j \urcorner = v(\sigma\vartheta)$.
   (e) $v \notin \mathcal{D}om(\sigma) \cup \mathcal{D}om(\vartheta)$. Then $(v\sigma)\vartheta = v = v(\sigma\vartheta)$.
   Let now t be $f(s_1, \dots, s_{n''})$. Then we have $(t\sigma)\vartheta = f((s_1\sigma)\vartheta, \dots, (s_{n''}\sigma)\vartheta) = f(s_1(\sigma\vartheta), \dots, s_{n''}(\sigma\vartheta)) = t(\sigma\vartheta)$.
   Finally, let t be $\overline{f}(s_1, \dots, s_{n''})$. We have the following cases:
   (a) $\overline{f} = \overline{f_i}$ for $1 \le i \le r$ and $\{\overline{f_1^i}, \dots, \overline{f_{l_i}^i}\} \cap \mathcal{D}om(\vartheta) = \emptyset$. Then

$$\begin{aligned}(t\sigma)\vartheta &= \ulcorner \overline{f_1^i}(s_1\sigma, \dots, s_{n''}\sigma), \dots, \overline{f_{l_i}^i}(s_1\sigma, \dots, s_{n''}\sigma) \urcorner \vartheta\\ &= \ulcorner \overline{f_1^i}((s_1\sigma)\vartheta, \dots, (s_{n''}\sigma)\vartheta), \dots, \overline{f_{l_i}^i}((s_1\sigma)\vartheta, \dots, (s_{n''}\sigma)\vartheta) \urcorner\\ &= \ulcorner \overline{f_1^i}(s_1(\sigma\vartheta), \dots, s_{n''}(\sigma\vartheta)), \dots, \overline{f_{l_i}^i}(s_1(\sigma\vartheta), \dots, s_{n''}(\sigma\vartheta)) \urcorner\\ &= \overline{f}(s_1, \dots, s_{n''})(\sigma\vartheta)\\ &= t(\sigma\vartheta)\end{aligned}$$

(b) $\overline{f} = \overline{f_i}$ for $1 \le i \le r$ and $\{\overline{f^i_1}, \ldots, \overline{f^i_{l_i}}\} \cap \mathcal{D}om(\vartheta) \notin \emptyset$. Assume without loss of generality that $\overline{f^i_1} = \overline{g_j}$ for some $1 \le j \le r'$. Then we have

$$\begin{aligned}
(t\sigma)\vartheta &= \ulcorner \overline{f^i_1}(s_1\sigma, \ldots, s_{n''}\sigma), \ldots, \overline{f^i_{l_i}}(s_1\sigma, \ldots, s_{n''}\sigma)\urcorner\vartheta \\
&= \ulcorner \overline{g^j_1}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta), \ldots, \overline{g^j_{l'_j}}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta), \\
&\quad \overline{f^i_2}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta), \ldots, \overline{f^i_{l_i}}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta)\urcorner \\
&= \ulcorner \overline{g^j_1}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta)), \ldots, \overline{g^j_{l'_j}}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta)), \\
&\quad \overline{f^i_2}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta)), \ldots, \overline{f^i_{l_i}}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta))\urcorner \\
&= \overline{f}(s_1, \ldots, s_{n''})(\sigma\vartheta) \\
&= t(\sigma\vartheta)
\end{aligned}$$

(c) $\overline{f} = \overline{g_j}$ for $1 \le j \le r'$ with $\overline{g_j} \notin \mathcal{D}om(\sigma)$. Then

$$\begin{aligned}
(t\sigma)\vartheta &= \overline{f}(s_1\sigma, \ldots, s_{n''}\sigma)\vartheta \\
&= \ulcorner \overline{g^j_1}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta), \ldots, \overline{g^j_{l'_j}}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta)\urcorner \\
&= \ulcorner \overline{g^j_1}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta)), \ldots, \overline{g^j_{l'_j}}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta))\urcorner \\
&= \overline{f}(s_1, \ldots, s_{n''})(\sigma\vartheta) \\
&= t(\sigma\vartheta)
\end{aligned}$$

(d) $\overline{f} \notin \mathcal{D}om(\sigma) \cup \mathcal{D}om(\vartheta)$. Then

$$\begin{aligned}
(t\sigma)\vartheta &= \overline{f}((s_1\sigma)\vartheta, \ldots, (s_{n''}\sigma)\vartheta) \\
&= \overline{f}(s_1(\sigma\vartheta), \ldots, s_{n''}(\sigma\vartheta)) \\
&= \overline{f}(s_1, \ldots, s_{n''})(\sigma\vartheta) \\
&= t(\sigma\vartheta).
\end{aligned}$$

□

Definition 12. *A substitution σ is called* standardizing away *from a finite set of sequence function symbols $\mathcal{Q}$ iff the following three conditions hold:*

1. $\mathcal{C}od(\sigma) \cap \mathcal{Q} = \emptyset$.
2. *For all $\overline{f}, \overline{g} \in \mathcal{D}om(\sigma) \cap \mathcal{Q}$, if $\overline{f} \neq \overline{g}$, then $\{\overline{f}\sigma\} \cap \{\overline{g}\sigma\} = \emptyset$.*
3. *If $\overline{f} \mapsto \ulcorner \overline{g_1} \ldots, \overline{g_n}\urcorner \in \sigma$ and $\overline{f} \in \mathcal{Q}$, then for all $1 \le i < j \le n$ we have $\overline{g_i} \neq \overline{g_j}$.*

Intuitively, a substitution standardizing away from $\mathcal{Q}$ either leaves a sequence function symbol in $\mathcal{Q}$ "unchanged", or "moves it away from" $\mathcal{Q}$, binding it with a sequence of distinct sequence function symbols that do not occur in $\mathcal{Q}$, and maps different sequence function symbols to disjoint sequences.

Let E be a set of equations (called identities) over $\mathcal{F}$ and $\mathcal{V}$. By $\simeq_E$ we denote the least congruence relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ that is closed under substitution application and contains E. More precisely, $\simeq_E$ is defined as follows:

1. $E \subseteq \simeq_E$.
2. (Reflexivity.) For all $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $t \simeq_E t$.
3. (Symmetry.) For all $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $s \simeq_E t$ implies $t \simeq_E s$.
4. (Transitivity.) For all $s, t, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $s \simeq_E t$ and $t \simeq_E r$ imply $s \simeq_E r$.
5. (Congruence.) For all $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, if $s_1 \simeq_E t_1, \ldots, s_n \simeq_E t_n$, then $f(s_1, \ldots, s_n) \simeq_E f(t_1, \ldots, t_n)$, where either $f \in \mathcal{F}^{\mathrm{Flex}}$, or $f \in \mathcal{F}^{\mathrm{Fix}}$ and $\mathcal{A}rity(F) = n$.
6. (Substitutivity.) For all $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, if $s \simeq_E t$ and σ is a substitution such that $s\sigma, t\sigma \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, then $s\sigma \simeq_E t\sigma$.

Substitutivity requires that $s\sigma$ and $t\sigma$ must be single terms, not arbitrary sequences of terms. The set $\simeq_E$ is called an *equational theory* defined by E.

If $s \simeq_E t$, then we say that the term s is equal modulo E to the term t. In the following, we will often slightly abuse the notion of an equational theory by also calling a set E an equational theory, or E-theory. For a given set of equations E, we denote by $\mathcal{S}ig(E)$ the *signature* of E, i.e., the set $\mathcal{IF}un(E) \cup \mathcal{SF}un(E)$ of all function symbols occurring in E. Solving equations in an E-theory is called *E-unification.* The fact that the equation $s \simeq t$ has to be solved in an E-theory is written as $s \simeq^?_E t$.

Definition 13. *Let E be an equational theory and $\mathcal{F}$ be a signature containing $\mathcal{S}ig(E)$. An E-*unification problem *over $\mathcal{F}$ is a finite multiset of equations $\Gamma = \{s_1 \simeq^?_E t_1, \ldots, s_n \simeq^?_E t_n\}$ over $\mathcal{F}$ and $\mathcal{V}$. An E-*unifier *of Γ is a substitution σ such that σ is standardizing away from $\mathcal{SF}un(\Gamma)$ and for all $1 \le i \le n$, $s_i\sigma \simeq_E t_i\sigma$. The set of all E-unifiers of Γ is denoted by $\mathcal{U}_E(\Gamma)$, and Γ is E-*unifiable *iff $\mathcal{U}_E(\Gamma) \neq \emptyset$.*

Note that if $\Gamma = \{s_1 \simeq^?_E t_1, \ldots, s_n \simeq^?_E t_n\}$ is a unification problem, then all $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F}, \mathcal{V})$.

Example 4. Let $E = \emptyset$.

1. $\Gamma = \{f(\overline{g}(\overline{x}, \overline{y}, a)) \simeq^?_E f(\overline{g}(\overline{c}, b, x))\}$. Then $\{\overline{x} \mapsto \overline{c_1}, \overline{y} \mapsto \ulcorner \overline{c_2}, b \urcorner, x \mapsto a, \overline{c} \mapsto \ulcorner \overline{c_1}, \overline{c_2} \urcorner\} \in \mathcal{U}_E(\Gamma)$.
2. $\Gamma = \{f(\overline{g}(\overline{x}, \overline{y}, a)) \simeq^?_E f(\overline{h}(\overline{c}, x))\}$. Then $\mathcal{U}_E(\Gamma) = \emptyset$. If we did not require the E-unifiers of a unification problem to be standardizing away from the sequence function symbol set of the problem, then Γ would have E-unifiers, e.g., $\{\overline{x} \mapsto \overline{c_1}, \overline{y} \mapsto \ulcorner \overline{c_2}, b \urcorner, x \mapsto a, \overline{g} \mapsto \overline{h}, \overline{c} \mapsto \ulcorner \overline{c_1}, \overline{c_2} \urcorner\}$ would be one of them.

In the sequel, if not otherwise stated, E stands for an equational theory, $\mathcal{X}$ for a finite set of variables and $\mathcal{Q}$ for a finite set of sequence function symbols.

Definition 14. *A substitution σ is called* erasing *on $\mathcal{X}$ modulo E iff either there exist $f \in \mathcal{S}ig(E)$ and $v \in \mathcal{X}$ such that $f(v)\sigma \simeq_E f()$, or there exists $\overline{x} \in \mathcal{X}$ such that $\overline{x} \mapsto \ulcorner\urcorner \in \sigma$. We say that σ is* non-erasing *on $\mathcal{X}$ modulo E iff σ is not erasing on $\mathcal{X}$ modulo E.*

Example 5.

1. Let $E = \emptyset$ and $\mathcal{X} = \{x, \overline{x}\}$. Then any substitution that contains the variable binding $\overline{x} \mapsto \ulcorner\urcorner$ is erasing on $\mathcal{X}$ modulo E. For instance, $\{x \mapsto a, \overline{x} \mapsto \ulcorner\urcorner\}$ is erasing on $\mathcal{X}$ modulo E.
2. Let $E = \{f(\overline{x}, f(\overline{y}), \overline{z}) \simeq f(\overline{x}, \overline{y}, \overline{z})\}$ and $\mathcal{X} = \{x, \overline{x}\}$. Then any substitution that contains $x \mapsto f()$, or $\overline{x} \mapsto \ulcorner\urcorner$, or a variable binding of the form $\overline{x} \mapsto \ulcorner t_1, \dots, t_n \urcorner$ with $n \geq 1$ and $t_1 = \cdots = t_n = f()$, is erasing on $\mathcal{X}$ modulo E. For instance, the substitutions $\{x \mapsto f()\}$, $\{\overline{x} \mapsto f()\}$, $\{\overline{x} \mapsto \ulcorner\urcorner\}$, $\{\overline{x} \mapsto \ulcorner f(), f(), f(), f() \urcorner\}$ are erasing on $\mathcal{X}$ modulo E.

Definition 15. *A substitution σ* agrees *with a substitution ϑ on $\mathcal{X}$ and $\mathcal{Q}$ modulo E, denoted $\sigma =_E^{\mathcal{X},\mathcal{Q}} \vartheta$, iff*

1. *for all $\overline{x} \in \mathcal{X}$, there exist $t_1, \dots, t_n, s_1, \dots, s_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $n \geq 0$, such that $\overline{x}\sigma = \ulcorner t_1, \dots, t_n \urcorner$, $\overline{x}\vartheta = \ulcorner s_1, \dots, s_n \urcorner$ and $t_i \simeq_E s_i$ for each $1 \leq i \leq n$;*
2. *for all $x \in \mathcal{X}$, $x\sigma \simeq_E x\vartheta$;*
3. *for all $\overline{f} \in \mathcal{Q}$, $\overline{f}\sigma = \overline{f}\vartheta$.*

Definition 16. *A substitution φ* matches *a substitution σ with a substitution ϑ on $\mathcal{X}$ and $\mathcal{Q}$ modulo E, iff $\sigma\varphi =_E^{\mathcal{X},\mathcal{Q}} \vartheta$.*

Example 6. Let $\sigma = \{\overline{x} \mapsto \overline{a}\}$, $\vartheta = \{\overline{x} \mapsto \ulcorner \overline{b}, \overline{c} \urcorner, \overline{a} \mapsto \ulcorner \overline{b}, \overline{c} \urcorner\}$ and $\varphi = \{\overline{x} \mapsto \ulcorner \overline{b}, \overline{c} \urcorner, \overline{a} \mapsto \ulcorner \overline{b}, \overline{c} \urcorner\}$. Let also $\mathcal{X} = \{\overline{x}\}$, $\mathcal{Q} = \{\overline{a}\}$ and $E = \emptyset$. Then $\sigma\varphi =_E^{\mathcal{X},\mathcal{Q}} \vartheta$.

Definition 17. *A substitution σ is* more general *(resp.* strongly more general*) than a substitution ϑ on $\mathcal{X}$ and $\mathcal{Q}$ modulo E, denoted $\sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \vartheta$ (resp. $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \vartheta$) , iff there exists a substitution (resp. a substitution non-erasing on $\mathcal{X}$ modulo E) φ such that $\sigma\varphi =_E^{\mathcal{X},\mathcal{Q}} \vartheta$.*

From Definition 17 it immediately follows that $\preceq_E^{\mathcal{X},\mathcal{Q}} \subseteq \lessdot_E^{\mathcal{X},\mathcal{Q}}$.

Example 7. Let $\sigma = \{\overline{x} \mapsto \overline{y}\}$, $\vartheta = \{\overline{x} \mapsto \ulcorner a, b \urcorner, \overline{y} \mapsto \ulcorner a, b \urcorner\}$, $\eta = \{\overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner\urcorner\}$, $\mathcal{X} = \{\overline{x}, \overline{y}\}$, $\mathcal{Q} = \emptyset$, $E = \emptyset$. Then $\sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \vartheta$, $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \vartheta$, $\sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \eta$, $\sigma \not\preceq_E^{\mathcal{X},\mathcal{Q}} \eta$.

A substitution ϑ is an *E-instance* (resp. *strong E-instance*) of σ on $\mathcal{X}$ and $\mathcal{Q}$ iff $\sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \vartheta$ (resp. $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \vartheta$). The equivalence associated with $\lessdot_E^{\mathcal{X},\mathcal{Q}}$ (i.e., $\lessdot_E^{\mathcal{X},\mathcal{Q}} \cap \gtrdot_E^{\mathcal{X},\mathcal{Q}}$) is denoted by $\doteq_E^{\mathcal{X},\mathcal{Q}}$. The equivalence associated with $\preceq_E^{\mathcal{X},\mathcal{Q}}$ is denoted by $\dot{\approx}_E^{\mathcal{X},\mathcal{Q}}$. The strict part of $\lessdot_E^{\mathcal{X},\mathcal{Q}}$ (resp. $\preceq_E^{\mathcal{X},\mathcal{Q}}$) is denoted by $\lessdot_E^{\mathcal{X},\mathcal{Q}}$ (resp. $\prec_E^{\mathcal{X},\mathcal{Q}}$). Both $\lessdot_E^{\mathcal{X},\mathcal{Q}}$ and $\preceq_E^{\mathcal{X},\mathcal{Q}}$ are quasi-orderings.

If φ matches σ with ϑ on $\mathcal{X}$ and $\mathcal{Q}$ modulo E, then $\sigma\varphi \doteq_E^{\mathcal{X},\mathcal{Q}} \vartheta$. If, in addition, φ is non-erasing on $\mathcal{X}$ and $\mathcal{Q}$ modulo E, then $\sigma\varphi \dot{\approx}_E^{\mathcal{X},\mathcal{Q}} \vartheta$.

Definition 18. *A set of substitutions $\mathcal{S}$ is called* minimal *(resp.* almost minimal*) with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E iff two distinct elements of $\mathcal{S}$ are incomparable with respect to $\lessdot_E^{\mathcal{X},\mathcal{Q}}$ (resp. $\preceq_E^{\mathcal{X},\mathcal{Q}}$), i.e., for all σ, $\vartheta \in \mathcal{Q}$, $\sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \vartheta$ (resp. $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \vartheta$) implies $\sigma = \vartheta$.*

Given $\mathcal{Q}$, $\mathcal{X}$ and E, every set of substitutions that is minimal with respect to $\mathcal{Q}$ and $\mathcal{X}$ modulo E is also almost minimal with respect to $\mathcal{Q}$ and $\mathcal{X}$ modulo E. The inverse statement is not true: Consider σ, η, $\mathcal{X}$, $\mathcal{Q}$ and E from Example 7. Then the set $\{\sigma, \eta\}$ is almost minimal but not minimal with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E.

Definition 19. *Let Γ be a E-unification problem over $\mathcal{F}$, $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. A* complete set of E-unifiers *of Γ is a set $\mathcal{S}$ of substitutions such that*

1. *$\mathcal{S} \subseteq \mathcal{U}_E(\Gamma)$, i.e., each element of $\mathcal{S}$ is an E-unifier of Γ,*
2. *for each $\vartheta \in \mathcal{U}_E(\Gamma)$ there exists $\sigma \in \mathcal{S}$ such that $\sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \vartheta$.*

The set $\mathcal{S}$ is a minimal *(resp.* almost minimal*)* complete set of E-unifiers *of Γ, denoted $mcu_E(\Gamma)$ (resp. $amcu_E(\Gamma)$), iff it is a complete set that is minimal (resp. almost minimal) with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E.*

Proposition 2. *An E-unification problem Γ has an almost minimal complete set of E-unifiers iff it has a minimal complete set of E-unifiers.*

Proof. ($\Rightarrow$) Let $\mathcal{X} = \mathcal{V}ar(\Gamma)$, $\mathcal{Q} = \mathcal{SF}un(\Gamma)$ and let $\mathcal{S}$ be an almost minimal complete set of E-unifiers of Γ. Then the set

$$\mathcal{S} \setminus \{\vartheta \mid \vartheta \in \mathcal{S} \text{ and there exists } \sigma \in \mathcal{S}, \text{ with } \sigma \neq \vartheta \text{ and } \sigma \lessdot_E^{\mathcal{X},\mathcal{Q}} \vartheta\}$$

is a minimal complete set of E-unifiers of Γ.

($\Leftarrow$) Every minimal complete set of E-unifiers of Γ is itself an almost minimal complete set of E-unifiers of Γ. □

A substitution σ is a *most general E-unifier*, of a unification problem Γ iff $\{\sigma\}$ is a (minimal) complete set of E-unifiers of Γ. If Γ is not E-unifiable, then $mcu_E(\Gamma) = amcu_E(\Gamma) = \emptyset$. A minimal (resp. almost minimal) complete set of E-unifiers of Γ, if it exists, is unique up to the equivalence $\doteq_E^{\mathcal{X},Q}$ (resp. $\dot{\approx}_E^{\mathcal{X},Q}$), where $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$.

Example 8. Let $E = \emptyset$.

1. $\Gamma = \{f(\overline{x}) \simeq_E^? f(\overline{y})\}$. Then

$$mcu_E(\Gamma) = \{\{\overline{x} \mapsto \overline{y}\}\}.$$
$$amcu_E(\Gamma) = \{\{\overline{x} \mapsto \overline{y}\}, \{\overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner\urcorner\}\}.$$

2. $\Gamma = \{f(\overline{x}, x, \overline{y}) \simeq_E^? f(f(\overline{x}), x, a, b)\}$. Then

$$mcu_E(\Gamma) = \{\{x \mapsto f(), \overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner f(), a, b\urcorner\}\}.$$
$$amcu_E(\Gamma) = mcu_E(\Gamma).$$

3. $\Gamma = \{f(\overline{x}, x, \overline{y}) \simeq_E^? f(a, x, b)\}$. Then

$$mcu_E(\Gamma) = \{\{\overline{x} \mapsto a, \overline{y} \mapsto b\}, \{x \mapsto a, \overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner a, b\urcorner\},$$
$$\{x \mapsto b, \overline{x} \mapsto \ulcorner a, b\urcorner, \overline{y} \mapsto \ulcorner\urcorner\}\}.$$
$$amcu_E(\Gamma) = mcu_E(\Gamma).$$

4. $\Gamma = \{f(a, \overline{x}) \simeq_E^? f(\overline{x}, a)\}$. Then

$$mcu_E(\Gamma) = \{\{\overline{x} \mapsto \ulcorner\urcorner\}, \{\overline{x} \mapsto a\}, \{\overline{x} \mapsto \ulcorner a, a\urcorner\}, \ldots\}$$
$$amcu_E(\Gamma) = mcu_E(\Gamma).$$

5. $\Gamma = \{f(\overline{x}, \overline{y}, x) \simeq_E^? f(\overline{c}, a)\}$. Then

$$mcu_E(\Gamma) = \{\{\overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \overline{c}, x \mapsto a\}, \{\overline{x} \mapsto \overline{c}, \overline{y} \mapsto \ulcorner\urcorner, x \mapsto a\},$$
$$\{\overline{x} \mapsto \overline{c_1}, \overline{y} \mapsto \overline{c_2}, x \mapsto a, \overline{c} \mapsto \ulcorner\overline{c_1}, \overline{c_2}\urcorner\}\}.$$
$$amcu_E(\Gamma) = mcu_E(\Gamma).$$

6. $\Gamma = \{f(\overline{a}) \simeq_E^? f(\overline{b})\}$. Then $mcu_E(\Gamma) = amcu_E(\Gamma) = \emptyset$.

Definition 20. *A set of substitutions $\mathcal{S}$ is called* disjoint *(resp.* almost disjoint*) with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E iff two distinct elements of $\mathcal{S}$ have no common E-instance (resp. strong E-instance) on $\mathcal{X}$ and $\mathcal{Q}$, i.e., for all $\sigma, \vartheta \in \mathcal{S}$, if there exists φ such that $\sigma \leq_E^{\mathcal{X},\mathcal{Q}} \varphi$ (resp. $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \varphi$) and $\vartheta \leq_E^{\mathcal{X},\mathcal{Q}} \varphi$ (resp. $\vartheta \preceq_E^{\mathcal{X},\mathcal{Q}} \varphi$), then $\sigma = \vartheta$.*

Given $\mathcal{X}$, $\mathcal{Q}$ and E, every set of substitutions that is disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E is also almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. The inverse statement is not true: Consider again σ, η, $\mathcal{X}$, $\mathcal{Q}$ and E from Example 7. Then the set $\{\sigma, \eta\}$ is almost disjoint but not disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E.

Proposition 3. *If a set of substitutions $\mathcal{S}$ is disjoint (resp. almost disjoint) with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E, then it is minimal (resp. almost minimal) with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E.*

Proof. Assume $\sigma, \vartheta \in \mathcal{S}$ such that $\sigma \leq_E^{\mathcal{X},\mathcal{Q}} \vartheta$. Since $\vartheta \leq_E^{\mathcal{X},\mathcal{Q}} \vartheta$, by disjointness of $\mathcal{S}$ with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E we get $\sigma = \vartheta$, which implies minimality of $\mathcal{S}$ with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E. Almost minimality can be proved in the same way. ◻

However, almost disjointness does not imply minimality: Again, take σ, η, $\mathcal{X}$, $\mathcal{Q}$ and E from Example 7. Then the set $\{\sigma, \eta\}$ is almost disjoint but not minimal with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E.

On the other hand, minimality does not imply almost disjointness: Let $\sigma = \{x \mapsto f(a, y)\}$, $\vartheta = \{x \mapsto f(y, b)\}$, $\mathcal{X} = \{x\}$, $\mathcal{Q} = \emptyset$ and $E = \emptyset$. Then $\{\sigma, \vartheta\}$ is minimal but not almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E, because $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \varphi$ and $\vartheta \preceq_E^{\mathcal{X},\mathcal{Q}} \varphi$, with $\varphi = \{x \mapsto f(a, b)\}$, but $\sigma \neq \vartheta$. The same example can be used to show that almost minimality does not imply almost disjointness either. From these observations we can also conclude that neither minimality nor almost minimality imply disjointness.

Definition 21. *A substitution σ is* disjointness preserving *(resp.* almost-disjointness preserving*), with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E iff for any two substitutions ϑ_1 and ϑ_2, disjointness (resp. almost disjointness) of the set $\{\vartheta_1, \vartheta_2\}$ with respect to the set of variables $\cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma)$ and sequence function symbols $\cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma)$ modulo E implies disjointness (resp. almost disjointness) of the set $\{\sigma\vartheta_1, \sigma\vartheta_2\}$ with respect to $\mathcal{X}$ and $\mathcal{Q}$ modulo E.*

Definition 22. *Let E be an equational theory, and let Γ be an E-unification problem over $\mathcal{F}$. The problem Γ has* type unitary *(*finitary, infinitary*) iff it has a minimal complete set of E-unifiers of cardinality 1 (finite cardinality, infinite cardinality). If Γ does not have a minimal complete set of E-unifiers, then it is of* type zero, *or* nullary*. We abbreviate type unitary by 1, type finitary by ω, type infinitary by ∞, and type nullary by 0, and order these types as follows: $1 < \omega < \infty < 0$.*

The unification type of E with respect to the signature $\mathcal{F}$ is the maximal type of an E-unification problem over $\mathcal{F}$.

Definition 23. *Let E be an equational theory, and let Γ be an E-unification problem over $\mathcal{F}$.*

- *Γ is an* elementary *E-unification problem iff $\mathcal{F} = \mathcal{S}ig(E)$.*
- *Γ is an E-unification problem* with constants *iff $\mathcal{F} \setminus \mathcal{S}ig(E)$ is a set of constant symbols.*
- *Γ is a* general *E-unification problem iff $\mathcal{F} \setminus \mathcal{S}ig(E)$ contains arbitrary function symbols.*

The equational theory $E = \emptyset$ is called the *free theory with sequence variables and sequence function symbols*. We will call unification in the free theory the *syntactic sequence unification*, or sometimes shortly as *syntactic unification*.

Below we will answer three questions about the syntactic sequence unification:

Decidability: Is it decidable whether a unification problem is solvable?
Unification Type: What is the unification type?
Unification procedure: How can we obtain a (preferably minimal) unification procedure?

The equational theory $E = \{f(\overline{x}, f(\overline{y}), \overline{z}) \simeq f(\overline{x}, \overline{y}, \overline{z})\}$ that we first encountered in Example 5 is called the flat theory with sequence variables and sequence function symbols, where f is the flat flexible arity individual function symbol. We call unification in the flat theory the flat sequence (or simply flat) unification, or the F-unification. Below we use certain properties of this theory in proving decidability of the syntactic sequence unification.

3 Decidability

We show decidability of a syntactic sequence unification problem in three steps: First, we reduce the problem by unifiability preserving transformation to a unification problem containing no sequence function symbols. Second, applying yet another unifiability preserving transformation we get rid of all free flexible arity (individual) function symbols, obtaining a unification problem whose signature consists of fixed arity individual function symbols and one flat flexible arity individual function symbol. Finally, we show decidability of the reduced problem.

3.1 First Reduction

Let Γ be a syntactic sequence unification problem and let $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Assume $\mathcal{Q} \neq \emptyset$, otherwise we can continue immediately with the second reduction step.

Theorem 1. *If Γ is unifiable, then there exists a syntactic unifier σ of Γ such that $\mathcal{FD}om(\sigma) = \emptyset$.*

Proof. Let ϑ_0 be a syntactic unifier of Γ. Assume without loss of generality that $\mathcal{FD}om(\vartheta_0) \subseteq \mathcal{Q}$. Let $\overline{f} \mapsto \ulcorner \overline{g_1}, \dots, \overline{g_n} \urcorner \in \vartheta_0$. Since ϑ_0 is standardizing away from $\mathcal{Q}$, the sequence function symbols $\overline{g_1}, \dots, \overline{g_n} \notin \mathcal{Q}$. Removing every sequence term with heads $\overline{g_2}, \dots, \overline{g_n}$ from $\mathcal{C}od(\vartheta_0)$, replacing the sequence function symbol $\overline{g_1}$ with $\overline{f}$ everywhere in $\mathcal{C}od(\vartheta_0)$ and removing the binding $\overline{f} \mapsto \ulcorner \overline{g_1}, \dots, \overline{g_n} \urcorner$ from ϑ_0 we get a new substitution ϑ_1 that is again a unifier of Γ and $\mathcal{D}om(\vartheta_1) \subseteq \mathcal{D}om(\vartheta_0)$. Repeating this process $m-1$ times, where m is the number of sequence function symbol bindings in ϑ_0, we arrive at the substitution $\sigma = \vartheta_m$ such that $\mathcal{FD}om(\sigma) = \emptyset$ and σ is a syntactic unifier of Γ. □

We transform Γ performing the following steps:

1. Introduce for each n-ary $\overline{f} \in \mathcal{Q}$ a new n-ary individual function symbol $g_{\overline{f}}$.
2. Introduce for each flexible arity $\overline{f} \in \mathcal{Q}$ a new flexible arity individual function symbol $g_{\overline{f}}$.
3. Replace each sequence function symbol $\overline{f}$ in Γ with the corresponding $g_{\overline{f}}$.

The transformation yields a new unification problem Λ that does not contain sequence function symbols. We impose a individual variable restriction on Λ, called the *first restriction on individual variables* demanding that for any syntactic unifier λ of Λ and for any $x \in \mathcal{V}_{\mathrm{Ind}}$, the term $x\lambda$ must not have any of the newly introduced individual function symbols as its head.

Theorem 2. *Γ is syntactically unifiable iff Λ with the first restriction on individual variables is syntactically unifiable.*

Proof. ($\Rightarrow$) By Theorem 1, there exists a syntactic sequence unifier γ of Γ such that $\mathcal{FD}om(\gamma) = \emptyset$. From γ we can get a syntactic unifier λ of Λ with the first restriction on individual variables, by replacing each sequence function symbol $\overline{f} \in \mathcal{Q}$, in $\mathcal{C}od(\gamma)$ by $g_{\overline{f}}$.

($\Leftarrow$) From an syntactic unifier λ of Λ with the first restriction on individual variables we can get a syntactic unifier of Γ by replacing each introduced individual function symbol $g_{\overline{f}}$ by the corresponding $\overline{f} \in \mathcal{Q}$. □

Remark 1. Unifiability of Λ without the first restriction on individual variables does not imply unifiability of Γ: Let Γ be $\{f(x) \simeq^?_\emptyset f(\overline{c})\}$. Then $\Lambda = \{f(x) \simeq^?_\emptyset f(c_{\overline{c}})\}$. It is clear that Γ is not unifiable, while $\{x \mapsto c_{\overline{c}}\}$ is a unifier of Λ, because the individual variable x can be bound with the individual term $c_{\overline{c}}$.

Next, our goal is to construct a general syntactic sequence unification problem (without sequence function symbols) that is unifiable without restrictions iff Λ with the first restriction on individual variables is syntactically unifiable.

We construct a finite set of individual terms $\mathcal{I}$ consisting of the following elements:

1. A new individual constant c.
2. Exactly one individual term of the form $h(y_1, \ldots, y_n)$ for each fixed arity function symbol $h \in \mathcal{IF}un(\Gamma)$ such that $n = \mathcal{A}rity(h)$ and $y_1, \ldots, y_n$ are distinct individual variables new for $\mathcal{I}$ and Λ.
3. Exactly one individual term of the form $h(\overline{x})$ for each flexible arity $h \in \mathcal{IF}un(\Gamma)$ such that $\overline{x}$ is a sequence variable new for $\mathcal{I}$ and Λ.

Theorem 3. *Let Λ have the form $\{s \simeq^?_\emptyset t\}$ with $\mathcal{IV}ar(\Lambda) = \{x_1, \ldots, x_n\}$ and $g \in \mathcal{F}^{\mathrm{Fix}}$ be a new symbol with $\mathcal{A}rity(g) = n+1$. Then Λ with the first restriction on individual variables is syntactically unifiable iff there exist $r_1, \ldots, r_n \in \mathcal{I}$ such that the general syntactic unification problem (without sequence function symbols) $\{g(s, x_1, \ldots, x_n) \simeq^?_\emptyset g(t, r_1, \ldots, r_n)\}$ is syntactically unifiable.*

Proof. Let $\mathcal{N} \subseteq \mathcal{IF}un(\Lambda)$ be the set of individual function symbols in Λ on which the first restriction on individual variables does not apply, i.e., these are all those symbols from $\mathcal{IF}un(\Lambda)$ that can be heads of terms that bind individual variables in a syntactic unifier of Λ. In fact, $\mathcal{N} = \mathcal{IF}un(\Gamma)$.

($\Rightarrow$) Let λ be a syntactic unifier of Λ with the first restriction on individual variables. We assume without loss of generality that $\mathcal{C}od(\lambda) \subseteq \mathcal{T}(\mathcal{IF}un(\Lambda), \mathcal{V})$. Then for each variable $x \in \mathcal{IV}ar(\Lambda)$ we have either $x\lambda \in \mathcal{V}_{\mathrm{Ind}}$ or $\mathcal{H}ead(x\lambda) \in \mathcal{N}$.

Let $\sigma = \{x \mapsto c \,|\, x \in \mathcal{IV}ar(\Lambda) \cup \mathcal{IV}ar(\mathcal{R}an(\lambda))\}$, where $c \in \mathcal{I}$. Then $\lambda\sigma$ is a syntactic unifier of Λ such that for each $x \in \mathcal{IV}ar(\Lambda)$ we have $\mathcal{H}ead(x\lambda\sigma) \in \mathcal{N} \cup \{c\}$. Since $\mathcal{IV}ar(\Lambda) = \{x_1, \ldots, x_n\}$, for each $1 \leq i \leq n$ there exists a term $t_i \in \mathcal{I}$ and a substitution φ_i with $\mathcal{D}om(\varphi_i) = \mathcal{IV}ar(t_i)$ such that $x_i\lambda\sigma = t_i\varphi_i$.

Moreover, we have that

1. $(\mathcal{D}om(\lambda\sigma) \cup \mathcal{R}an(\lambda\sigma)) \cap \mathcal{V}ar(\mathcal{I}) = \emptyset$,
2. $(\mathcal{D}om(\varphi_i) \cup \mathcal{R}an(\varphi_i)) \cap \mathcal{D}om(\varphi_j) = \emptyset$ for all $1 \leq i < j \leq n$.

It implies, on one hand, that for each $1 \leq i \leq n$ we have $x_i\vartheta\sigma\varphi_i = t_i\vartheta\sigma\varphi_i$, and, on the other hand, $\varphi_1 \cdots \varphi_n = \varphi_{\pi(1)} \cdots \varphi_{\pi(n)}$ for any permutation π of $1, \ldots, n$. Let ψ be $\varphi_1 \cdots \varphi_n$. Then we get $x_i\vartheta\sigma\psi = t_i\vartheta\sigma\psi$ for each $1 \leq i \leq n$.

Let r_i be t_i for each $1 \leq i \leq n$. Then the substitution $\vartheta\sigma\psi$ is a syntactic unifier of $\{g(s, x_1 \ldots, x_n) \simeq^?_\emptyset g(t, t_1, \ldots, t_n)\}$.

($\Leftarrow$) Let $r_1, \ldots, r_n \in \mathcal{I}$ be terms such that the problem $\{g(s, x_1, \ldots, x_n) \simeq^?_\emptyset g(t, r_1, \ldots, r_n)\}$ is syntactically unifiable with a unifier ϑ. Then, obviously, ϑ is a syntactic unifier of Λ. Moreover, $\mathcal{H}ead(x\vartheta) \in \mathcal{N} \cup \{c\}$ for each $x \in \mathcal{IV}ar(\Lambda)$.

Hence, ϑ is a syntactic unifier of Λ with the first restriction on individual variables. $\square$

Thus, we can decide unifiability of a general syntactic unification problem iff we can decide unifiability of a general syntactic unification problem without sequence function symbols.

3.2 Second Reduction

Let Δ be a syntactic unification problem containing no sequence function symbols. We assume that $\mathcal{F}lex(\Delta) \neq \emptyset$, otherwise Δ would be a Robinson unification problem. We transform Δ performing the following steps:

1. Introduce a new flat flexible arity individual function symbol *seq*.
2. Introduce a new unary individual function symbol g_f for each function symbol $f \in \mathcal{F}lex(\Delta)$.
3. Replace each term $f(r_1, \dots, r_m)$, $m \geq 0$, in Δ by $g_f(seq(r_1, \dots, r_m))$.

The transformation yields a new unification problem Θ that is a general flat unification problem. Sequence variables occur in Θ only as arguments of terms with the head *seq*. We impose the *second restriction on individual variables* on Θ demanding that, for any F-unifier ϑ of Θ and for any $x \in \mathcal{V}_{\mathrm{Ind}}$, the term $x\vartheta$ must not have *seq* as the head.

Theorem 4. *Δ is syntactically unifiable iff Θ with the second restriction on individual variables is F-unifiable.*

Proof. ($\Rightarrow$) From a syntactic unifier γ of Δ we can get an F-unifier of Θ with individual variable restrictions by replacing each term $f(r_1, \dots, r_m)$, $m \geq 0$, $f \in \mathcal{F}lex(\Delta)$, in $\mathcal{C}od(\gamma)$ by $g_f(seq(r_1, \dots, r_m))$.

($\Leftarrow$) From an F-unifier ϑ of Θ with individual variable restrictions we can get a syntactic unifier of Δ by

1. replacing each introduced unary function symbol g_f by the corresponding $f \in \mathcal{F}lex(\Delta)$,
2. replacing each term of the form $seq(r_1, \dots, r_m)$, $m \geq 0$, by the sequence of terms $r_1, \dots, r_m$.

Remark 2. F-unifiability of Θ without the second restriction on individual variables does not imply syntactic unifiability of Δ: Let Δ be $\{f(x) \simeq^?_\emptyset f(a, b)\}$, where $f \in \mathcal{F}^{\mathrm{Flex}}$. Then $\Theta = \{g_f(seq(x)) \simeq^?_F g_f(seq(a, b))\}$. It is clear that Δ is not unifiable, while $\{x \mapsto seq(a, b)\}$ is an F-unifier of Θ, because $seq(seq(a, b)) \simeq_F seq(a, b)$.

Next, our goal is to construct a general F-unification problem that is F-unifiable (without restrictions) iff Θ with individual variable restrictions is F-unifiable.

First, we construct a finite set $\mathcal{J}$ of individual terms consisting of the following elements:

1. A new individual constant d.
2. Exactly one term of the form $h(y_1, \ldots, y_n)$ for each $h \in \mathcal{F}ix(\Theta)$ such that $n = \mathcal{A}rity(h)$ and $y_1, \ldots, y_n$ are distinct individual variables occurring neither in the other terms from $\mathcal{J}$ nor in Θ.

Theorem 5. *Let Θ be $\{s \simeq_F^? t\}$ with $\mathcal{IV}ar(\Theta) = \{x_1, \ldots, x_n\}$ and $h \in \mathcal{F}^{\mathrm{Fix}}$ be a new symbol with $\mathcal{A}rity(h) = n + 1$. Then Θ with the second restriction on individual variables is F-unifiable iff there exist $r_1, \ldots, r_n \in \mathcal{J}$ such that the general F-unification problem $\{h(s, x_1, \ldots, x_n) \simeq_F^? h(t, r_1, \ldots, r_n)\}$ is F-unifiable.*

Proof. ($\Rightarrow$) Let ϑ be an F-unifier of Θ with individual variable restrictions. We can assume without loss of generality that $\mathcal{C}od(\vartheta) \subseteq \mathcal{T}(\mathcal{F}ix(\Theta) \cup \mathcal{F}lex(\Theta), \mathcal{V})$. Then for each variable $x \in \mathcal{IV}ar(\Theta)$ we have either $x\vartheta \in \mathcal{V}_{\mathrm{Ind}}$ or $\mathcal{H}ead(x\vartheta) \in \mathcal{F}ix(\Theta)$.

Let $\sigma = \{x \mapsto d \,|\, x \in \mathcal{IV}ar(\Theta) \cup \mathcal{IV}ar(\mathcal{R}an(\vartheta))\}$. Then $\vartheta\sigma$ is an F-unifier of Θ such that for each $x \in \mathcal{IV}ar(\Theta)$ we have $\mathcal{H}ead(x\vartheta\sigma) \in \mathcal{F}ix(\Theta) \cup \{d\}$. Since $\mathcal{IV}ar(\Theta) = \{x_1, \ldots, x_n\}$, for each $1 \le i \le n$ there exists a term $t_i \in \mathcal{J}$ and a substitution φ_i with $\mathcal{D}om(\varphi_i) = \mathcal{IV}ar(t_i)$ such that $x_i\vartheta\sigma = t_i\varphi_i$.

Moreover, we have that

1. $(\mathcal{D}om(\vartheta\sigma) \cup \mathcal{R}an(\vartheta\sigma)) \cap \mathcal{V}ar(\mathcal{J}) = \emptyset$,
2. $(\mathcal{D}om(\varphi_i) \cup \mathcal{R}an(\varphi_i)) \cap \mathcal{D}om(\varphi_j) = \emptyset$ for all $1 \le i < j \le n$.

It implies, on one hand, that for each $1 \le i \le n$ we have $x_i\vartheta\sigma\varphi_i = t_i\vartheta\sigma\varphi_i$, and, on the other hand, $\varphi_1 \cdots \varphi_n = \varphi_{\pi(1)} \cdots \varphi_{\pi(n)}$ for any permutation π of $1, \ldots, n$. Let ψ be $\varphi_1 \cdots \varphi_n$. Then we get $x_i\vartheta\sigma\psi = t_i\vartheta\sigma\psi$ for each $1 \le i \le n$.

Let r_i be t_i for each $1 \le i \le n$. Then the substitution $\vartheta\sigma\psi$ is an F-unifier of $h(s, x_1 \ldots, x_n) \simeq_F^? h(t, t_1, \ldots, t_n)$.

($\Leftarrow$) Let $r_1, \ldots, r_n \in \mathcal{J}$ be terms such that the problem $h(s, x_1, \ldots, x_n) \simeq_F^? h(t, r_1, \ldots, r_n)$ is F-unifiable with an F-unifier ϑ. Then, obviously, ϑ is an F-unifier of Θ. Moreover, $\mathcal{H}ead(x\vartheta) \in \mathcal{F}ix(\Theta) \cup d$ for each $x \in \mathcal{IV}ar(\Theta)$. Hence, ϑ is an F-unifier of Θ with individual variable restrictions. $\square$

3.3 Decidability of the Reduced Problem

We have to prove that unifiability of an F-unification problem Φ, whose signature consists of fixed arity individual function symbols and the only flexible arity flat individual function symbol *seq*, is decidable.

Let Ψ be an F-unification problem obtained from Φ by replacing each $\overline{x} \in \mathcal{SV}ar(\Phi)$ with a new individual variable x_Ψ. It is easy to see that Φ is unifiable iff Ψ is. Indeed, replacing each variable binding $\overline{x} \mapsto \ulcorner s_1, \ldots, s_n \urcorner$ in a unifier of Φ with $x_\Psi \mapsto seq(s_1, \ldots, s_n)$ yields a unifier of Ψ, and vice versa.

To prove decidability of Ψ we will use the combination method introduced by [BS96]:

Theorem 6 (Combination Method). *Let $E_1, \ldots, E_n$ be equational theories over disjoint signatures such that solvability of E_i-unification problems with linear constant restriction is decidable for each $1 \leq i \leq n$. Then solvability of elementary unification problems is decidable for the combined theory $E_1 \cup \ldots \cup E_n$.*

Linear constant restrictions are induced by a linear order $<$ on the set of variables and constants, demanding that, for a unifier σ, a constant c and a variable x, c must not occur in $x\sigma$ if $c > x$.

Theorem 7. *F-unifiability of Ψ is decidable.*

Proof. Let $\mathcal{F}_1 = \{seq\}$ and $\mathcal{F}_2 = \mathcal{F}ix(\Psi)$ be two disjoint signatures. Let E_1 be a flat equational theory over $\mathcal{F}_1$ and $\mathcal{V}_{\mathrm{Ind}}$, and let E_2 be a free equational theory over $\mathcal{F}_2$ and $\mathcal{V}_{\mathrm{Ind}}$. Then Ψ can be considered as an elementary unification problem in the combined theory $E_1 \cup E_2$. Then, by Theorem 6, we need to prove that solvability of E_1- and E_2-unification problems with linear constant restrictions is decidable.

E_1-unification problems are, in fact, word equations, while E_2-unification problems are Robinson unification problems. Decidability of word equations with linear constant restrictions, and Robinson unification with linear constant restrictions was proved by [BS91]. $\square$

4 Unification Type

The unification problem $\{f(a, \overline{x}) \simeq^?_\emptyset f(\overline{x}, a)\}$ has an infinite minimal complete set of unifiers $\{\{\overline{x} \mapsto \ulcorner\urcorner\}, \{\overline{x} \mapsto a\}, \{\overline{x} \mapsto \ulcorner a, a\urcorner\}, \ldots\}$. It implies that the syntactic sequence unification is at least infinitary. Now we show that it is not nullary. Note that by Definition 22 and Proposition 2 it is sufficient to show that every syntactic sequence unification problem has an almost minimal set of unifiers.

First we define the notion of norm for terms, sequences of terms, sequences of sequence function symbols and substitutions.

Definition 24. *The* norm of a term t*, denoted $\|t\|$, is defined as follows:*

1. *If $t \in \mathcal{V}_{\mathrm{Seq}}$, then $\|t\| = 1$.*
2. *If $t \in \mathcal{V}_{\mathrm{Ind}}$, then $\|t\| = 2$.*
3. *If $t = f(s_1, \ldots, s_n)$, $n \geq 0$, $f \in \mathcal{F}$, then $\|t\| = \sum_{i=1}^{n} \|s_i\| + 3$.*

The norm of a sequence of terms $t_1, \ldots, t_n$*, denoted $\|t_1, \ldots, t_n\|$, is defined as $\|t_1, \ldots, t_n\| = \sum_{i=1}^{n} \|t_i\|$. The norm of the empty sequence of terms is 0.*

The norm of a sequence of sequence function symbols $\overline{f_1}, \ldots, \overline{f_n}$ *is the length of the sequence: $\|\overline{f_1}, \ldots, \overline{f_n}\| = n$.*

From the definition it immediately follows that

1. $\|t\| \leq \|t\sigma\|$ for any term t and a substitution σ that is non-erasing on $\mathcal{SV}ar(t)$ modulo the free theory.
2. $\|\overline{f}\| \leq \|\overline{f}\sigma\|$ for any sequence function symbol $\overline{f}$ and a substitution σ.

Definition 25. *The* norm of a substitution σ *on* $\mathcal{X}$ *and* $\mathcal{Q}$*, denoted by* $\|\sigma\|_{\mathcal{X},\mathcal{Q}}$*, is defined as* $\|\sigma\|_{\mathcal{X},\mathcal{Q}} = \sum_{v\in\mathcal{X}} \|v\sigma\| + \sum_{\overline{f}\in\mathcal{Q}} \|\overline{f}\sigma\|$.

Definition 25 implies the following proposition:

Proposition 4. *For any* σ*,* ϑ*,* $\mathcal{X}$ *and* $\mathcal{Q}$*, if* $\sigma \preceq_E^{\mathcal{X},\mathcal{Q}} \vartheta$*, then* $\|\sigma\|_{\mathcal{X},Q} \leq \|\vartheta\|_{\mathcal{X},Q}$.

Let now $\mathcal{VCod}_{\mathrm{m}}(\sigma)$ be a multiset of terms $\{\!|v\sigma \mid v \in \mathcal{Dom}(\sigma)|\!\}$, $\mathcal{FCod}_{\mathrm{m}}(\sigma)$ be a multiset of sequence function symbols $\{\!|\overline{f}\sigma \mid \overline{f} \in \mathcal{Dom}(\sigma)|\!\}$, and $\mathcal{Ran}_{\mathrm{m}}(\sigma)$ be a multiset of variables occurring in $\mathcal{VCod}_{\mathrm{mult}}(\sigma)$. Then we define the following multiset of positive integers:

$$\begin{aligned}\mathcal{M}(\sigma) = \{\!|n \mid\ & \text{there exists a variable } v \text{ occurring } n \text{ times in } \mathcal{Ran}_{\mathrm{m}}(\sigma) \text{ or}\\ & \text{there exists a sequence function symbol } \overline{f} \text{ occurring}\\ & n \text{ times in } \mathcal{FCod}_{\mathrm{m}}(\sigma), \text{ and } n > 0|\!\}.\end{aligned}$$

Example 9. Let σ be

$$\begin{aligned}\{\ & x \mapsto y, y \mapsto f(\overline{x}, \overline{x}, x), \overline{x} \mapsto \ulcorner \overline{y}, \overline{x} \urcorner, \overline{z} \mapsto \ulcorner \overline{y}, \overline{x} \urcorner,\\ & \overline{f} \mapsto \ulcorner \overline{f_1}, \overline{f_2} \urcorner, \overline{g} \mapsto \ulcorner \overline{f_1}, \overline{f_2} \urcorner\}.\end{aligned}$$

Then

1. $\mathcal{VCod}_{\mathrm{m}}(\sigma) = \{\!|\underline{y}, \underline{f(\overline{x}, \overline{x}, x)}, \overline{y}, \overline{x}, \overline{y}, \overline{x}|\!\}$.
2. $\mathcal{FCod}_{\mathrm{m}}(\sigma) = \{\!|\overline{f_1}, \overline{f_2}, \overline{f_1}, \overline{f_2}|\!\}$.
3. $\mathcal{Ran}_{\mathrm{m}}(\sigma) = \{\!|y, x, \overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}|\!\}$.
4. $\mathcal{M}(\sigma) = \{\!|1, 1, 4, 2, 2, 2|\!\}$.

Theorem 8. *Let* Γ *be a syntactic unification problem, and let* $\mathcal{X} = \mathcal{Var}(\Gamma)$ *and* $\mathcal{Q} = \mathcal{SFun}(\Gamma)$*. Then every strictly decreasing chain of substitutions*

$$\sigma_1 \succ_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_2 \succ_{\emptyset}^{\mathcal{X},\mathcal{Q}} \cdots$$

in $\mathcal{U}_{\emptyset}(\Gamma)$ *is finite.*

Proof. Assume by contradiction that there exists a strictly decreasing infinite chain of substitutions $\sigma_1 \succ_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_2 \succ_{\emptyset}^{\mathcal{X},\mathcal{Q}} \cdots$ in $\mathcal{U}_{\emptyset}(\Gamma)$. Then for all $i > 0$ we have $\|\sigma_i\|_{\mathcal{X},\mathcal{Q}} \geq \|\sigma_{i+1}\|_{\mathcal{X},\mathcal{Q}}$ and, therefore, there must exists a k such that for all $j > k$ we have $\|\sigma_j\|_{\mathcal{X},\mathcal{Q}} = \|\sigma_{j+1}\|_{\mathcal{X},\mathcal{Q}}$.

Let for all $i \geq 0$ the substitution φ_i be non-erasing on $\mathcal{X}$ modulo the free theory such that $\sigma_{i+1}\varphi_i =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_i$. Then for all $j > k$ we have

- $x\varphi_j \in \mathcal{V}_{\mathrm{Ind}}$ for each individual variable $x \in \mathcal{Ran}(\sigma_{j+1}|_{\mathcal{X}})$.
- $\overline{x}\varphi_j \in \mathcal{V}_{\mathrm{Seq}}$ for each sequence variable $\overline{x} \in \mathcal{Ran}(\sigma_{j+1}|_{\mathcal{X}})$.
- $\overline{f}\varphi_j \in \mathcal{F}_{\mathrm{Seq}}$ for each sequence function symbol $\overline{f} \in \mathcal{FCod}(\sigma_{j+1}|_{\mathcal{Q}})$.

Otherwise we would have $\|\sigma_j\|_{\mathcal{X},\mathcal{Q}} > \|\sigma_{j+1}\|_{\mathcal{X},\mathcal{Q}}$.

Moreover, for all $j > k$ the substitution φ_j should satisfy at least one of the following properties:

1. It binds two distinct variables from $\mathcal{R}an(\sigma_{j+1}|_{\mathcal{X}})$ with the same variable.
2. It binds a variable from the set $\mathcal{R}an(\sigma_{j+1}|_{\mathcal{X}})$ with another variable from $\mathcal{R}an(\sigma_{j+1}|_{\mathcal{X}}) \setminus \mathcal{D}om(\varphi_j)$.
3. It binds two distinct sequence function symbols from $\mathcal{FC}od(\sigma_{j+1}|_{\mathcal{Q}})$ with the same sequence function symbol.
4. It binds a sequence function symbol from $\mathcal{FC}od(\sigma_{j+1}|_{\mathcal{Q}})$ with another sequence function symbol from $\mathcal{FC}od(\sigma_{j+1}|_{\mathcal{Q}}) \setminus \mathcal{D}om(\varphi_j)$.

Otherwise we would have $\sigma_j \dot{\approx}_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_{j+1}$.

Then we get that $\mathcal{M}(\sigma_j|_{\mathcal{X}\cup\mathcal{Q}}) >_m \mathcal{M}(\sigma_{j+1}|_{\mathcal{X}\cup\mathcal{Q}})$ for all $j > k$, where $>_m$ is a multiset extension of the ordering $>$ on positive integers. Therefore, we have a strictly decreasing infinite chain of multisets:

$$\mathcal{M}(\sigma_{k+1}|_{\mathcal{X}\cup\mathcal{Q}}) >_m \mathcal{M}(\sigma_{k+2}|_{\mathcal{X}\cup\mathcal{Q}}) >_m \cdots$$

But it is a contradiction, because $>_m$ is well-founded. $\square$

Theorem 8 implies that an almost minimal complete set of unifiers of a syntactic unification problem Γ exists: Just collect fixed representatives with respect to $\dot{\approx}_{\emptyset}^{\mathcal{V}ar(\Gamma),\mathcal{SF}un(\Gamma)}$ of almost minimal substitutions in $\mathcal{U}_{\emptyset}(\Gamma)$. Therefore, by Proposition 2, Γ has a minimal complete set of syntactic unifiers and, hence, the syntactic sequence unification can not be nullary. It implies that the syntactic sequence unification is infinitary.

5 Unification Procedure

We now present inference rules for deriving solutions for unification problems.

A *system* is either the symbol $\bot$ (representing failure), or a pair $\langle \Gamma; \sigma \rangle$, where Γ is a unification problem and σ is a substitution.

The inference system $\mathfrak{U}$ consists of the transformation rules on systems listed below. The function symbol g in the rule Partial Decomposition 2 is a new flexible arity individual function symbol. The sequence function symbols $\overline{f_1}$ and $\overline{f_2}$ in Splitting rule are new sequence function symbols of the same arity as $\overline{f}$ in the same rule. We assume that the indices $n, m, k, l \geq 0$.

Projection (P):

$$\langle \Gamma;\ \sigma \rangle \Longrightarrow \langle \Gamma\vartheta;\ \sigma\vartheta \rangle$$

where $\vartheta \neq \varepsilon$, $\mathcal{D}om(\vartheta) \subseteq \mathcal{SV}ar(\Gamma)$ and $\mathcal{C}od(\vartheta) = \emptyset$.

Trivial (T):

$$\langle \{s \simeq_{\emptyset}^{?} s\} \cup \Gamma';\ \sigma \rangle \Longrightarrow \langle \Gamma';\ \sigma \rangle.$$

Total Decomposition (TD):

$$\langle\{f(s_1,\ldots,s_n) \simeq^?_\emptyset f(t_1,\ldots,t_n\} \cup \Gamma';\ \sigma\rangle \Longrightarrow$$
$$\langle\{s_1 \simeq^?_\emptyset t_1,\ldots,s_n \simeq^?_\emptyset t_n\} \cup \Gamma';\ \sigma\rangle$$

if $f(s_1,\ldots,s_n) \neq f(t_1,\ldots,t_n)$, and $s_i, t_i \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F},\mathcal{V})$ for all $1 \leq i \leq n$.

Partial Decomposition 1 (PD1):

$$\langle\{f(s_1,\ldots,s_n) \simeq^?_\emptyset f(t_1,\ldots,t_m\} \cup \Gamma';\ \sigma\rangle \Longrightarrow$$
$$\langle\{s_1 \simeq^?_\emptyset t_1,\ldots,s_{k-1} \simeq^?_\emptyset t_{k-1}, f(s_k,\ldots,s_n) \simeq^?_\emptyset f(t_k,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle$$

if $f(s_1,\ldots,s_n) \neq f(t_1,\ldots,t_m)$, for some $1 < k \leq min(n,m)$, $s_k \in \mathcal{T}_{\mathrm{Seq}}(\mathcal{F},\mathcal{V})$ or $t_k \in \mathcal{T}_{\mathrm{Seq}}(\mathcal{F},\mathcal{V})$, and $s_i, t_i \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F},\mathcal{V})$ for all $1 \leq i < k$.

Partial Decomposition 2 (PD2):

$$\langle\{f(\overline{f}(r_1,\ldots,r_k),s_1,\ldots,s_n) \simeq^?_\emptyset f(\overline{f}(q_1,\ldots,q_l),t_1,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow$$
$$\langle\{g(r_1,\ldots,r_k) \simeq^?_\emptyset g(q_1,\ldots,q_l), f(s_1,\ldots,s_n) \simeq^?_\emptyset f(t_1,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle.$$

if $f(\overline{f}(r_1,\ldots,r_k),s_1,\ldots,s_n) \neq f(\overline{f}(q_1,\ldots,q_l),t_1,\ldots,t_m)$.

Orient 1 (O1):

$$\langle\{s \simeq^?_\emptyset x\} \cup \Gamma';\sigma\rangle \Longrightarrow \langle\{x \simeq^?_\emptyset s\} \cup \Gamma';\sigma\rangle$$
if $s \notin \mathcal{V}_{\mathrm{Ind}}$.

Orient 2 (O2):

$$\langle\{f(t,s_1,\ldots,s_n) \simeq^?_\emptyset f(\overline{x},t_1,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow$$
$$\langle\{f(\overline{x},s_1,\ldots,s_n) \simeq^?_\emptyset f(t,t_1,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle$$

if $t \notin \mathcal{V}_{\mathrm{Seq}}$.

Solve (S):

$$\langle\{x \simeq^?_\emptyset t\} \cup \Gamma';\sigma\rangle \Longrightarrow \langle\Gamma'\vartheta;\ \sigma\vartheta\rangle$$

if $x \notin \mathcal{IV}ar(t)$ and $\vartheta = \{x \mapsto t\}$.

Sequence Variable Elimination 1 (SVE1):

$$\langle\{f(\overline{x},s_1,\ldots,s_n) \simeq^?_\emptyset f(\overline{x},t_1,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow$$
$$\langle\{f(s_1,\ldots,s_n) \simeq^?_\emptyset f(t_1,\ldots,t_m)\} \cup \Gamma';\ \sigma\rangle$$

if $f(\overline{x},s_1,\ldots,s_n) \neq f(\overline{x},t_1,\ldots,t_m)$.

Sequence Variable Elimination 2 (SVE2):

$$\begin{array}{l}\langle\{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\} \cup \Gamma';\ \sigma\rangle \quad \Longrightarrow \\ \langle\{f(s_1, \ldots, s_n)\vartheta \simeq^?_\emptyset f(t_1, \ldots, t_m)\vartheta\} \cup \Gamma'\vartheta;\ \sigma\vartheta\rangle\end{array}$$

if $\overline{x} \notin \mathcal{SV}ar(t)$ and $\vartheta = \{\overline{x} \mapsto t\}$.

Widening 1 (W1):

$$\begin{array}{l}\langle\{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\} \cup \Gamma';\ \sigma\rangle \qquad \Longrightarrow \\ \langle\{f(\overline{x}, s_1\vartheta, \ldots, s_n\vartheta) \simeq^?_\emptyset f(t_1\vartheta, \ldots, t_m\vartheta)\} \cup \Gamma'\vartheta;\ \sigma\vartheta\rangle\end{array}$$

if $\overline{x} \notin \mathcal{SV}ar(t)$ and $\vartheta = \{\overline{x} \mapsto \ulcorner t, \overline{x}\urcorner\}$.

Widening 2 (W2):

$$\begin{array}{l}\langle\{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(\overline{y}, t_1, \ldots, t_m)\} \cup \Gamma';\ \sigma\rangle \qquad \Longrightarrow \\ \langle\{f(s_1\vartheta, \ldots, s_n\vartheta) \simeq^?_\emptyset f(\overline{y}, t_1\vartheta, \ldots, t_m\vartheta)\} \cup \Gamma'\vartheta;\ \sigma\vartheta\rangle\end{array}$$

where $\vartheta = \{\overline{y} \mapsto \ulcorner \overline{x}, \overline{y}\urcorner\}$.

Splitting (Sp):

$$\begin{array}{l}\langle\{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(\overline{f}(r_1, \ldots, r_k), t_1, \ldots, t_m)\} \cup \Gamma';\ \sigma\rangle \qquad \Longrightarrow \\ \langle\{f(s_1, \ldots, s_n)\vartheta \simeq^?_\emptyset f(\overline{f_2}(r_1, \ldots, r_k), t_1, \ldots, t_m)\vartheta\} \cup \Gamma'\vartheta;\ \sigma\vartheta\rangle\end{array}$$

if $\overline{x} \notin \mathcal{SV}ar(\overline{f}(r_1, \ldots, r_k))$ and $\vartheta = \{\overline{x} \mapsto \overline{f_1}(r_1, \ldots, r_k)\}\{\overline{f} \mapsto \ulcorner \overline{f_1}, \overline{f_2}\urcorner\}$.

We call the rules above except Projection the *basic transformation rules*. If we want to specify which rule was used to transform a system, we may use the rule name abbreviations as subscripts, e.g., we may write $\langle\Gamma_1;\ \sigma_1\rangle \Longrightarrow_{\mathsf{P}} \langle\Gamma_2;\ \sigma_2\rangle$ to indicate that the system $\langle\Gamma_1;\ \sigma_1\rangle$ was transformed to the system $\langle\Gamma_2;\ \sigma_2\rangle$ by Projection. We may also write $\langle\Gamma_1;\ \sigma_1\rangle \Longrightarrow_{\mathsf{BT}} \langle\Gamma_2;\ \sigma_2\rangle$ to indicate that the system $\langle\Gamma_1;\ \sigma_1\rangle$ was transformed to the system $\langle\Gamma_2;\ \sigma_2\rangle$ by some basic transformation rule.

We denote the transitive and reflexive-transitive closure of $\Longrightarrow$ respectively by $\Longrightarrow^+$ and $\Longrightarrow^*$.

Projection rule can be applied to the same system in (finitely many) different ways.

Example 10. Projection rule transforms the system $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\}; \varepsilon\rangle$ in the following three different ways:

1. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{P}} \langle\{f(a) \simeq^?_\emptyset f(a, \overline{y})\}; \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle$.
2. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{P}} \langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a)\}; \{\overline{y} \mapsto \ulcorner\urcorner\}\rangle$.
3. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{P}} \langle\{f(a) \simeq^?_\emptyset f(a)\}; \{\overline{x} \mapsto \ulcorner\urcorner,\ \overline{y} \mapsto \ulcorner\urcorner\}\rangle$.

Sequence Variable Elimination 2, Widening 1 and 2, and Splitting are non-deterministic basic transformation rules.

Example 11. The system $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\};\ \varepsilon\rangle$ can be transformed by Sequence Variable Elimination 2 and Widening 1 rules:

1. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{SVE2}} \langle\{f(a) \simeq^?_\emptyset f(\overline{y})\}; \{\overline{x} \mapsto a\}\rangle$.
2. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(a, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{W1}} \langle\{f(\overline{x}, a) \simeq^?_\emptyset f(\overline{y})\}; \{\overline{x} \mapsto \ulcorner a, \overline{x}\urcorner\}\rangle$.

The system $\langle\{f(\overline{x}, \overline{y}, \overline{a}) \simeq^?_\emptyset f(\overline{a}, \overline{z})\};\ \varepsilon\rangle$ can be transformed by Sequence Variable Elimination 2, Widening 1 and Splitting rules:

1. $\langle\{f(\overline{x}, \overline{y}, \overline{a}) \simeq^?_\emptyset f(\overline{a}, \overline{z})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{SVE2}} \langle\{f(\overline{y}, \overline{a}) \simeq^?_\emptyset f(\overline{z})\}; \{\overline{x} \mapsto \overline{a}\}\rangle$.
2. $\langle\{f(\overline{x}, \overline{y}, \overline{a}) \simeq^?_\emptyset f(\overline{a}, \overline{z})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{W1}} \langle\{f(\overline{x}, \overline{y}, \overline{a}) \simeq^?_\emptyset f(\overline{z})\};\ \{\overline{x} \mapsto \ulcorner\overline{a}, \overline{x}\urcorner\}\rangle$.
3. $\langle\{f(\overline{x}, \overline{y}, \overline{a}) \simeq^?_\emptyset f(\overline{a}, \overline{z})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{Sp}} \langle\{f(\overline{y}, \overline{a_1}, \overline{a_2}) \simeq^?_\emptyset f(\overline{a_2}, \overline{z})\};\ \{\overline{x} \mapsto \overline{a_1}, \overline{a} \mapsto \ulcorner\overline{a_1}, \overline{a_2}\urcorner\}\rangle$.

The system $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(\overline{z}, \overline{y})\};\ \varepsilon\rangle$ can be transformed by Sequence Variable Elimination 2, Widening 1 and Widening 2 rules:

1. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(\overline{z}, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{SVE2}} \langle\{f(a) \simeq^?_\emptyset f(\overline{y})\}; \{\overline{x} \mapsto \overline{z}\}\rangle$.
2. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(\overline{z}, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{W1}} \langle\{f(\overline{x}, a) \simeq^?_\emptyset f(\overline{y})\};\ \langle\{\overline{x} \mapsto \ulcorner\overline{z}, \overline{x}\urcorner\}\rangle$.
3. $\langle\{f(\overline{x}, a) \simeq^?_\emptyset f(\overline{z}, \overline{y})\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{W2}} \langle\{f(a) \simeq^?_\emptyset f(\overline{z}, \overline{y})\};\ \langle\{\overline{z} \mapsto \ulcorner\overline{x}, \overline{z}\urcorner\}\rangle$.

A *derivation* is a sequence of system transformations

$$\langle\Gamma_1; \sigma_1\rangle \Longrightarrow \langle\Gamma_2; \sigma_2\rangle \Longrightarrow \langle\Gamma_3; \sigma_3\rangle \Longrightarrow \cdots.$$

A derivation is said to be *fair* if any transformation rule which is continuously enabled is eventually applied. Any finite fair derivation $S_1 \Longrightarrow S_2 \Longrightarrow \cdots \Longrightarrow S_n$ is maximal, i.e., no further transformation rule can be applied on the system S_n.

Definition 26. *A* syntactic unification procedure *with sequence variables and flexible arity symbols is any program that takes a system $\langle\Gamma;\ \varepsilon\rangle$ as an input and uses the transformation rules of the inference system $\mathfrak{U}$ to generate a tree of fair derivations, called the* unification tree for Γ*, in the following way:*

1. *The root of the tree is labelled with $\langle\Gamma;\ \varepsilon\rangle$;*
2. *Each branch of the tree is a fair derivation either of the form*

$$\langle\Gamma;\ \varepsilon\rangle \Longrightarrow_{\mathsf{P}} \langle\Gamma_1;\ \sigma_1\rangle \Longrightarrow_{\mathsf{BT}} \langle\Gamma_2;\ \sigma_2\rangle \Longrightarrow_{\mathsf{BT}} \cdots$$

or of the form

$$\langle\Gamma;\ \varepsilon\rangle \Longrightarrow_{\mathsf{BT}} \langle\Gamma_1;\ \sigma_1\rangle \Longrightarrow_{\mathsf{BT}} \langle\Gamma_2;\ \sigma_2\rangle \Longrightarrow_{\mathsf{BT}} \cdots.$$

The nodes in the tree are systems.

3. *If several transformation rules, or different instances of the same transformation rule are applicable to a node in the tree, they are applied concurrently.*
4. *The decision procedure is applied to the root and to each node generated by a non-deterministic transformation rule, to decide whether the node contains a solvable unification problem. If the unification problem Δ in a node $\langle\Delta;\ \delta\rangle$ is unsolvable, then the branch is extended by $\langle\Delta;\ \delta\rangle \Longrightarrow_{\mathsf{DP}} \bot$.*

We denote the syntactic unification tree for Γ with $\mathcal{UT}ree_\emptyset(\Gamma)$.

The leaves of the unification tree of Γ, $\mathcal{UT}ree_\emptyset(\Gamma)$, are labeled either with the systems of the form $\langle\emptyset;\ \sigma\rangle$ or with the system $\bot$. The branches of $\mathcal{UT}ree_\emptyset(\Gamma)$ that end with leaves of the form $\langle\emptyset;\ \sigma\rangle$ are called *successful branches*, and those with the leaves $\bot$ are *failed branches*. We denote by $\mathcal{S}ol_\emptyset(\Gamma)$ the solution set for Γ, i.e., the set of all substitutions σ such that $\langle\emptyset;\ \sigma\rangle$ is a leaf of $\mathcal{UT}ree_\emptyset(\Gamma)$.

Application of a decision procedure can be a costly operation. Therefore, we give below a "lighter" version that can detect some failing cases. It can serve as a pre-filter before applying the full scale decision procedure. This version consists of the following eight rules:

Individual Variable Occurrence Check (IVOC):

$$\langle\{x \simeq^?_\emptyset t\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

if $x \neq t$ and $x \in \mathcal{IV}ar(t)$.

Symbol Clash 1 (SC1):

$$\langle\{f(s_1,\dots,s_n) \simeq^?_\emptyset g(t_1,\dots,t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

if $f \neq g$.

Symbol Clash 2 (SC2):

$$\langle\{f(\overline{f}(r_1,\dots,r_k), s_1,\dots,s_n) \simeq^?_\emptyset f(\overline{g}(q_1,\dots,q_l), t_1,\dots,t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

if $\overline{f} \neq \overline{g}$.

Arity Disagreement (AD):

$$\langle\{f(s_1,\dots,s_n) \simeq^?_\emptyset f(t_1,\dots,t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

if for all $1 \leq i \leq n$ and for all $1 \leq j \leq m$, $s_i \notin \mathcal{V}_{\mathrm{Seq}}$ and $t_j \notin \mathcal{V}_{\mathrm{Seq}}$.

Empty 1 (E1):

$$\langle\{f() \simeq^?_\emptyset f(t, t_1,\dots,t_n)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

Empty 2 (E2):

$$\langle\{f(s, s_1,\dots,s_n) \simeq^?_\emptyset f()\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

Sequence Variable Occurrence Check (SVOC):

$$\langle\{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

if $\overline{x} \neq t$ and $\overline{x} \in \mathcal{SV}ar(t)$.

Prefix Failure (PF):

$$\langle\{f(s, s_1, \ldots, s_n) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\} \cup \Gamma';\ \sigma\rangle \Longrightarrow \bot$$

if $s \in \mathcal{T}_{\mathrm{Seq}}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{V}_{\mathrm{Seq}}$ and $t \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F}, \mathcal{V})$, or $s \in \mathcal{T}_{\mathrm{Ind}}(\mathcal{F}, \mathcal{V})$ and $t \in \mathcal{T}_{\mathrm{Seq}}(\mathcal{F}, \mathcal{V}) \setminus \mathcal{V}_{\mathrm{Seq}}$.

These rules can detect many failing cases but not all. For instance, none of them apply to the unsolvable unification problem $\{f(\overline{x}) \simeq^?_\emptyset f(c, \overline{x})\}$.

Example 12. Let $\Gamma = \{f(\overline{x}) \simeq^?_\emptyset f(f(\overline{x}))\}$. Then the unification procedure immediately terminates because the pre-filter decision procedure (SVOC rule) shows that Γ is unsolvable:

$$\langle\{f(\overline{x}) \simeq^?_\emptyset f(f(\overline{x}))\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{DP}} \bot$$

Thus, $\mathcal{S}ol_\emptyset(\Gamma) = \emptyset$.

Example 13. Let $\Gamma = \{f(\overline{x}, x, \overline{y}) \simeq^?_\emptyset f(f(\overline{x}), x, a, b)$. Then the unification procedure generates the following seven derivations:

$$\begin{array}{ll}
\langle\{f(\overline{x}, x, \overline{y}) \simeq^?_\emptyset f(f(\overline{x}), x, a, b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x) \simeq^?_\emptyset f(f(), x, a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}} \bot
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x}, x, \overline{y}) \simeq^?_\emptyset f(f(\overline{x}), x, a, b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x, \overline{y}) \simeq^?_\emptyset f(f(), x, a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset f(), f(\overline{y}) \simeq^?_\emptyset f(x, a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(f(), a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f()\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f() \simeq^?_\emptyset f(a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto f()\}\rangle & \Longrightarrow_{\mathsf{DP}} \bot
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x}, x, \overline{y}) \simeq^?_\emptyset f(f(\overline{x}), x, a, b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x, \overline{y}) \simeq^?_\emptyset f(f(), x, a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset f(), f(\overline{y}) \simeq^?_\emptyset f(x, a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(f(), a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f()\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(a, b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f() \simeq^?_\emptyset f(b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}} \bot
\end{array}$$

$$
\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(f(\overline{x}),x,a,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{x \simeq^?_\emptyset f(), f(\overline{y}) \simeq^?_\emptyset f(x,a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(f(),a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f()\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a, \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f() \simeq^?_\emptyset f()\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a, b\urcorner\}\rangle & \Longrightarrow_{\mathsf{T}} \\
\langle\emptyset;\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a, b\urcorner\}\rangle &
\end{array}
$$

$$
\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(f(\overline{x}),x,a,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{x \simeq^?_\emptyset f(), f(\overline{y}) \simeq^?_\emptyset f(x,a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(f(),a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f()\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(f(),a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f()\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(a,b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(b)\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a, \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f()\};\ \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a, b, \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}} \perp
\end{array}
$$

$$
\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(f(\overline{x}),x,a,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(f(\overline{x}),x,a,b)\};\ \{\overline{y} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}} \perp
\end{array}
$$

$$
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(f(\overline{x}),x,a,b)\};\ \varepsilon\rangle \Longrightarrow_{\mathsf{DP}} \perp
$$

Therefore, $\mathcal{S}ol_\emptyset(\mathit{\Gamma}) = \{\{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto f(), \overline{y} \mapsto \ulcorner f(), a, b\urcorner\}\}$.

Example 14. Let $\mathit{\Gamma} = \{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\}$. Then the unification procedure generates the following nine derivations:

$$
\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x) \simeq^?_\emptyset f(a,x,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}} \perp
\end{array}
$$

$$
\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset a, f(\overline{y}) \simeq^?_\emptyset f(x,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(a,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto a\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f() \simeq^?_\emptyset f(b)\}; \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto a, \overline{y} \mapsto a\}\rangle & \Longrightarrow_{\mathsf{DP}} \quad \perp
\end{array}
$$

$$
\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset a, f(\overline{y}) \simeq^?_\emptyset f(x,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(a,b)\}; \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto a\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(b)\}; \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto a, \overline{y} \mapsto \ulcorner a, \overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f() \simeq^?_\emptyset f()\}; \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto a, \overline{y} \mapsto \ulcorner a, b\urcorner\}\rangle & \Longrightarrow_{\mathsf{T}} \\
\langle\emptyset; \{\overline{x} \mapsto \ulcorner\urcorner, x \mapsto a, \overline{y} \mapsto \ulcorner a, b\urcorner\}\rangle &
\end{array}
$$

$$\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\{\overline{x}\mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset a, f(\overline{y}) \simeq^?_\emptyset f(x,b)\};\{\overline{x}\mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(a,b)\};\{\overline{x}\mapsto \ulcorner\urcorner, x\mapsto a\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(b)\};\{\overline{x}\mapsto \ulcorner\urcorner, x\mapsto a, \overline{y}\mapsto \ulcorner a,\overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f()\};\{\overline{x}\mapsto \ulcorner\urcorner, x\mapsto a, \overline{y}\mapsto \ulcorner a,b,\overline{y}\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}}\ \bot
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(a,x,b)\};\{\overline{y}\mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f(x) \simeq^?_\emptyset f(x,b)\};\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto a\}\rangle & \Longrightarrow_{\mathsf{DP}}\ \bot
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(a,x,b)\};\{\overline{y}\mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(x,b)\};\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,\overline{x}\urcorner\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f(x) \simeq^?_\emptyset f(b)\};\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,x\urcorner\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset b\};\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,x\urcorner\}\rangle & \Longrightarrow_{\mathsf{S}} \\
\langle\emptyset;\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,b\urcorner, x\mapsto b\}\rangle &
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(a,x,b)\};\{\overline{y}\mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(x,b)\};\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,\overline{x}\urcorner\}\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{x},x) \simeq^?_\emptyset f(b)\};\{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,x,\overline{x}\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}}\ \bot
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f(x,\overline{y}) \simeq^?_\emptyset f(x,b)\};\{\overline{x}\mapsto a\}\rangle & \Longrightarrow_{\mathsf{PD1}} \\
\langle\{x \simeq^?_\emptyset x, f(\overline{y}) \simeq^?_\emptyset f(b)\};\{\overline{x}\mapsto a\}\rangle & \Longrightarrow_{\mathsf{T}} \\
\langle\{f(\overline{y}) \simeq^?_\emptyset f(b)\};\{\overline{x}\mapsto a, \overline{y}\mapsto b\}\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f() \simeq^?_\emptyset f()\};\{\overline{x}\mapsto a, \overline{y}\mapsto b\}\rangle & \Longrightarrow_{\mathsf{T}} \\
\langle\emptyset;\{\overline{x}\mapsto a, \overline{y}\mapsto b\}\rangle &
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(a,x,b)\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{W1}} \\
\langle\{f(\overline{x},x,\overline{y}) \simeq^?_\emptyset f(x,b)\};\{\overline{x}\mapsto \ulcorner a,\overline{x}\urcorner\}\rangle & \Longrightarrow_{\mathsf{DP}}\ \bot
\end{array}$$

$\mathcal{S}ol_\emptyset(\Gamma) = \{\{\overline{x}\mapsto \ulcorner\urcorner, x\mapsto a, \overline{y}\mapsto \ulcorner a,b\urcorner\}, \{\overline{y}\mapsto \ulcorner\urcorner, \overline{x}\mapsto \ulcorner a,b\urcorner, x\mapsto b\}, \{\overline{x}\mapsto a, \overline{y}\mapsto b\}\}$.

Example 15. Let $\Gamma = \{f(\overline{x},a) \simeq^?_\emptyset f(a,\overline{x})\}$. Then the unification procedure generates the following derivations:

$$\begin{array}{ll}
\langle\{f(\overline{x},a) \simeq^?_\emptyset f(a,\overline{x})\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{P}} \\
\langle\{f(a) \simeq^?_\emptyset f(a)\};\{\overline{x}\mapsto \ulcorner\urcorner\}\rangle & \Longrightarrow_{\mathsf{T}} \\
\langle\emptyset;\{\overline{x}\mapsto \ulcorner\urcorner\}\rangle &
\end{array}$$

$$\begin{array}{ll}
\langle\{f(\overline{x},a) \simeq^?_\emptyset f(a,\overline{x})\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{SVE2}} \\
\langle\{f(a) \simeq^?_\emptyset f(a)\};\{\overline{x}\mapsto a\}\rangle & \Longrightarrow_{\mathsf{T}} \\
\langle\emptyset;\{\overline{x}\mapsto a\}\rangle &
\end{array}$$

$$\begin{array}{ll}\langle\{f(\overline{x},a) \simeq^?_\emptyset f(a,\overline{x})\};\ \varepsilon\rangle & \Longrightarrow_{\mathsf{W1}}\\ \langle\{f(\overline{x},a) \simeq^?_\emptyset f(a,\overline{x})\}; \{\overline{x} \mapsto \ulcorner a,\overline{x}\urcorner\}\rangle & \Longrightarrow_{\mathsf{SVE2}}\\ \langle\{f(a) \simeq^?_\emptyset f(a)\}; \{\overline{x} \mapsto \ulcorner a,a\urcorner\}\rangle & \Longrightarrow_{\mathsf{T}}\\ \langle\emptyset; \{\overline{x} \mapsto \ulcorner a,a\urcorner\}\rangle & \\ & \\ \ldots & \end{array}$$

Then $\mathcal{S}ol_\emptyset(\Gamma)$ is infinite: $\mathcal{S}ol_\emptyset(\Gamma) = \{\{\overline{x} \mapsto \ulcorner\,\urcorner\}, \{\overline{x} \mapsto a\}, \{\overline{x} \mapsto \ulcorner a,a\urcorner\}, \ldots\}$.

5.1 Soundness

The following lemmata will be used to show that the inference system $\mathfrak{U}$ is sound.

Lemma 1. *If* $\langle\Gamma;\ \sigma\rangle \Longrightarrow \langle\Delta;\ \sigma\vartheta\rangle$, *then* $\mathcal{U}_\emptyset(\Delta) = \mathcal{U}_\emptyset(\Gamma\vartheta)$.

Proof. The non-trivial cases concern Solve, Sequence Variable Elimination 2, Widening 1 and 2 and Splitting rules.

Solve: If $x \notin \mathcal{IV}ar(t)$, then $x\theta \simeq t\theta$ for $\theta = \{x \mapsto t\}$ and $\Gamma\theta = \{x\theta \simeq^?_\emptyset t\theta\} \cup \Gamma'\theta$ and $\Delta = \Gamma'\theta$ have the same set of unifiers.

Sequence Variable Elimination 2 is similar to Solve.

Widening 1: Let $\Gamma = \{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\}$ such that $\overline{x} \notin \mathcal{SV}ar(t)$. Then, by W1, $\Delta = \{f(\overline{x}, s_1\vartheta, \ldots, s_n\vartheta) \simeq^?_\emptyset f(t_1\vartheta, \ldots, t_m\vartheta)\}$, where $\vartheta = \{\overline{x} \mapsto \ulcorner t, \overline{x}\urcorner\}$. Then $\Gamma\vartheta = \{f(t, \overline{x}, s_1\vartheta, \ldots, s_n\vartheta) \simeq^?_\emptyset f(t, t_1\vartheta, \ldots, t_m\vartheta)\}$. Then $\Gamma\vartheta$ and Δ have exactly the same set of unifiers.

Widening 2 is similar to widening 1.

Splitting. Let $\Gamma = \{f(\overline{x}, s_1, \ldots, s_n) \simeq^?_\emptyset f(\overline{g}(r_1, \ldots, r_k), t_1, \ldots, t_m)\}$ such that $\overline{x} \notin \mathcal{SV}ar(\overline{g}(r_1, \ldots, r_k))$. Then by Sp we get $\Delta = \{f(s_1, \ldots, s_n)\vartheta \simeq^?_\emptyset f(\overline{g_2}(r_1, \ldots, r_k), t_1, \ldots, t_m)\vartheta\}$, where $\vartheta = \{\overline{x} \mapsto \overline{g_1}(r_1, \ldots, r_k)\}\{\overline{g} \mapsto \ulcorner\overline{g_1}, \overline{g_2}\urcorner\}$. Then $\Gamma\theta$ and Δ have exactly the same set of unifiers, where

$$\begin{aligned}\Gamma\vartheta = \{\ & f(\overline{g_1}(r_1, \ldots, r_k), s_1, \ldots, s_n)\vartheta \simeq^?_\emptyset \\ & f(\overline{g_1}(r_1, \ldots, r_k), \overline{g_2}(r_1, \ldots, r_k), t_1, \ldots, t_m)\vartheta\}.\end{aligned}$$

$\square$

Corollary 1. *If* $\langle\Gamma;\ \sigma\rangle \Longrightarrow \langle\emptyset;\ \sigma\vartheta\rangle$, *then* $\vartheta \in \mathcal{U}_\emptyset(\Gamma)$.

Proof. By Lemma 1, $\mathcal{U}_\emptyset(\emptyset) = \mathcal{U}_\emptyset(\Gamma\vartheta)$, which implies that any substitution is a unifier of $\Gamma\vartheta$. In particular, $\varepsilon \in \mathcal{U}_\emptyset(\Gamma\vartheta)$ and, therefore, $\vartheta \in \mathcal{U}_\emptyset(\Gamma)$. $\square$

Lemma 2. *If* $\mathcal{U}_\emptyset(\Gamma\sigma) = \mathcal{U}_\emptyset(\Delta)$ *and* $\mathcal{U}_\emptyset(\Delta\vartheta) = \mathcal{U}_\emptyset(\Phi)$, *then* $\mathcal{U}_\emptyset(\Gamma\sigma\vartheta) = \mathcal{U}_\emptyset(\Phi)$.

Proof. $\varphi \in \mathcal{U}_\emptyset(\Gamma\sigma\vartheta)$ iff $\vartheta\varphi \in \mathcal{U}_\emptyset(\Gamma\sigma)$ iff $\vartheta\varphi \in \mathcal{U}_\emptyset(\Delta)$ iff $\varphi \in \mathcal{U}_\emptyset(\Delta\vartheta)$ iff $\varphi \in \mathcal{U}_\emptyset(\Phi)$. $\square$

Now we prove soundness.

Theorem 9 (Soundness). *If* $\langle\Gamma;\ \varepsilon\rangle \Longrightarrow^+ \langle\emptyset;\ \vartheta\rangle$, *then* $\vartheta \in \mathcal{U}_\emptyset(\Gamma)$.

Proof. If the derivation length is 1, then Corollary 1 implies the theorem. Assume the derivation length is greater than 1. Then the derivation has the form $\langle \Gamma;\ \varepsilon\rangle \Longrightarrow \langle \Delta_1;\ \delta_1\rangle \Longrightarrow \cdots \Longrightarrow \langle \Delta_n;\ \delta_1\delta_2\cdots\delta_n\rangle$ where $\Delta_n = \emptyset$, $\delta_1\delta_2\cdots\delta_n = \vartheta$ and $n > 1$. Applying Lemma 1 n times we get $\mathcal{U}_\emptyset(\Gamma\delta_1) = \mathcal{U}_\emptyset(\Delta_1)$, $\mathcal{U}_\emptyset(\Delta_1\delta_2) = \mathcal{U}_\emptyset(\Delta_2)$, $\ldots$, $\mathcal{U}_\emptyset(\Delta_{n-1}\delta_n) = \mathcal{U}_\emptyset(\Delta_n)$. After that, applying Lemma 2 $n-1$ times we get $\mathcal{U}_\emptyset(\Gamma\delta_1\cdots\delta_n) = \mathcal{U}_\emptyset(\Delta_n) = \mathcal{U}_\emptyset(\emptyset)$. Therefore, $\vartheta = \delta_1\cdots\delta_n \in \mathcal{U}_\emptyset(\Gamma)$. □

5.2 Completeness

Proving completeness is more involved than the soundness proof. In this section we prove completeness by showing that for any unifier ϑ of a unification problem Γ there exists a derivation from $\langle \Gamma;\ \varepsilon\rangle$ that terminates with success and the substitution in the last system of the derivation is strongly more general than ϑ.

First we construct a derivation from $\langle \Gamma;\ \varepsilon\rangle$ such that every substitution in the systems of the derivation is strongly more general than ϑ. To prove termination of the derivation, we need to define a complexity measure on the systems, introduce a well-founded ordering on the measures and show that every step in the derivation strictly decreases the measure.

By default, we work modulo the free theory. Therefore we will not mention it explicitly below.

We start with introducing notions needed later to define the complexity measure of a system.

Definition 27. *The* length of the image *of a variable v with respect to a substitution σ, denoted $\mathcal{ILen}(v, \sigma)$, is 1 if $v \in \mathcal{V}_{\mathrm{Ind}}$, and is the length of the sequence $v\sigma$ if $v \in \mathcal{V}_{\mathrm{Seq}}$.*

The length of the image *of $\mathcal{X}$ with respect to σ, denoted $\mathcal{ILen}(\mathcal{X}, \sigma)$, is defined as $\sum_{v\in\mathcal{X}} \mathcal{ILen}(v, \sigma)$.*

Lemma 3. *For any σ, ϑ, $\mathcal{X}$ and $\mathcal{Q}$, if $\sigma \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$ then $\mathcal{ILen}(\mathcal{X}, \sigma) \le \mathcal{ILen}(\mathcal{X}, \vartheta)$.*

Proof. From $\sigma \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$ we have that there exists a substitution φ, non-erasing on $\mathcal{X}$, such that $\sigma\varphi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. It implies $\mathcal{ILen}(\mathcal{SVar}(\mathcal{X}), \sigma) \le \mathcal{ILen}(\mathcal{SVar}(\mathcal{X}), \vartheta)$. On the other hand, $\mathcal{ILen}(\mathcal{IVar}(\mathcal{X}), \sigma) = \mathcal{ILen}(\mathcal{IVar}(\mathcal{X}), \vartheta)$. Therefore, we get that $\mathcal{ILen}(\mathcal{X}, \sigma) \le \mathcal{ILen}(\mathcal{X}, \vartheta)$. □

$\mathcal{LDif}(\mathcal{X}, \vartheta, \sigma)$ denotes $\mathcal{ILen}(\mathcal{X}, \vartheta) - \mathcal{ILen}(\mathcal{X}, \sigma)$, where $\mathcal{LDif}$ stands for *lengths difference.* It is easy to see that $\mathcal{LDif}(\mathcal{X}, \vartheta, \sigma) = \mathcal{LDif}(\mathcal{SVar}(\mathcal{X}), \vartheta, \sigma)$.

Lemma 4. *Let Γ be a general syntactic unification problem, $\mathcal{X} = \mathcal{Var}(\Gamma)$, $\mathcal{Q} = \mathcal{SFun}(\Gamma)$ and $\vartheta \in \mathcal{U}_\emptyset(\Gamma)$. If ϑ is non-erasing on $\mathcal{X}$, then there exists a derivation of the form*

$$\langle \Gamma_1;\ \sigma_1\rangle \Longrightarrow_{\mathsf{BT}} \langle \Gamma_2;\ \sigma_2\rangle \Longrightarrow_{\mathsf{BT}} \cdots \Longrightarrow_{\mathsf{BT}} \langle \emptyset, \sigma_k\rangle$$

with $\Gamma_1 = \Gamma$ and $\sigma_1 = \varepsilon$ such that $\sigma_k \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Proof. We prove the lemma in two steps. First, we construct a derivation of the form $\langle \Gamma_1;\ \sigma_1\rangle \Longrightarrow_{\mathsf{BT}} \langle \Gamma_2;\ \sigma_2\rangle \Longrightarrow_{\mathsf{BT}} \cdots$ with $\Gamma_1 = \Gamma$ and $\sigma_1 = \varepsilon$ such that for all $i \geq 1$ we have $\sigma_i \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, and then we show that the derivation terminates with success.

We construct the derivation recursively. We take $\Gamma_1 = \Gamma$, $\sigma_1 = \varepsilon$ and start the derivation from $\langle \Gamma_1;\ \sigma_1\rangle$. Obviously, $\sigma_1 \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Now, assume $\langle \Gamma_n;\ \sigma_n\rangle$, $n \geq 1$, $\Gamma_n \neq \emptyset$ belongs to the derivation and find a system $\langle \Gamma_{n+1};\ \sigma_{n+1}\rangle$ such that $\langle \Gamma_n;\ \sigma_n\rangle \Longrightarrow_{\mathsf{BT}} \langle \Gamma_{n+1};\ \sigma_{n+1}\rangle$ and $\sigma_{n+1} \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Since $\langle \Gamma_n;\ \sigma_n\rangle$ belongs to the derivation, we have $\sigma_n \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, i.e., there exists φ, non-erasing on $\mathcal{X}$, such that $\sigma_n\varphi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. Thus, we have $\sigma_n\varphi \in \mathcal{U}_{\emptyset}(\Gamma_1)$ and, therefore, $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_1\sigma_n)$. Moreover, using Lemma 2 we can show that $\mathcal{U}_{\emptyset}(\Gamma_1\sigma_n) = \mathcal{U}_{\emptyset}(\Gamma_n)$, which implies $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_n)$.

Without loss of generality, pick an arbitrary equation $s \simeq_{\emptyset}^{?} t$ from Γ_n and represent Γ_n as $\{s \simeq_{\emptyset}^{?} t\} \cup \Gamma_n'$. The pair $\langle s, t\rangle$ can have one of the following four forms:

1. A pair of identical terms;
2. A pair of distinct individual variables;
3. A pair of an individual variable and a non-variable term;
4. A pair of distinct non-variable terms.

Consider each of these cases separately:

1. We extend the derivation with the step $\langle \Gamma_n;\ \sigma_n\rangle \Longrightarrow_{\mathsf{T}} \langle \Gamma_n';\ \sigma_n\rangle$. Therefore, $\sigma_{n+1} = \sigma_n \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.
2. Let $s = x$ and $t = y$. Then $x\varphi \simeq_{\emptyset} y\varphi$. Let $\psi = \{x \mapsto y\}$. Then $\psi\varphi = \varphi$. Thus, $\sigma_n\psi\varphi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, which implies $\sigma_n\psi \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. Therefore, we can take $\Gamma_{n+1} = \Gamma_n'$, $\sigma_{n+1} = \sigma_n\psi$ and extend the derivation with the step $\langle \Gamma_n;\ \sigma_n\rangle \Longrightarrow_{\mathsf{S}} \langle \Gamma_{n+1};\ \sigma_{n+1}\rangle$.
3. If $s = x$ and t is a non-variable term that does not contain x, then we proceed as in the previous case, extending the derivation with Solve rule. The case when t contains x is impossible, because it leads to the contradiction $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_n) = \emptyset$. If $t = x$ and s is a non-variable term, then we take $\Gamma_{n+1} = \{x \simeq_{\emptyset}^{?} s\} \cup \Gamma_n'$, $\sigma_{n+1} = \sigma_n$ and extend the derivation with the step $\langle \Gamma_n;\ \sigma_n\rangle \Longrightarrow_{\mathsf{O1}} \langle \Gamma_{n+1};\ \sigma_{n+1}\rangle$.
4. Let $s = f(s_1, \ldots, s_k)$ and $t = f(t_1, \ldots, t_m)$ be distinct non-variable terms with the same heads such that the argument lists of the terms are non-empty (The cases when the heads are different, or when the list of arguments of one term is empty, lead to the contradiction $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_n) = \emptyset$.).
   If none of the arguments of s and t is a sequence term, then $k = m$ (otherwise we get a contradiction) and the derivation can be extended with the Total Decomposition rule, taking $\sigma_{n+1} = \sigma_n$.
   If there are sequence terms among the arguments of s and t, we have the following two alternatives:
   (a) Neither s_1 nor t_1 are sequence terms. Then we take $\sigma_{n+1} = \sigma_n$ and extend the derivation with Partial Decomposition 1 rule.

(b) s_1 or t_1 is a sequence term.

Let first assume that neither s_1 nor t_1 is a sequence variable. Then they both must be sequence terms with the same head (otherwise we get a contradiction) and we can proceed we Partial Decomposition 2 rule taking again $\sigma_{n+1} = \sigma_n$.

If s_1 and t_1 are the same sequence variable, then we extend the derivation with Sequence variable Elimination 1 rule again having $\sigma_{n+1} = \sigma_n$.

Next, assume only t_1 is a sequence variable. Then we perform the Orient 2 step with $\sigma_{n+1} = \sigma_n$.

Now, let s_1 be a sequence variable $\overline{x}$ and t_1 be a term different from s_1. We can have the following three cases:

i. t_1 is a sequence variable $\overline{y}$. Then we define substitutions ψ and ρ in three different ways as follows:
   - $\psi = \{\overline{x} \mapsto \overline{y}\}$ and $\rho = \varphi$, if $f(\overline{x})\varphi \simeq_\emptyset f(\overline{y})\varphi$;
   - $\psi = \{\overline{x} \mapsto \ulcorner \overline{y}, \overline{x} \urcorner\}$ and $\rho = \{\overline{x} \mapsto \ulcorner r_1, \ldots, r_l \urcorner\} \cup \varphi'$, where $\varphi' = \varphi|_{\mathcal{D}om(\varphi)\setminus\{\overline{x}\}}$, if there exists a non-empty sequence of terms $r_1, \ldots, r_l$ such that $f(\overline{x}\varphi) \simeq_\emptyset f(\overline{y}\varphi, r_1, \ldots, r_l)$;
   - $\psi = \{\overline{y} \mapsto \ulcorner \overline{x}, \overline{y} \urcorner\}$ and $\rho = \{\overline{y} \mapsto \ulcorner r_1, \ldots, r_l \urcorner\} \cup \varphi'$, where $\varphi' = \varphi|_{\mathcal{D}om(\varphi)\setminus\{\overline{y}\}}$, if there exists a non-empty sequence of terms $r_1, \ldots, r_l$ such that $f(\overline{y}\varphi) \simeq_\emptyset f(\overline{x}\varphi, r_1, \ldots, r_l)$.

   Since $\varphi \in \mathcal{U}_\emptyset(\Gamma_n)$, these three cases for ψ are the only possibilities to get Γ_{n+1} from Γ_n. Thus, we can take $\sigma_{n+1} = \sigma_n\psi$ and extend the derivation either by Sequence Variable Elimination 2, Widening 1 or Widening 2 rules, depending on the cases for ψ. Since $\psi\rho = \varphi$, we have $\sigma_n\psi\rho =_\emptyset^{\mathcal{X},\mathcal{Q}} \vartheta$ and, thus, $\sigma_{n+1}\rho =_\emptyset^{\mathcal{X},\mathcal{Q}} \vartheta$. Moreover, ρ is non-erasing on the set $\mathcal{X}$, which implies that $\sigma_{n+1} \preceq_\emptyset^{\mathcal{X},\mathcal{Q}} \vartheta$.

ii. t_1 is an individual term. Then t_1 does not contain $\overline{x}$ and we can proceed as in the previous case, with the only difference that now ψ can have only two alternatives instead of three. Therefore, we can extend the derivation either by Sequence Variable Elimination 2 or Widening 1 rule ensuring $\sigma_{n+1} = \sigma_n\psi \preceq_\emptyset^{\mathcal{X},\mathcal{Q}} \vartheta$.

iii. t_1 is a sequence term $\overline{f}(r_1, \ldots, r_l)$. Then we define substitutions ψ and ρ in three different ways as follows:
   - $\psi = \{\overline{x} \mapsto t\}$ and $\rho = \varphi$, if $f(\overline{x})\varphi \simeq_\emptyset f(t)\varphi$;
   - $\psi = \{\overline{x} \mapsto \ulcorner t, \overline{x} \urcorner\}$ and $\rho = \{\overline{x} \mapsto \ulcorner q_1, \ldots, q_j \urcorner\} \cup \varphi'$, where $\varphi' = \varphi|_{\mathcal{D}om(\varphi)\setminus\{\overline{x}\}}$, if there exists a non-empty sequence of terms $q_1, \ldots, q_j$ such that $f(\overline{x})\varphi \simeq_\emptyset f(t\varphi, q_1, \ldots, q_j)$;
   - $\psi = \{\overline{x} \mapsto \overline{f_1}(r_1, \ldots, r_l)\}\{\overline{f} \mapsto \ulcorner \overline{f_1}, \overline{f_2} \urcorner\}$, $\rho = \varphi|_{\mathcal{D}om(\varphi)\setminus\{\overline{f}\}}$, if for some sequence function symbols $\overline{f_1}$ and $\overline{f_2}$, $f(\overline{f}(r_1, \ldots, r_l))\varphi = f(\overline{f_1}(r_1, \ldots, r_l), \overline{f_2}(r_1, \ldots, r_l))\varphi$ and $f(\overline{x})\varphi = f(\overline{f_1}(r_1, \ldots, r_l))\varphi$.

   Since $\varphi \in \mathcal{U}_\emptyset(\Gamma_n)$, these three cases for ψ are the only possibilities to get Γ_{n+1} from Γ_n. Thus, we can take $\sigma_{n+1} = \sigma_n\psi$ and get Γ_{n+1} from Γ_n either by Sequence Variable Elimination 2, Widening 1 or Splitting rules, depending on the cases for ψ. Since $\psi\rho = \varphi$, we have

$\sigma_n\psi\rho =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$ and, thus, $\sigma_{n+1}\rho =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. Moreover, ρ is non-erasing on the set $\mathcal{X}$, which implies that $\sigma_{n+1} \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Now, we have to show that the constructed derivation terminates with success. Define a complexity measure (with respect to a given substitution and a set of variables) on systems as a 7-tuple of integers, ordered by the lexicographic ordering on tuples of integers as follows: the tuple $\langle m_1, m_2, m_3, m_4, m_5, m_6, m_7\rangle$ is a complexity measure of a system $\langle \Delta;\ \sigma\rangle$ with respect to a substitution λ and a set of variables $\mathcal{Y}$, if

m_1 = The number of distinct variables in Δ;
$m_2 = \mathcal{LD}if(\mathcal{Y}, \lambda, \sigma)$.
m_3 = The number of symbols in Δ;
m_4 = The number of occurrences of sequence function symbols in Δ;
m_5 = The number of subterms in Δ of the form $f(s_1, \ldots, s_n)$, where s_1 is not a sequence term;
m_6 = The number of equations in Δ of the form $t \simeq_{\emptyset}^{?} x$, where t is not an individual variable;
m_7 = The number of equations in Δ that have the form $f(s, s_1, \ldots, s_n) \simeq_{\emptyset}^{?} f(\overline{x}, t_1, \ldots, t_m)$, where s is not a sequence variable.

First, observe that all these numbers except m_2 are, indeed, natural numbers. As for m_2, depending on λ and σ, it can also be negative. But for each system $\langle \Gamma_i;\ \sigma_i\rangle$ in the derivation we constructed above, for the substitution ϑ and for the set $\mathcal{X}$, by Lemma 3 we have $\mathcal{LD}if(\mathcal{X}, \vartheta, \sigma_i) \geq 0$, i.e., m_2 is a natural number for each $\langle \Gamma_i;\ \sigma_i\rangle$. Thus, the ordering on complexity measures of systems (with respect to ϑ and $\mathcal{X}$) in the derivation is well-founded.

Now we show that each step in the derivation strictly reduces the complexity measure of the system with respect to $\mathcal{X}$ and ϑ.

- Trivial rule does not increase m_1, does not change m_2 and decreases m_3.
- Total Decomposition leaves m_1 and m_2 unchanged and decreases m_3.
- Partial Decomposition 1 leaves m_1, m_2, m_3 and m_4 unchanged and decreases m_5.
- Partial Decomposition 2 leaves m_1, m_2 and m_3 unchanged and decreases m_4.
- Orient 1 decreases m_6 and leaves the others unchanged.
- Orient 2 decreases m_7 and leaves the others unchanged.
- Solve decreases m_1.
- Sequence Variable Elimination 1 does not increase m_1, does not change m_2 and decreases m_3.
- Sequence Variable Elimination 2 decreases m_1.
- Widening 1 does not change m_1 but decreases m_2: Let the corresponding step in the derivation be $\langle \Gamma_i;\ \sigma_i\rangle \Longrightarrow_{\mathsf{W1}} \langle \Gamma_{i+1};\ \sigma_{i+1}\rangle$ where $\Gamma_i = \{f(\overline{x}, s_1, \ldots, s_n) \simeq_{\emptyset}^{?} f(t, t_1, \ldots, t_m)\} \cup \Gamma_i'$, $\Gamma_{i+1} = \{f(\overline{x}, s_1\varphi, \ldots, s_n\varphi) \simeq_{\emptyset}^{?} f(t_1\varphi, \ldots, t_m\varphi)\} \cup \Gamma_i'\varphi$ and $\sigma_{i+1} = \sigma_i\varphi$, with $\overline{x} \notin \mathcal{SV}ar(t)$ and $\varphi = \{\overline{x} \mapsto \ulcorner t, \overline{x}\urcorner\}$. It is clear that the step does not change the number of distinct

variables in the systems. Moreover, $\overline{x} \in \mathcal{X}$, because $\overline{x} \in \mathcal{SV}ar(\Gamma_i)$ and $\mathcal{SV}ar(\Gamma_i) \subseteq \mathcal{X}$ (The last inclusion follows from the simple observation that no rule in the inference system $\mathfrak{U}$ introduces a new variable.). For all $\overline{y} \in \mathcal{X} \setminus \{\overline{x}\}$, $\mathcal{ILen}(\overline{y}, \sigma_{i+1}) \geq \mathcal{ILen}(\overline{y}, \sigma_i)$. As for $\overline{x}$ itself, if $\overline{x} \in \mathcal{VD}om(\sigma_i)$, then $\overline{x} \in \{\overline{x}\sigma_i\}$ (otherwise it would have been impossible to have $\overline{x}$ in $\mathcal{SV}ar(\Gamma_{i+1})$) and therefore $\mathcal{ILen}(\overline{x}, \sigma_{i+1}) > \mathcal{ILen}(\overline{x}, \sigma_i)$. If $\overline{x} \notin \mathcal{VD}om(\sigma_i)$, then $\mathcal{ILen}(\overline{x}, \sigma_{i+1}) = 2 > 1 = \mathcal{ILen}(\overline{x}, \sigma_i)$. It means that $\mathcal{ILen}(\mathcal{X}, \sigma_{i+1}) > \mathcal{ILen}(\mathcal{X}, \sigma_i)$, i.e., $\mathcal{LD}if(\mathcal{X}, \vartheta, \sigma_i) > \mathcal{LD}if(\mathcal{X}, \vartheta, \sigma_{i+1})$.
- Widening 2, like Widening 1, does not change m_1 but decreases m_2.
- Splitting decreases m_1.

Thus, every step reduces the complexity of the systems in the derivation. Therefore, the derivation terminates. Let $\langle \Gamma_k, \sigma_k \rangle$ be the last system in the derivation. Then, on the one hand, $\sigma_k \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, on the other hand, $\Gamma_k = \emptyset$ (otherwise we could make another basic transformation step), which finishes the proof. $\square$

Lemma 5. *Let Γ be a general syntactic unification problem, $\mathcal{X} = \mathcal{V}ar(\Gamma)$, $\mathcal{Q} = \mathcal{SF}un(\Gamma)$, and $\vartheta \in \mathcal{U}_{\emptyset}(\Gamma)$. If ϑ is erasing on $\mathcal{X}$, then there exists a derivation of the form $\langle \Gamma_0;\ \sigma_0 \rangle \Longrightarrow_{\mathsf{P}} \langle \Gamma_1;\ \sigma_1 \rangle \Longrightarrow_{\mathsf{BT}} \langle \Gamma_2;\ \sigma_2 \rangle \Longrightarrow_{\mathsf{BT}} \cdots \Longrightarrow_{\mathsf{BT}} \langle \emptyset;\ \sigma_n \rangle$ with $\Gamma_0 = \Gamma$ and $\sigma_0 = \varepsilon$ such that $\sigma_n \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.*

Proof. Assume that $\overline{x_1} \mapsto \ulcorner\urcorner, \ldots, \overline{x_k} \mapsto \ulcorner\urcorner$, where $k > 0$ and $\overline{x_1}, \ldots, \overline{x_k} \in \mathcal{X}$, are all the bindings in ϑ that bind a sequence variable from $\mathcal{X}$ with the empty sequence. Let $\sigma_1 = \{\overline{x_1} \mapsto \ulcorner\urcorner, \ldots, \overline{x_k} \mapsto \ulcorner\urcorner\}$, $\Gamma_1 = \Gamma_0\sigma_1$, and make Projection the first step of derivation $\langle \Gamma_0;\ \sigma_0 \rangle \Longrightarrow_{\mathsf{P}} \langle \Gamma_1;\ \sigma_1 \rangle$.

We have $\sigma_1\sigma_1 = \sigma_1$. Let φ be $\vartheta|_{\mathcal{D}om(\vartheta)\setminus\mathcal{D}om(\sigma_1)}$. Then $\vartheta = \sigma_1\varphi = \sigma_1\sigma_1\varphi \in \mathcal{U}_{\emptyset}(\Gamma_0)$. Therefore, $\sigma_1\varphi \in \mathcal{U}_{\emptyset}(\Gamma_0\sigma_1)$, i.e., $\vartheta \in \mathcal{U}_{\emptyset}(\Gamma_1)$. Moreover, ϑ is non-erasing on $\mathcal{V}ar(\Gamma_1)$. Then, by Lemma 4, there exists a derivation $\langle \Delta_1;\ \delta_1 \rangle \Longrightarrow_{\mathsf{BT}} \langle \Delta_2;\ \delta_2 \rangle \Longrightarrow_{\mathsf{BT}} \cdots \Longrightarrow_{\mathsf{BT}} \langle \emptyset;\ \delta_n \rangle$ such that $\Delta_1 = \Gamma_1$, $\delta_1 = \varepsilon$ and $\delta_n \preceq_{\emptyset}^{\mathcal{X}',\mathcal{Q}'} \vartheta$, where $\mathcal{X}' = \mathcal{V}ar(\Gamma_1)$ and $\mathcal{Q}' = \mathcal{SF}un(\Gamma_1)$, i.e., there exists a substitution ψ that is non-erasing on $\mathcal{X}'$ such that $\delta_n\psi =_{\emptyset}^{\mathcal{X}',\mathcal{Q}'} \vartheta$. Since $\mathcal{Q}' = \mathcal{Q}$, we, in fact, have $\delta_n\psi =_{\emptyset}^{\mathcal{X}',\mathcal{Q}} \vartheta$. Moreover, since the rules in $\mathfrak{U}$ do not introduce new variables, $\mathcal{VD}om(\delta_n) \subseteq \mathcal{X}'$ and $\mathcal{R}an(\delta_n) \subseteq \mathcal{X}'$. From the latter we can assume that ψ is non-erasing on any finite superset of $\mathcal{X}'$, in particular, on $\mathcal{X}$. From the former we have $\sigma_1\delta_n = \sigma_1 \cup \delta_n$ and, finally, $\sigma_1\delta_n\psi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Note that for all $i \geq 1$, if $\langle \Delta_{i+1};\ \delta_{i+1} \rangle$ is obtained from $\langle \Delta_i;\ \delta_i \rangle$ by some rule of the system $\mathfrak{U}$, then $\langle \Delta_{i+1};\ \sigma_1\delta_{i+1} \rangle$ can be obtained from $\langle \Delta_i;\ \sigma_1\delta_i \rangle$ exactly by the same rule. Thus, taking $\Gamma_i = \Delta_i$ and $\sigma_i = \sigma_1\delta_i$ for all $1 < i \leq n$, we get the derivation $\langle \Gamma;\ \varepsilon \rangle \Longrightarrow_{\mathsf{P}} \langle \Gamma_1;\ \sigma_1 \rangle \Longrightarrow_{\mathsf{BT}} \langle \Gamma_2;\ \sigma_2 \rangle \Longrightarrow_{\mathsf{BT}} \cdots \Longrightarrow_{\mathsf{BT}} \langle \emptyset;\ \sigma_n \rangle$ such that $\sigma_n \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. $\square$

From Theorem 9, Lemma 4, Lemma 5 and the fact that $\preceq_{E}^{\mathcal{X},\mathcal{Q}} \subseteq \leq_{E}^{\mathcal{X},\mathcal{Q}}$, by Definition 26 and Definition 19 we get the completeness theorem:

Theorem 10 (Completeness). *Let Γ be a general syntactic unification problem. Then $\mathcal{S}ol_{\emptyset}(\Gamma)$ is a complete set of syntactic unifiers of Γ.*

5.3 Almost Minimality

The set $\mathcal{S}ol_\emptyset(\Gamma)$, in general, is not minimal with respect to $\mathcal{V}ar(\Gamma)$ and $\mathcal{SF}un(\Gamma)$. Just consider $\Gamma = \{f(\overline{x}) \simeq^?_\emptyset f(\overline{y})\}$, then $\mathcal{S}ol_\emptyset(\Gamma) = \{\{\overline{x} \mapsto \overline{y}\},\ \{\overline{x} \mapsto \ulcorner\urcorner, \overline{y} \mapsto \ulcorner\urcorner\}\}$. However, it can be shown that $\mathcal{S}ol_\emptyset(\Gamma)$ is almost minimal with respect to $\mathcal{V}ar(\Gamma)$ and $\mathcal{SF}un(\Gamma)$. In fact, we will prove a stronger statement: $\mathcal{S}ol_\emptyset(\Gamma)$ is almost disjoint with respect to $\mathcal{V}ar(\Gamma)$ and $\mathcal{SF}un(\Gamma)$.

First we introduce some notation. Let $\Gamma = \{s \simeq^?_\emptyset t\} \cup \Gamma'$ be a general syntactic unification problem where the equation $s \simeq^?_\emptyset t$ is selected to perform a transformation step. We denote by $\mathcal{S}ub(\Gamma)$ the set of substitutions $\{\delta \mid \langle\{s \simeq^?_\emptyset t\} \cup \Gamma';\ \varepsilon\rangle \Longrightarrow \langle\Delta;\ \delta\rangle$ for some $\Delta\}$ and by $\mathcal{P}roj(\Gamma)$ the set of projecting substitutions $\{\pi \mid \langle\{s \simeq^?_\emptyset t\} \cup \Gamma';\ \varepsilon\rangle \Longrightarrow_{\mathsf{P}} \langle\{s\pi \simeq^?_\emptyset t\pi\} \cup \Gamma'\pi;\ \pi\rangle\}$.

Lemma 6. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then $\mathcal{P}roj(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. Assume by contradiction that $\mathcal{P}roj(\Gamma)$ is not almost disjoint. Then there exist two substitutions σ, $\vartheta \in \mathcal{P}roj(\Gamma)$ and a substitution φ such that $\sigma \preceq^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$ and $\vartheta \preceq^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$. Since $\sigma \neq \vartheta$, we assume without loss of generality that there exists $\overline{x} \in \mathcal{X}$ such that $\overline{x} \mapsto \ulcorner\urcorner \in \vartheta$ and $\overline{x} \mapsto \ulcorner\urcorner \notin \sigma$. But then $\overline{x} \mapsto \ulcorner\urcorner \in \varphi$, and therefore $\overline{x} \mapsto \ulcorner\urcorner \in \delta$ for any substitution δ such that $\sigma\delta =^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$. It contradicts the fact that $\sigma \preceq^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$. □

Lemma 7. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then $\mathcal{S}ub(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ iff $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. ($\Rightarrow$) Since $\mathcal{S}ub(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$, any subset of $\mathcal{S}ub(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$.

($\Leftarrow$) Since, by the assumption and Lemma 6, respectively, $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ and $\mathcal{P}roj(\Gamma)$ are almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$, what remains to show is that no $\sigma \in \mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ and $\vartheta \in \mathcal{P}roj(\Gamma)$ can not have a common strong $\emptyset$-instance on $\mathcal{X}$ and $\mathcal{Q}$. Assume by contradiction that there exists φ such that $\sigma \preceq^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$ and $\vartheta \preceq^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$. Then there exists $\overline{x} \in \mathcal{X}$ such that $\overline{x} \mapsto \ulcorner\urcorner \in \vartheta$ and $\overline{x} \mapsto \ulcorner\urcorner \notin \sigma$. But then $\overline{x} \mapsto \ulcorner\urcorner \in \varphi$, and therefore $\overline{x} \mapsto \ulcorner\urcorner \in \delta$ for any substitution δ such that $\sigma\delta =^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$. It contradicts the fact that $\sigma \preceq^{\mathcal{X},\mathcal{Q}}_\emptyset \varphi$. □

Lemma 8. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{V}ar(\Gamma)$. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. A singleton or the empty set of substitutions is trivially almost disjoint with respect to any finite sets of variables and function symbols. Therefore, we consider only the cases when $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ can contain more than one element. It leads to assuming that the equation selected from Γ for transformation has a form $\{f(s, s_1, \ldots, s_n) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\}$, where s is a sequence variable and t is a term different from s. We can have the following cases:

1. Let $s = \overline{x}$ and $t = \overline{y}$. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{\overline{x} \mapsto \overline{y}\}$, $\sigma_2 = \{\overline{x} \mapsto \ulcorner \overline{y}, \overline{x} \urcorner\}$ and $\sigma_3 = \{\overline{y} \mapsto \ulcorner \overline{x}, \overline{y} \urcorner\}$. Assume by contradiction that $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ is not almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Then there must exist $1 \leq i < j \leq 3$ such that σ_i and σ_j have a common strong $\emptyset$-instance, i.e., there exist ϑ and φ such that $\sigma_i \vartheta =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_j \varphi$, and ϑ and φ are non-erasing on $\mathcal{X}$.
   (a) Assume $i = 1$ and $j = 2$. Then we have $\overline{x}\sigma_1\vartheta = \overline{y}\vartheta$, $\overline{x}\sigma_2\varphi = \ulcorner \overline{y}\varphi, \overline{x}\varphi \urcorner$, $\overline{y}\sigma_1\vartheta = \overline{y}\vartheta$, $\overline{y}\sigma_2\varphi = \overline{y}\varphi$. It implies that $\overline{x}\varphi$ should be the empty sequence, i.e., φ must contain the binding $\overline{x} \mapsto \ulcorner\urcorner$, but this contradicts the fact that φ is non-erasing on $\mathcal{X}$.
   (b) Assume $i = 1$ and $j = 3$. Then $\overline{x}\sigma_1\vartheta = \overline{y}\vartheta$, $\overline{x}\sigma_3\varphi = \overline{x}\varphi$, $\overline{y}\sigma_1\vartheta = \overline{y}\vartheta$, $\overline{y}\sigma_3\varphi = \ulcorner \overline{x}\varphi, \overline{y}\varphi \urcorner$. It implies that $\overline{y}\varphi$ should be the empty sequence, i.e., φ must contain the binding $\overline{y} \mapsto \ulcorner\urcorner$, but this contradicts the fact that φ is non-erasing on $\mathcal{X}$.
   (c) Assume $i = 2$ and $j = 3$. Then $\overline{x}\sigma_2\vartheta = \ulcorner \overline{y}\vartheta, \overline{x}\vartheta \urcorner$, $\overline{x}\sigma_3\varphi = \overline{x}\varphi$, $\overline{y}\sigma_2\vartheta = \overline{y}\vartheta$, $\overline{y}\sigma_3\varphi = \ulcorner \overline{x}\varphi, \overline{y}\varphi \urcorner$. It implies that $\overline{y}\varphi$ and $\overline{x}\vartheta$ should be the empty sequences, i.e., φ must contain the binding $\overline{y} \mapsto \ulcorner\urcorner$ and ϑ must contain the binding $\overline{x} \mapsto \ulcorner\urcorner$, but this contradicts the fact that φ and ϑ are non-erasing on $\mathcal{X}$.
2. If s occurs in t, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \emptyset$ and is trivially almost disjoint. Otherwise, if t is not a sequence term, we can proceed similarly to the case above, having only two elements in $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$. If t is a non-variable sequence term, assume that $s = \overline{x}$ and $t = \overline{f}(r_1, \ldots, r_k)$, $k \geq 0$. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{\overline{x} \mapsto \overline{f}(r_1, \ldots, r_k)\}$, $\sigma_2 = \{\overline{x} \mapsto \ulcorner \overline{f}(r_1, \ldots, r_k), \overline{x} \urcorner\}$ and $\sigma_3 = \{\overline{x} \mapsto \overline{f_1}(r_1, \ldots, r_k)\}\{\overline{f} \mapsto \ulcorner \overline{f_1}, \overline{f_2} \urcorner\}$. We can show as in the case (1a) above that σ_1 and σ_2 can not have a common strong $\emptyset$-instance. As for σ_1 and σ_3, they can not even have a common $\emptyset$-instance, as well as σ_2 and σ_3, because the sequence function symbol $\overline{f_2}$ can not be eliminated from σ_3. It implies that $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. $\square$

From Lemma 6, Lemma 7 and Lemma 8 we get the following theorem:

Theorem 11. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then $\mathcal{S}ub(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.*

Now, we show that substitutions from $\mathcal{S}ub(\Gamma)$ preserve almost disjointness. First, we prove the two lemmata below.

Lemma 9. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then every substitution $\sigma \in \mathcal{P}roj(\Gamma)$ is almost-disjointness preserving on $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. Let ϑ_1 and ϑ_2 be two substitutions such that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to the set of variables $\cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma) = \mathcal{X} \setminus \mathcal{VD}om(\sigma)$ and the set of sequence function symbols $\cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma) = \mathcal{Q}$. Assume by contradiction that the set $\{\sigma\vartheta_1, \sigma\vartheta_2\}$ is not almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Then there exist substitutions φ_1 and φ_2, both non-erasing on $\mathcal{X}$, such that $\sigma\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma\vartheta_2\varphi_2$. But then $\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X}\setminus\mathcal{VD}om(\sigma),\mathcal{Q}} \vartheta_2\varphi_2$, which contradicts the fact that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X} \setminus \mathcal{VD}om(\sigma)$ and $\mathcal{Q}$. □

Lemma 10. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then every substitution $\sigma \in \mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ is almost-disjointness preserving on $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. Without loss of generality, pick an arbitrary equation $s \simeq_{\emptyset}^{?} t$ from Γ and represent Γ as $\{s \simeq_{\emptyset}^{?} t\} \cup \Gamma'$. The pair $\langle s, t\rangle$ can have one of the following four forms:

1. A pair of identical terms;
2. A pair of distinct individual variables;
3. A pair of an individual variable and a non-variable term;
4. A pair of distinct non-variable terms.

We consider each of these cases.

1. In this case $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ contains only ε, that trivially preserves almost disjointness.
2. Let $s = x$ and $t = y$. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \mathcal{S}ub(\Gamma) = \{\sigma\}$, where $\sigma = \{x \mapsto y\}$. We have $\cup_{v\in\mathcal{X}}\mathcal{V}ar(v\sigma) = \mathcal{X} \setminus \{x\}$ and $\cup_{\overline{f}\in\mathcal{Q}}\mathcal{SF}un(\overline{f}\sigma) = \mathcal{Q}$. Let ϑ_1 and ϑ_2 be two substitutions such that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X} \setminus \{x\}$ and $\mathcal{Q}$ and show that $\{\sigma\vartheta_1, \sigma\vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Assume by contradiction that it is not. Then there exist substitutions φ_1 and φ_2 such that $\sigma\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma\vartheta_2\varphi_2$. It implies that $u\sigma\vartheta_1\varphi_1 \simeq_{\emptyset} u\sigma\vartheta_2\varphi_2$ for all $u \in \mathcal{X} \setminus \{x\}$. But since $u\sigma = u$ for all $u \in \mathcal{X} \setminus \{x\}$, we get $u\vartheta_1\varphi_1 \simeq_{\emptyset} u\vartheta_2\varphi_2$ for all $u \in \mathcal{X} \setminus \{x\}$ that contradicts the fact that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X} \setminus \{x\}$ and $\mathcal{Q}$.
3. If $t = x$, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$ contains only $\{\varepsilon\}$ that preserves almost disjointness. If $s = x$ and $x \in \mathcal{V}ar(t)$, then $\mathcal{S}ub(\Gamma) = \emptyset$ and the theorem is trivially true. Assume $s = x$ and $x \notin \mathcal{V}ar(t)$. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\sigma\}$, where $\sigma = \{x \mapsto t\}$. Moreover, $\cup_{v\in\mathcal{X}}\mathcal{V}ar(v\sigma) = \mathcal{X} \setminus \{x\}$ and $\cup_{\overline{f}\in\mathcal{Q}}\mathcal{SF}un(\overline{f}\sigma) = \mathcal{Q}$ and we can proceed as in the previous case.
4. Let $s = f(s_1, \ldots, s_k)$ and $t = f(t_1, \ldots, t_m)$ be distinct non-variable terms with the same heads such that the argument lists of the terms are non-empty (The cases when the heads are different, or when the list of arguments of one term is empty, we have $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \emptyset$ and the theorem is trivially true.).
   If none of the arguments of s and t is a sequence term, then $k \neq m$ gives $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \emptyset$, and $k = m$ gives $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\varepsilon\}$. In both cases the theorem is true.
   If there are sequence terms among the arguments of s and t, we have the following two alternatives:
   (a) Neither s_1 nor t_1 are sequence terms. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\varepsilon\}$ and the theorem is true.

(b) s_1 or t_1 is a sequence term.
Let first assume that neither s_1 nor t_1 is a sequence variable. If they both are sequence terms with different heads, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \emptyset$. If they both are sequence terms with the same head, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\varepsilon\}$. The theorem is true in either case.
If s_1 and t_1 are the same sequence variable, or only t_1 is a sequence variable, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\varepsilon\}$ and the theorem is again true.
Now, let s_1 be a sequence variable $\overline{x}$ and t_1 be a term different from s_1. Then we have the following three cases:
i. t_1 is a sequence variable $\overline{y}$. Then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{\overline{x} \mapsto \overline{y}\}$, $\sigma_2 = \{\overline{x} \mapsto \ulcorner \overline{y}, \overline{x} \urcorner\}$ and $\sigma_3 = \{\overline{y} \mapsto \ulcorner \overline{x}, \overline{y} \urcorner\}$. We have

$$\cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma_1) = \mathcal{X} \setminus \{\overline{x}\}$$
$$\cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma_2) = \cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma_3) = \mathcal{X}$$
$$\cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma_1) = \cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma_2) = \cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma_3) = \mathcal{Q}$$

We will show that each of σ_1, σ_2 and σ_3 preserves almost disjointness on $\mathcal{X}$ and $\mathcal{Q}$.
For σ_1 we can proceed similarly to the case (2) above.
As for σ_2, let ϑ_1 and ϑ_2 be two substitutions such that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Assume by contradiction that $\{\sigma_2\vartheta_1, \sigma_2\vartheta_2\}$ is not almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Then there exist φ_1 and φ_2, both non-erasing on $\mathcal{X}$, such that $\sigma_2\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_2\vartheta_2\varphi_2$. But then $\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta_2\varphi_2$, that contradicts the fact that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.
The case with σ_3 is similar to the case with σ_2.
ii. t_1 is an individual term. If $\overline{x}$ occurs in t_1, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \emptyset$ and the theorem is trivially true. Otherwise, we proceed as in the case above with t_1 a sequence variable, having only two elements in $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma)$.
iii. t_1 is a sequence term $\overline{f}(r_1, \ldots, r_k)$. If $\overline{x}$ occurs in t_1, then $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \emptyset$ and the theorem is trivially true. Otherwise $\mathcal{S}ub(\Gamma) \setminus \mathcal{P}roj(\Gamma) = \{\sigma_1, \sigma_2, \sigma_3\}$, where

$$\sigma_1 = \{\overline{x} \mapsto \overline{f}(r_1, \ldots, r_k)\}$$
$$\sigma_2 = \{\overline{x} \mapsto \ulcorner \overline{f}(r_1, \ldots, r_k), \overline{x} \urcorner\}$$
$$\sigma_3 = \{\overline{x} \mapsto \overline{f_1}(r_1, \ldots, r_k), \overline{f} \mapsto \ulcorner \overline{f_1}, \overline{f_2} \urcorner\}$$

Moreover, we have

$$\cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma_1) = \cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma_3) = \mathcal{X} \setminus \{\overline{x}\}$$
$$\cup_{v \in \mathcal{X}} \mathcal{V}ar(v\sigma_2) = \mathcal{X}$$
$$\cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma_1) = \cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma_2) = \mathcal{Q}$$
$$\cup_{\overline{f} \in \mathcal{Q}} \mathcal{SF}un(\overline{f}\sigma_3) = \mathcal{Q} \cup \{\overline{f_1}, \overline{f_2}\} \setminus \{\overline{f}\}.$$

We will show that each of σ_1, σ_2 and σ_3 preserves almost disjointness on $\mathcal{X}$ and $\mathcal{Q}$.

For σ_1 and σ_2 we can repeat the reasoning used in the corresponding cases above where both s_1 and t_1 were sequence variables.

As for σ_3, let ϑ_1 and ϑ_2 be two substitutions such that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}\setminus\{\overline{x}\}$ and $\mathcal{Q}\cup\{\overline{f_1}, \overline{f_2}\}\setminus\{\overline{f}\}$. Assume by contradiction that $\{\sigma_3\vartheta_1, \sigma_3\vartheta_2\}$ is not almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Then there exist φ_1 and φ_2, both non-erasing on $\mathcal{X}$, such that $\sigma_3\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_3\vartheta_2\varphi_2$. But then $\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X}',\mathcal{Q}'} \vartheta_2\varphi_2$, where $\mathcal{X}' = \mathcal{X} \setminus \{\overline{x}\}$ and $\mathcal{Q}' = \mathcal{Q} \cup \{\overline{f_1}, \overline{f_2}\} \setminus \{\overline{f}\}$, that contradicts the fact that $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to $\mathcal{X} \setminus \{\overline{x}\}$ and $\mathcal{Q} \cup \{\overline{f_1}, \overline{f_2}\} \setminus \{\overline{f}\}$. $\square$

From Lemma 9 and Lemma 10 we get the following theorem.

Theorem 12. *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then every substitution $\sigma \in \mathcal{S}ub(\Gamma)$ is almost-disjointness preserving on $\mathcal{X}$ and $\mathcal{Q}$.*

Lemma 11. *Let σ_1 and σ_2 be two substitutions such that $\{\sigma_1, \sigma_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Let ϑ_1 and ϑ_2 be two non-erasing substitutions on $\mathcal{X}$. Then $\{\sigma_1\vartheta_1, \sigma_2\vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. Assume by contradiction that $\{\sigma_1\vartheta_1, \sigma_2\vartheta_2\}$ is not almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. Then there exist substitutions φ_1 and φ_2, both non-erasing on $\mathcal{X}$, such that $\sigma_1\vartheta_1\varphi_1 =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \sigma_2\vartheta_2\varphi_2$, which contradicts the fact that $\{\sigma_1, \sigma_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. $\square$

The next result is the almost disjointness theorem:

Theorem 13 (Almost Disjointness). *Let Γ be a general syntactic unification problem and let $\mathcal{X} = \mathcal{V}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$. Then $\mathcal{S}ol_{\emptyset}(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.*

Proof. Let σ_1, σ_2 be two substitutions from $\mathcal{S}ol_{\emptyset}(\Gamma)$ and let

$$\begin{aligned}\langle\Gamma,\varepsilon\rangle \Longrightarrow \langle\Gamma_1,\gamma_1\rangle \Longrightarrow \ldots \Longrightarrow \langle\Gamma_n,\gamma_1\gamma_2\cdots\gamma_n\rangle \Longrightarrow \\ \Longrightarrow \langle\Delta_1,\delta_1\rangle \Longrightarrow \ldots \Longrightarrow \langle\Delta_m,\delta_m\rangle \Longrightarrow \langle\emptyset,\sigma_1\rangle\end{aligned}$$

and

$$\begin{aligned}\langle\Gamma,\varepsilon\rangle \Longrightarrow \langle\Gamma_1,\gamma_1\rangle \Longrightarrow \ldots \Longrightarrow \langle\Gamma_n,\gamma_1\gamma_2\cdots\gamma_n\rangle \Longrightarrow \\ \Longrightarrow \langle\Phi_1,\varphi_1\rangle \Longrightarrow \ldots \Longrightarrow \langle\Phi_k,\varphi_k\rangle \Longrightarrow \langle\emptyset,\sigma_2\rangle\end{aligned}$$

be two derivations, leading respectively to $\langle\emptyset,\sigma_1\rangle$ and $\langle\emptyset,\sigma_2\rangle$ and having the common part

$$\langle\Gamma,\varepsilon\rangle \Longrightarrow \langle\Gamma_1,\gamma_1\rangle \Longrightarrow \langle\Gamma_2,\gamma_1\gamma_2\rangle \Longrightarrow \ldots \Longrightarrow \langle\Gamma_n,\gamma_1\gamma_2\cdots\gamma_n\rangle.$$

Let $\gamma_0 = \varepsilon$. First, we show that $\gamma_0\gamma_1 \cdots \gamma_n$ is almost-disjointness preserving with respect to $\mathcal{X}$ and $\mathcal{Q}$.

We use induction on n. If $n = 0$, by Definition 21, $\gamma_0 = \varepsilon$ is almost-disjointness preserving with respect to $\mathcal{X}$ and $\mathcal{Q}$. Then, assume $\gamma_0\gamma_1 \cdots \gamma_{n_0}$ is almost-disjointness preserving with respect to $\mathcal{X}$ and $\mathcal{Q}$ for $n_0 > 0$, and show that $\gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1}$ also preserves almost disjointness with respect to $\mathcal{X}$ and $\mathcal{Q}$.

Let ϑ_1, ϑ_2 be two substitutions such that the set $\{\vartheta_1, \vartheta_2\}$ is almost disjoint with respect to the set of variables $\cup_{v\in\mathcal{X}}\mathcal{V}ar(v\gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1})$ and the set of sequence function symbols $\cup_{\overline{f}\in\mathcal{Q}}\mathcal{SF}un(\overline{f}\gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1})$. We have to prove that the set $\{\gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1}\vartheta_1,\ \gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1}\vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. The substitution $\gamma_{n_0+1} \in \mathcal{S}ub(\Gamma_{n_0+1})$, therefore, by Theorem 12, γ_{n_0+1} is almost-disjointness preserving with respect to $\mathcal{V}ar(\Gamma_{n_0+1})$ and $\mathcal{SF}un(\Gamma_{n_0+1})$. It implies that the set $\{\gamma_{n_0+1}\vartheta_1, \gamma_{n_0+1}\vartheta_2\}$ is almost disjoint with respect to the sets $\mathcal{V}ar(\Gamma_{n_0+1})$ and $\mathcal{SF}un(\Gamma_{n_0+1})$. From the induction hypothesis and the facts $\mathcal{V}ar(\Gamma_{n_0+1}) = \cup_{v\in\mathcal{X}}\mathcal{V}ar(v\gamma_0\gamma_1 \cdots \gamma_{n_0})$ and $\mathcal{SF}un(\Gamma_{n_0+1}) = \cup_{\overline{f}\in\mathcal{Q}}\mathcal{SF}un(\overline{f}\gamma_0\gamma_1 \cdots \gamma_{n_0})$, by Definition 21, we conclude that the set of substitutions $\{\gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1}\vartheta_1,\ \gamma_0\gamma_1 \cdots \gamma_{n_0}\gamma_{n_0+1}\vartheta_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$.

Thus, the substitution $\gamma_0\gamma_1 \ldots \gamma_n$ is almost-disjointness preserving with respect to $\mathcal{X}$ and $\mathcal{Q}$. Now we will show that the set $\{\delta_1 \cdots \delta_m, \varphi_1 \cdots \varphi_k\}$ is almost disjoint with respect to $\mathcal{V}ar(\Gamma_n)$ and $\mathcal{SF}un(\Gamma_n)$.

The substitutions $\delta_1, \varphi_1 \in \mathcal{S}ub(\Gamma_n)$ and, by Theorem 11, the set $\{\delta_1, \varphi_1\}$ is almost disjoint with respect to $\mathcal{V}ar(\Gamma_n)$ and $\mathcal{SF}un(\Gamma_n)$. The substitutions $\delta_2 \cdots \delta_m$ and $\varphi_2 \cdots \varphi_k$ are non-erasing on $\mathcal{V}ar(\Gamma_n)$. Therefore, by Lemma 11, the set $\{\delta_1\delta_2 \cdots \delta_m, \varphi_1\varphi_2 \cdots \varphi_k\}$ is almost disjoint with respect to $\mathcal{V}ar(\Gamma_n)$ and $\mathcal{SF}un(\Gamma_n)$. Since we have $\mathcal{V}ar(\Gamma_n) = \cup_{v\in\mathcal{X}}\mathcal{V}ar(v\gamma_0\gamma_1 \cdots \gamma_n)$ and $\mathcal{SF}un(\Gamma_n) = \cup_{\overline{f}\in\mathcal{Q}}\mathcal{SF}un(\overline{f}\gamma_0\gamma_1 \cdots \gamma_n)$, therefore, by Definition 21, we obtain that the set $\{\gamma_0\gamma_1 \cdots \gamma_n\delta_1 \cdots \delta_m, \gamma_0\gamma_1 \cdots \gamma_n\varphi_1 \cdots \varphi_k\}$ is almost disjoint with respect to the sets $\mathcal{X}$ and $\mathcal{Q}$. Since $\gamma_0\gamma_1 \cdots \gamma_n\delta_1 \cdots \delta_m = \sigma_1$ and $\gamma_0\gamma_1 \cdots \gamma_n\varphi_1 \cdots \varphi_k = \sigma_2$, we obtained that $\{\sigma_1, \sigma_2\}$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$, i.e., an arbitrary two element subset of $\mathcal{S}ol_\emptyset(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. By Definition 20, we conclude that $\mathcal{S}ol_\emptyset(\Gamma)$ is almost disjoint with respect to $\mathcal{X}$ and $\mathcal{Q}$. $\square$

Theorem 10, Theorem 13 and Proposition 3 imply the main result of this section:

Theorem 14 (Main Theorem). *Let Γ be a general syntactic unification problem. Then $\mathcal{S}ol_\emptyset(\Gamma) = amcu_\emptyset(\Gamma)$.*

6 Special Cases

6.1 Sequence Variables Occur only in the Last Argument Position

The case when sequence variables occur only in the last argument positions in terms is quite interesting. It turns out that for this case unification becomes unitary, there is no need to apply the decision procedure, and we can use a modified version of the inference system $\mathfrak{U}$. We denote this version by $\mathfrak{U}'$. It consists of the following inference rules: Trivial, Total Decomposition, Partial Decomposition 1 and 2, Individual variable occurrence check, Symbol Clash 1 and 2, Arity Disagreement, Prefix Failure, Orient 1 and 2, and Solve. In addition, we have the inference rules Empty' 1 and 2, Sequence Variable Occurrence Check', Sequence Variable Deletion' 1 and 2, and Sequence Variable Elimination', defined as follows:

Empty' 1 (E'1):

$$\langle \{f() \simeq^?_\emptyset f(t, t_1, \ldots, t_n)\};\ \sigma\rangle \Longrightarrow \bot$$

if $t \notin \mathcal{V}_{\mathrm{Seq}}$.

Empty' 2 (E'2):

$$\langle \{f(s, s_1, \ldots, s_n) \simeq^?_\emptyset f()\};\ \sigma\rangle \Longrightarrow \bot$$

if $s \notin \mathcal{V}_{\mathrm{Seq}}$.

Sequence Variable Occurrence Check' (SVOC'):

$$\langle \{f(\overline{x}) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\};\ \sigma\rangle \Longrightarrow \bot$$

if $\overline{x} \neq t$ and $\overline{x} \in \mathcal{SV}ar(t, t_1, \ldots, t_m)$.

Sequence Variable Deletion 1 (SVD1):

$$\langle \{f(\overline{x}) \simeq^?_\emptyset f()\};\ \sigma\rangle \Longrightarrow \langle \emptyset;\ \sigma\vartheta\rangle$$

where $\vartheta = \{\overline{x} \mapsto \ulcorner\urcorner\}$.

Sequence Variable Deletion 2 (SVD2):

$$\langle \{f() \simeq^?_\emptyset f(\overline{x})\};\ \sigma\rangle \Longrightarrow \langle \emptyset;\ \sigma\vartheta\rangle$$

where $\vartheta = \{\overline{x} \mapsto \ulcorner\urcorner\}$.

Sequence Variable Elimination' (SVE'):

$$\langle \{f(\overline{x}) \simeq^?_\emptyset f(t, t_1, \ldots, t_m)\};\ \sigma\rangle \Longrightarrow \langle \emptyset;\ \sigma\vartheta\rangle$$

if $\overline{x} \notin \mathcal{SV}ar(t, t_1, \ldots, t_m)$ and $\vartheta = \{\overline{x} \mapsto \ulcorner t, t_1, \ldots, t_m\urcorner\}$.

Remark 3. Note that every rule in $\mathfrak{U}'$ is deterministic, i.e., for each system $\langle \Gamma;\ \sigma\rangle$, where $\Gamma \neq \emptyset$, there is only one rule in $\mathfrak{U}'$ that can be applied on $\langle \Gamma;\ \sigma\rangle$. No rules can be applied on $\bot$ and $\langle \emptyset;\ \sigma\rangle$.

First, we show that every sequence of transformations in $\mathfrak{U}'$ terminates.

Lemma 12. *Let Γ be a unification problem such that all sequence variables occur in the last argument positions in subterms of Γ. Then every sequence of transformations in $\mathfrak{U}'$*

$$\langle \Gamma;\ \varepsilon\rangle \Longrightarrow \langle \Gamma_1;\ \sigma_1\rangle \Longrightarrow \langle \Gamma_2;\ \sigma_2\rangle \Longrightarrow \cdots$$

terminates either with $\perp$ or $\langle \emptyset; \sigma\rangle$ for some σ.

Proof. For each system $\langle \Delta;\ \sigma\rangle$, we define a complexity measure as a tuple of natural numbers $\langle n_1, n_2, n_3, n_4, n_5, n_6\rangle$, where

$n_1 =$ The number of distinct variables in Δ;
$n_2 =$ The number of symbols in Δ;
$n_3 =$ The number of occurrences of sequence function symbols in Δ;
$n_4 =$ The number of subterms in Δ of the form $f(s_1, \dots, s_n)$, where s_1 is not a sequence term;
$n_5 =$ The number of equations in Δ of the form $t \simeq^?_\emptyset x$, where t is not an individual variable;
$n_6 =$ The number of equations in Δ of the form $f(s_1, \dots, s_n) \simeq^?_\emptyset f(\overline{x})$, where $n, m \geq 0$ and s is not a sequence variable.

The complexity measures are ordered by the well-founded lexicographic ordering on natural numbers.

Now we prove that every rule in $\mathfrak{U}'$ reduces the complexity measure of systems. For the rules in $\mathfrak{U}'$ that belong also to $\mathfrak{U}$ it was shown in the proof of Lemma 4. As for the other rules, Empty' 1 and 2 and Sequence variable occurrence check' do not increase n_1 and reduce n_2, Sequence Variable Deletion' 1 and 2 and Sequence Variable Elimination' reduce n_1.

A rule in $\mathfrak{U}'$ can always be applied to a system different from $\perp$ and $\langle \emptyset;\ \sigma\rangle$ for any σ. Thus the sequence of transformations terminates either with $\perp$ or with $\langle \emptyset; \sigma\rangle$ for some σ. □

The first major result about $\mathfrak{U}'$ shows that it does indeed produce a unifier.

Theorem 15 (Soundness of $\mathfrak{U}'$). *Let Γ be a general syntactic sequence unification problem where sequence variables occur only in the last argument positions. Then if $\langle \Gamma;\ \varepsilon\rangle \Longrightarrow^+ \langle \emptyset;\ \vartheta\rangle$, then $\vartheta \in \mathcal{U}_\emptyset(\Gamma)$.*

Proof. The theorem can be proved in the similar way as the soundness result for the inference system $\mathfrak{U}$. The counterparts of the lemmata 1 and 2, for $\mathfrak{U}'$, needed for this proof, can be established easily. □

Another major result shows that $\mathfrak{U}'$ can calculate a most general unifier for a unifiable general syntactic unification problem with sequence variables in the last argument positions in subterms.

Theorem 16 (Completeness of $\mathfrak{U}'$). *Let Γ be a general syntactic sequence unification problem where sequence variables occur only in the last argument positions and ϑ be a unifier of Γ. Then any maximal sequence of transformations in $\mathfrak{U}'$ $\langle \Gamma; \varepsilon\rangle \Longrightarrow \cdots$ must end in some system $\langle \emptyset; \sigma\rangle$ such that $\sigma \lesssim^{\mathcal{X},\mathcal{Q}}_\emptyset \vartheta$, where $\mathcal{X} = \mathcal{SV}ar(\Gamma)$ and $\mathcal{Q} = \mathcal{SF}un(\Gamma)$.*

Proof. By Remark 3, for each system there exists only one maximal derivation starting from the system. We first construct such a maximal derivation

$$\langle \Gamma_0; \sigma_0 \rangle \Longrightarrow \langle \Gamma_1; \sigma_1 \rangle \Longrightarrow \cdots$$

with $\Gamma_0 = \Gamma$, $\sigma_0 = \varepsilon$ and show that $\sigma_i \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$ for all $i \geq 0$.

We construct the derivation recursively starting from $\langle \Gamma_0; \sigma_0 \rangle$. Obviously, $\sigma_0 = \varepsilon \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Now, assume $\langle \Gamma_n;\ \sigma_n \rangle$, $n \geq 0$ belongs to the derivation and show that for the system $\langle \Gamma_{n+1};\ \sigma_{n+1} \rangle$ that extends the derivation we have $\sigma_{n+1} \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$.

Since $\langle \Gamma_n;\ \sigma_n \rangle$ belongs to the derivation, we have $\sigma_n \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, i.e., there exists φ such that $\sigma_n \varphi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. Thus, we have $\sigma_n \varphi \in \mathcal{U}_{\emptyset}(\Gamma_0)$ and, therefore, $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_0 \sigma_n)$. Moreover, since the counterpart of Lemma 2 holds for $\mathfrak{U}'$, we can show that $\mathcal{U}_{\emptyset}(\Gamma_0 \sigma_n) = \mathcal{U}_{\emptyset}(\Gamma_n)$, which implies $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_n)$.

Without loss of generality, pick an arbitrary equation $s \simeq_{\emptyset}^{?} t$ from Γ_n and represent Γ_n as $\{s \simeq_{\emptyset}^{?} t\} \cup \Gamma_n'$. The pair $\langle s, t \rangle$ can have one of the following four forms:

1. A pair of identical terms;
2. A pair of distinct individual variables;
3. A pair of an individual variable and a non-variable term;
4. A pair of distinct non-variable terms.

The first three cases can be proved in the same way as the corresponding cases in Lemma 4. We consider the fourth one.

Let $s = f(s_1, \ldots, s_k)$ and $t = f(t_1, \ldots, t_m)$ be distinct non-variable terms with the same heads (The case when the heads are different leads to the contradiction $\varphi \in \mathcal{U}_{\emptyset}(\Gamma_n) = \emptyset$.).

If none of the arguments of s and t is a sequence term, then $k = m$ (otherwise we get a contradiction) and the derivation can be extended with the Total Decomposition rule, taking $\sigma_{n+1} = \sigma_n$.

If there are sequence terms among the arguments of s and t, we have the following four alternatives:

1. If $m = 0$, then $k = 1$ and $s_1 = \overline{x}$ is a sequence variable. Then $\overline{x} \mapsto \ulcorner\urcorner \in \varphi$, because φ is a unifier of Γ_n. Let $\psi = \{\overline{x} \mapsto \ulcorner\urcorner\}$. Then $\psi\varphi = \varphi$. Thus, $\sigma_n \psi \varphi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, which implies $\sigma_n \psi \preceq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. Then we extend the derivation with Sequence Variable Deletion 1 rule with $\sigma_{n+1} = \sigma_n \psi$.
2. If $n = 0$, then $m = 1$ with $t_1 = \overline{x}$ and we proceed as in the previous case, extending the derivation with Sequence Variable Deletion 2 rule.
3. Assume $n, m > 0$ and neither s_1 nor t_1 are sequence terms. Then we take $\sigma_{n+1} = \sigma_n$ and extend the derivation with Partial Decomposition 1 rule.
4. Assume again $n, m > 0$, but now let s_1 or t_1 be a sequence term.
   Let first assume that neither s_1 nor t_1 is a sequence variable. Then they both must be sequence terms with the same head (otherwise we get a contradiction) and we can proceed we Partial Decomposition 2 rule taking again $\sigma_{n+1} = \sigma_n$.

Next, assume only t_1 is a sequence variable. Then we perform the Orient 2 step with $\sigma_{n+1} = \sigma_n$.

Now, let s_1 be a sequence variable $\overline{x}$ and t_1 be a term different from s_1. Note that $\overline{x}$ does not occur in $t_1, \ldots, t_m$, since otherwise we would get a contradiction. Then $\overline{x}\varphi = \ulcorner t_1\varphi, \ldots, t_m\varphi \urcorner$. Let $\psi = \{\overline{x} \mapsto \ulcorner t_1, \ldots, t_m \urcorner\}$. Then $\psi\varphi = \varphi$. Thus, $\sigma_n\psi\varphi =_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$, which implies $\sigma_n\psi \leq_{\emptyset}^{\mathcal{X},\mathcal{Q}} \vartheta$. Then we extend the derivation with Sequence Variable Elimination' rule with $\sigma_{n+1} = \sigma_n\psi$.

Thus, every substitution in the derivation is more general than ϑ. By the construction of the derivation and Lemma 12, the derivation terminates with success and the substitution σ in the last system is more general than ϑ. $\square$

The Completeness Theorem shows that $\mathfrak{U}'$ is able to calculate a most general unifier for two unifiable terms.

Unification type for this case is unitary, since any unifiable problem has an mgu.

6.2 Equations in Unification Problems Have at Least One Ground Side

In this section we show that the unification procedure terminates if equations in the unification problem have at least one ground side.

Theorem 17. *Let Γ be a general syntactic unification problem with equations having at least one ground side. Then the unification procedure terminates on $\langle \Gamma;\ \varepsilon \rangle$.*

Proof. With each system $\langle \Delta;\ \sigma \rangle$ where Δ contains only the equations with at least one ground side, we associate a complexity measure as a 7-tuple of natural numbers $\langle n_1, n_2, n_3, n_4, n_5, n_6, n_7 \rangle$, where:

n_1 = The number of distinct variables in Δ;
m_2 = The total number of symbols in the ground sides of equations in Δ.
m_3 = The number of symbols in Δ;
m_4 = The number of occurrences of sequence function symbols in Δ;
m_5 = The number of subterms in Δ of the form $f(s_1, \ldots, s_n)$, where s_1 is not a sequence term;
m_6 = The number of equations in Δ of the form $t \simeq_{\emptyset}^{?} x$, where t is not an individual variable;
m_7 = The number of equations in Δ that have the form $f(s, s_1, \ldots, s_n) \simeq_{\emptyset}^{?} f(\overline{x}, t_1, \ldots, t_m)$, where s is not a sequence variable.

Measures will be compared lexicographically.

Every rule in the inference system $\mathfrak{U}$ strictly decreases the measure: Indeed, Projection, Sequence Variable Elimination 2, Solve and Splitting rules decrease n_1. Trivial rule does not increase m_1, does not change m_2 and decreases m_3.

Total Decomposition leaves m_1 and m_2 unchanged and decreases m_3. Partial Decomposition 1 leaves m_1, m_2, m_3 and m_4 unchanged and decreases m_5. Partial Decomposition 2 leaves m_1, m_2 and m_3 unchanged and decreases m_4. Orient 1 decreases m_6 and leaves the others unchanged. Orient 2 decreases m_7 and leaves the others unchanged. Widening 1 does not change m_1 but decreases m_2. Sequence Variable Elimination 1 and Widening 2 never apply.

It implies that every branch in $ut(\Gamma)$ is finite, i.e., the syntactic unification procedure terminates on Γ. □

This theorem has the following corollary:

Corollary 2. *General syntactic matching with sequence variables and sequence function symbols in finitary.*

It can be shown that if equations in a unification problem have at least one ground side, then there is no need in full-scale decision procedure. The pre-filter defined in Sect. 5 is sufficient to detect all failing cases.

7 Implementation

We implemented the unification procedure with sequence variables and sequence function symbols in Mathematica [Wol99] on the base of a rule-based programming system RhoLog [MK03]. A rule in RhoLog is a specification of a non-deterministic and partially defined computation. The system has primitive operators for defining elementary rules and for computing with unions, compositions, reflexive-transitive closures, rewrite rules, and normal forms of rules. With these tools sequence unification procedure with bounded depth, without the decision algorithm, was implemented quite easily. Within the bounded depth, the procedure, by default, uses the depth-first search method with backtracking provided with RhoLog. Options allow the user to modify the depth bound, use iterative deepening instead of depth-first method, and stop computation after obtaining a certain number of solutions.

The sequence unification package comes with RhoLog distribution that can be downloaded from the web:

`http://www.ricam.oeaw.ac.at/people/page/marin/RhoLog/`

8 Related Work

Solving equations involving sequence variables is needed in various fields, like databases, programming, rewriting, artificial intelligence, or theorem proving. In this section we briefly review just some of the methods related to our work.

Note that in the literature flexible arity symbols are also called "variadic", "polyadic", "variable arity", "varying-arity", or "multiple arity" symbols.

Sequence variables are part of Common Logic [Gro03] (a language designed for use in the interchange of knowledge among disparate computer systems), its simplified version SCL, a "concrete" instance of Common Logic language, Knowledge Interchange Format KIF [GPH$^+$98] and a simplified version of KIF, called SKIF [HM01]. In SKIF sequence variables are called row variables. Hayes and Menzel in [HM01] point out that unification with row variables is very difficult because two expressions can have infinitely many most general unifiers. They also remark that allowing row variables only in the last argument positions guarantee that unification patterns that create the difficulties with infinitely many most general unifiers cannot arise. Unification procedures for these languages are not discussed.

Probably the first attempt to design and implement unification with sequence variables (without sequence functions) was made in the system MVL [Gin89]. The implementation of unification was incomplete because of restricted use of widening technique. The restriction was imposed intentionally, for the efficiency reasons. No theoretical study of the unification algorithm of MVL, to the best of our knowledge, was undertaken.

Implementation of first-order logic in Isabelle [Pau90] is based on sequent calculus formulated using sequence variables (on the meta level). For example, a basic sequent is one that has a formula in common on the left and right sides. A scheme for basic sequents can be directly represented in Isabelle:

$$?\Gamma, ?P, ?\Gamma' \vdash ?\Delta, ?P, ?\Delta'$$

where $?\Gamma, ?\Gamma', ?\Delta, ?\Delta'$ are sequence meta-variables and $?P$ is an individual meta-variable. Unifying this scheme with the sequent $Q(?a), Q(f(?b)) \vdash Q(f(1))$ produces two unifiers. The first one binds $?P$ with $Q(f(1))$, $?a$ with $f(1)$, $?\Gamma'$ with $Q(f(?b))$, and $?\Gamma$, $?\Delta$ and $?\Delta'$ with the empty sequence of formulae. The second one binds $?\Gamma$ with $Q(?a)$, $?P$ with $Q(f(1))$, $?b$ with 1, and $?\Gamma'$, $?\Delta$ and $?\Delta'$ with the empty sequence of formulae. We are not aware of any formal description the sequence unification algorithm used in Isabelle implementation, but taking into account that in every unification problem no sequence meta-variable occurs more that once, we can conclude that, Isabelle sequence unification is finitary.

Richardson and Fuchs [RF97] describe another unification algorithm with sequence variables that they call vector variables. Vector variables come with their length attached, that makes unification finitary. Besides, applicative higher-order expressions are allowed. The algorithm was implemented but its properties have never been investigated.

Extensions of logic and functional programming, integrated in the RelFun system [Bol99], permit sequence variables in the last argument positions of flexible arity symbols. Unification for such terms is unitary. RelFun allows multiple-valued functions as well.

Word equations [AP90,Jaf90] and associative unification [Plo72] can be modelled by syntactic sequence unification without sequence functions using constants, sequence variables and one flexible arity function symbol.

The Set-Var prover [BF93] of Bledsoe and Feng has a construct called vector of (Skolem) functions that resembles our sequence functions. For instance, a vector of functions denoted $\overline{g}(A, \overline{s})$, where $\overline{s}$ is a vector of variables, abbreviates a sequence of functions $g_1(A, \overline{s}), \ldots, g_m(A, \overline{s})$. However, in unification it is not allowed to split vectors of functions between variables: such a vector of functions either entirely unifies with a variable, or with another vector of functions.

The programming language of MATHEMATICA uses pattern matching that, among the other kinds of patterns, supports sequence variables (represented in MATHEMATICA as identifiers with "triple blanks", e.g., `x___`) and flexible arity function symbols. The behavior of matching algorithm is explained in examples in the MATHEMATICA book [Wol99]. Our procedure (without sequence function symbols) can imitate this behavior. For a given matching problem, the output of the procedure would be identical to the set of all possible matchers MATHEMATICA matching algorithm computes (One can see all the MATHEMATICA matchers using the function `ReplaceList`, for instance.). On the other hand, when MATHEMATICA tries to match patterns in the left hand side of its assignments or rules to some expression, it tries first those matchers that assign the shortest sequences of arguments to the first triple blanks that appear in the pattern, and returns the first matcher it finds. We can simulate also this behavior, imposing an order of choosing successors in the Projection rule, applying Sequence Variable Elimination 2 before Widening 1, and stopping the procedure whenever the first solution appears.

Widera and Beierle [WB01] extend term rewriting scheme with flexible arity function symbols and three kinds of patterns. Patterns of the form $< i \ldots >$ correspond to sequence variables in our notation, while the other two kinds of patterns do not have direct counterparts in our language. Matching for terms involving these constructs is used to perform reduction.

"Sequence variables" are involved in the Sequence Datalog approach to develop a finite query language for sequence databases [MB95], But these variables are treated in a different manner than sequence variables in our framework. They behave like variables of type "sequence" and unify only with terms of that type.

Constraint logic programming with strings is considered in [Raj94]. The problem of infinitary string unification is solved by deferring full unification and performing partial unification at resolution step. By adapting this technique at each step a set of string equations are generated, which subsumes (possibly infinite) minimal complete sets unifiers. An algorithm for obtaining partially solved forms from any set of string equations is developed.

Buchberger introduced sequence functions in the THEOREMA system [BDJ+00] to Skolemize quantified sequence variables. In the equational prover of THEOREMA[Kut03] we implemented two proving methods: completion without failure [BDP89], extended with sequence variables that occur only in the last argument positions in terms (and, thus, with unary unification), and proving by simplification/rewriting, extended with sequence variables (without any restriction on their occurrence and, thus, with finitary matching). The unification procedure implemented in the prover largely follow the procedure described in this paper. MATHEMATICA pattern matching capabilities are exploited for matching.

In [Kut02] we described unification procedures for free, flat, restricted flat and orderless theories with sequence variables, but without sequence functions. The first three procedures have been implemented in RhoLog.

9 Conclusion

We proved that general unification in the free theory with sequence variables and sequence functions is decidable and has the infinitary type. We developed a rule-based unification procedure and proved its soundness, completeness and almost minimality.

Under certain restrictions sequence unification problems have finitely many solutions: sequence variables in the last argument positions, unification problems with at least one ground side (matching as a particular instance), all sequence variables on the top level with maximum one occurrence. It would be interested to identify more cases with finite or finitely representable solution sets.

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Index