PLANE PARTITIONS VI: STEMBRIDGE'S TSPP THEOREM — A DETAILED ALGORITHMIC PROOF

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Dedicated to the memory of David Robbins

Abstract. We provide a new proof of Stembridge's theorem which validated the Totally Symmetric Plane Partitions (TSPP) Conjecture. The overall strategy of our proof follows the same general pattern of determinant evaluation as discussed by the first named author in a series of papers. The resulting hypergeometric multiple sum identities turn out to be quite complicated. Their correctness is proved by applying new algorithmic methods from symbolic summation.

1. Introduction

The object of this paper is to provide a new proof of John Stembridge's theorem [Ste95] which validated the Totally Symmetric Plane Partitions (TSPP) Conjecture. We begin with a description of the objects in question.

A plane partition π is an array of nonnegative integers $(\pi_{ij})_{i\geq 1,j\geq 1}$ wherein $\pi_{ij} \geq \pi_{i'j'}$ if $i \geq i'$ and $j \geq j'$. We say this is a plane partition of n if

$$n = |\pi| := \sum_{i,j\geq 1} \pi_{ij}.$$

Obviously this means that all but finitely many of the π_{ij} are zero.

Often a plane partition is displayed in the fourth quadrant (rather than the expected first quadrant). So if $\pi_{1,1} = \pi_{1,2} = 4$, $\pi_{1,3} = 3$, $\pi_{1,4} = 1$, $\pi_{2,1} = 3$, $\pi_{2,2} = \pi_{2,3} = 2$, $\pi_{3,1} = \pi_{3,2} = 1$ and all other $\pi_{ij} = 0$, then π is represented by the diagram

$$\begin{array}{cccc} 4 & 4 & 3 & 1 \\ 3 & 2 & 2 & \\ 1 & 1 & & \end{array}$$

and π is a plane partition of 21 with 3 rows and 4 columns.

A plane partition is called *symmetric* if $\pi_{ij} = \pi_{ji}$. A plane partition is called *cyclically symmetric* if the ith row π (considered as an ordinary partition) is the conjugate of the ith column of π.

A plane partition is called *totally symmetric* if it is both symmetric and cyclically symmetric.

In [Sta86a], R. Stanley provides an account of many conjectures on plane partitions (many of which have been settled); of these, we are interested in

Conjecture 1 (cf. [Sta86b, Case 4])**.** *Let T_n equal the number of totally symmetric plane partitions with largest part $\leq n$. Then for $n \geq 1$*

$$T_n = \prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}. \tag{1}$$

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This assertion is now known as Stembridge's theorem: Stembridge's proof [Ste95] combines a variety of masterful steps involving the combinatorics of Pfaffians and reduction of such to known determinant representations from which (1) follows.

Prior to Stembridge's work, S. Okada had obtained an evaluation of T_n which is easily seen to be equivalent to the following:

Okada's Theorem ([Oka89, Sec. 4])**.** For $n \geq 3$,

$$T_{n-2}^2 = \begin{cases} \det\big(M(n)\big)x^{-1} & \text{if } n \text{ is odd,} \\ \det\big(M(n)\big) & \text{if } n \text{ is even,} \end{cases} \tag{2}$$

where

$$M(n) = \big(\mu_1(i,j)\big)_{0\leq i,j\leq n-1}, \tag{3}$$

$$\mu(i,j) = \begin{cases} 0 & \text{if } j \leq i, \\ 2^{j-1} + (-1)^{j-1} & \text{if } i = 0,\ i < j, \\ (-1)^{j-i-1} + \displaystyle\sum_{s=i}^{j-1} \binom{i+j-2}{s} & \text{if } 0 < i < j, \end{cases} \tag{4}$$

and

$$\mu_1(i,j) = \begin{cases} x & \text{if } i = j = 0, \\ (-1)^{j-1} & \text{if } i = 0,\ j > 0, \\ (-1)^{i} & \text{if } j = 0,\ i > 0, \\ 0 & \text{if } i = j > 0, \\ \mu(i-1,j-1) & \text{if } j > i \geq 1, \\ -\mu(j-1,i-1) & \text{if } 1 \leq j < i. \end{cases} \tag{5}$$

In this paper, we shall deduce Stembridge's theorem by exhibiting an upper triangular matrix $W(n)$ with 1's on the main diagonal for which we prove that

$$M(n)W(n) = \begin{pmatrix} A_{00} & 0 & 0 & \dots & 0 \\ A_{10} & A_{11} & 0 & \dots & 0 \\ A_{20} & A_{21} & A_{22} & \dots & 0 \\ \vdots & & & \ddots & \\ A_{n-1,0} & A_{n-1,1} & A_{n-1,2} & \dots & A_{n-1,n-1} \end{pmatrix} \tag{6}$$

is a lower triangular matrix with the A_{ij} explicitly determined. From the form of the A_{ii} and the immediate fact that

$$\det\big(M(n)\big) = A_{00}A_{11}\dots A_{n-1,n-1}, \tag{7}$$

we obtain Stembridge's theorem.

The entries of the matrix $W(n)$ will be presented in the next section. They are complicated to say the least. This then naturally suggests the question: Why provide a new proof of Stembridge's theorem under these circumstances?

We believe there are two compelling reasons. First, the five previous entries [And79a, And77, And79b, And87, And94] in this series have been devoted to determinant evaluations precisely along the lines of (6), and it is of interest to see that Stembridge's theorem fits this general pattern. Second, the proof of (6) requires the development of really new results concerning Zeilberger's telescoping and its extensions. The full treatment of these discoveries will be presented in [Sch04a] and [PS04]. Consequently the effort to prove (6) has yielded substantial improvement in the methods of symbolic summation.

The remainder of this paper is structured as follows. Section 2 introduces notation and basic definitions together with Theorem 1 which implies Stembridge's theorem as a corollary. Section 3 presents all the identities we need to verify in order to prove Theorem 1. In Section 4 we briefly describe the general proof method which we used to prove all the identities in question. The details of all the proof steps are given in the Sections 5 to 8. In Section 9 concluding remarks are made.

2. Notation and Definitions

We begin with the standard rising factorial

$$(x)_n = \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)\dots(x+n-1) & \text{if } n > 0, \end{cases}$$

and the binomial coefficient

$$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!}, \qquad \text{for } n \geq 0.$$

The remaining symbols are all specific to this paper and are not to be confused with other meanings in other settings:

$$\left\{ {x \atop n} \right\} = \frac{1}{2}\left(\binom{x}{n} + \binom{x-1}{n} \right);$$

$$t_1(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{(n+1)(n+3)\dots(3n-1)}{(n)_n} & \text{if } n > 0; \end{cases}$$

$$t(n) = \begin{cases} 1 & \text{if } n = 0, \\ \frac{t_1(n)}{t_1(n-1)} & \text{if } n > 0. \end{cases}$$

We remark in passing that if the right-hand side of (1) is denoted by τ_n, then for $n \geq 1$

$$\frac{\tau_n}{\tau_{n-1}} = \prod_{1 \leq i \leq j \leq n} \frac{i+j+n-1}{i+j+n-2} = \prod_{j=1}^{n} \frac{2j+n-1}{j+n-1} = t_1(n).$$

Therefore to prove (1) we need only prove equivalently

$$T_n^2 = \prod_{j=0}^{n} t_1(j)^2 = \tau_n^2$$

in light of the fact that each of the T_n and τ_n is positive.

Now we come to the more intricate components of $W(n)$:

$$r_3(s,j) = 4^{-s} \sum_{k=0}^{s} \frac{(j-k)(j)_k(-3j-1)_k}{jk!(-2j+\frac{1}{2})_k},$$

$$f_1(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s(-3j+1)_s(3j-3s-1)}{4^s s!(-2j+\frac{3}{2})_s(3j-1)},$$

$$f_2(c,j) = (-1)^c \sum_{s=0}^{\lfloor \frac{c}{2} \rfloor} (-1)^s \left\{ {j-s \atop c-2s} \right\} r_3(s,j),$$

$$r_2(j) = \begin{cases} \frac{t_1(j-1)}{2} & \text{if } j \text{ even}, \\ \frac{t_1(j-1)}{2} + \frac{f_2(j-2, \frac{j-1}{2})}{2} & \text{if } j \text{ odd}, \end{cases}$$

$$r_1(j) = \begin{cases} t_1(j-1) & \text{if } j \text{ even}, \\ 0 & \text{if } j \text{ odd}, \end{cases}$$

$$e_1(i,j)=\begin{cases}0 & \text{if } i>j,\\ 1 & \text{if } i=j,\\ r_1(j) & \text{if } i=0,\ i<j,\\ r_2(j) & \text{if } i=1,\ i<j,\\ f_1(j-i,\frac{j}{2}) & \text{if } 2\le i<j,\ j \text{ even},\\ f_2(j-i,\frac{j-1}{2}) & \text{if } 2\le i<j,\ j \text{ odd},\end{cases}$$

$$e(i,j)=\begin{cases}0 & \text{if } i>j,\\ 1 & \text{if } i=j,\\ e_1(i,j)-t(j-1)x^{(-1)^j}e_1(i,j-1) & \text{if } i<j,\end{cases}$$

$$W(n)=\big(e(i,j)\big)_{0\le i,j\le n-1}.$$

Given all of the above, our object in the remainder of the paper will be to prove the following

Theorem 1. *For each $n\ge 1$,*

$$M(n)W(n)=\begin{pmatrix} x & 0 & 0 & 0 & \dots & 0\\ A_{10} & \frac{t_1(0)^2}{x} & 0 & 0 & \dots & 0\\ A_{20} & A_{21} & t_1(1)^2x & 0 & \dots & 0\\ A_{30} & A_{31} & A_{32} & \frac{t_1(2)^2}{x} & \dots & 0\\ \vdots & & & & \ddots & \vdots\\ A_{n-1,0} & A_{n-1,1} & \dots & \dots & \dots & t_1(n-2)^2x^{\pm 1}\end{pmatrix} \tag{8}$$

where the entry $x^{\pm 1}$ in the lower triangular matrix has to be interpreted as x, if n is odd, and as x^{-1}, if n is even.

We note before proceeding that in light of the fact that

$$\det\big(W(n)\big)=1,$$

and (2) together with (8) we have the following

Corollary 1 (Stembridge's theorem [Ste95, Th. 0.2])**.** *For each $n\ge 2$,*

$$T_{n-2}^2=\det\big(M(n)_{x=1}\big)=\prod_{j=0}^{n-2}t_1(j)^2.$$

3. The Identities To Prove

3.1. **The odd case $j=2m+1$.** The following definitions will be convenient for proving Theorem 1.

Definition 1. For $i,j\ge 0$ denote the entry of the ith row and the jth column of the matrix $M(n)W(n)$ by $G(i,j)$; i.e.,

$$G(i,j)=\sum_{k=0}^{j}\mu_1(i,k)e(k,j).$$

Definition 2. Extend the definition of μ by

$$\mu(-1,k):=\begin{cases}0, & \text{if } k=0,\\ (-1)^k, & \text{if } k\ge 1.\end{cases}$$

For this subsection we fix $j=2m+1$ with $m\ge 1$.

Definition 3. For $i \geq 0$ and $m \geq 1$ define

$$a_1(i,m) := \sum_{k=1}^{i-2} \mu(k,i-1) f_2(2m-k,m), \qquad a_2(i,m) := \sum_{k=i}^{2m} \mu(i-1,k) f_2(2m-k,m),$$
$$b_1(i,m) := \sum_{k=1}^{i-2} \mu(k,i-1) f_1(2m-k-1,m), \qquad b_2(i,m) := \sum_{k=i}^{2m} \mu(i-1,k) f_1(2m-k-1,m).$$

The next fact is immediate from the definitions.

Lemma 1. *We have $f_1(0,j) = 1$ for all $j \geq 0$ and $f_2(0,j) = 1$ for all $j \geq 1$.*

From Lemma 1 and the definitions the following representation is straightforward.

Proposition 1. *For $m \geq 1$ and $0 \leq i \leq 2m+1$,*

$$G(i,2m+1) = \mu_1(i,1)\frac{t_1(2m) + f_2(2m-1,m)}{2} - a_1(i,m) + a_2(i,m)$$
$$- \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(\mu_1(i,0)t_1(2m-1) + \mu_1(i,1)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m)\Big).$$

Since Proposition 1 involves $\mu_1(i,0)$ and $\mu_1(i,1)$ we have to distinguish three cases: $i = 0$ (case A), $i = 1$ (case B), and $i \geq 2$ (case C).

3.1.1. *Case A: $i = 0$.* In this case, Proposition 1 turns into the following form.

Proposition 2. *For $m \geq 1$,*

$$G(0,2m+1) = \frac{-t_1(2m) + f_2(2m-1,m)}{2} + a_2(0,m) - \frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(\frac{t_1(2m-1)}{2} + b_2(0,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 2, it is immediate that showing $G(0,2m+1) = 0$ for all $m \geq 1$ is equivalent to proving the following identities for $m \geq 1$; namely,

$$\sum_{k=1}^{2m} (-1)^k f_2(2m-k,m) = \frac{t_1(2m) - f_2(2m-1,m)}{2} \tag{9}$$

and

$$\sum_{k=1}^{2m-1} (-1)^k f_1(2m-k-1,m) = -\frac{t_1(2m-1)}{2}. \tag{10}$$

Proposition 3. *For $m \geq 1$,*

$$\sum_{k=0}^{2m} (-1)^k f_2(2m-k,m) = \frac{1}{2} t_1(2m) \tag{11}$$

and

$$\sum_{k=0}^{2m-1} (-1)^k f_1(2m-k-1,m) = -\frac{1}{2} t_1(2m-1). \tag{12}$$

Proof: See Section 8. □

Lemma 2. *For $m \geq 1$,*

$$f_1(2m-1,m) = 0 \qquad \text{and} \qquad f_2(2m-1,m) = 2f_2(2m,m).$$

Proof: The first assertion is immediate. The proof of the second statement is elementary and is left to the reader. □

From Lemma 2 it is easily seen that (11) and (12) imply (9) and (10), respectively. These steps conclude the proof of

$$G(0,2m+1)=0 \text{ for } m\geq 1. \tag{13}$$

3.1.2. *Case B: $i=1$.* In this case, Proposition 1 turns into the following form.

Proposition 4. *For $m\geq 1$,*

$$G(1,2m+1)=a_2(1,m)-\frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(-t_1(2m-1)+b_2(1,m)\Big).$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 4, it is immediate that showing $G(1,2m+1)=0$ $(m\geq 1)$ is equivalent to proving the following identities for $m\geq 1$; namely,

$$\sum_{k=1}^{2m}(-1)^k f_2(2m-k,m)=\frac{1}{2}\sum_{k=1}^{2m}2^k f_2(2m-k,m) \tag{14}$$

and

$$\sum_{k=1}^{2m-1}(2^{k-1}-(-1)^k)f_1(2m-k-1,m)=t_1(2m-1). \tag{15}$$

Proposition 5. *For $m\geq 1$,*

$$\sum_{k=0}^{2m}2^k f_2(2m-k,m)=t_1(2m)-f_2(2m,m) \tag{16}$$

and

$$\sum_{k=0}^{2m-1}2^k f_1(2m-k-1,m)=t_1(2m-1). \tag{17}$$

Proof: See Section 8. □

It is easily seen that (16) and (17) together with (9), (10) and Lemma 2 imply (14) and (15), respectively. These steps conclude the proof of

$$G(1,2m+1)=0 \text{ for } m\geq 1. \tag{18}$$

3.1.3. *Case C: $i\geq 2$.* In this case, Proposition 1 turns into the following form.

Proposition 6. *For $m\geq 1$ and $2\leq i\leq 2m+1$,*

$$\begin{aligned}G(i,2m+1)=&-\big(2^{i-2}+(-1)^i\big)\frac{t_1(2m)+f_2(2m-1,m)}{2}-a_1(i,m)+a_2(i,m)\\&-\frac{1}{x}\frac{t_1(2m)}{t_1(2m-1)}\Big(-(2^{i-2}-(-1)^i)\frac{t_1(2m-1)}{2}-b_1(i,m)+b_2(i,m)\Big).\end{aligned}$$

Considering the "$\frac{1}{x}$–part" and the "$\frac{1}{x}$–free part" of Proposition 6, it is immediate that showing

$$\forall m\geq 1:\qquad G(i,2m+1)=\begin{cases}0, & \text{if } 2\leq i\leq 2m,\\ \frac{1}{x}t_1(2m)^2, & \text{if } i=2m+1\end{cases} \tag{19}$$

is equivalent to proving the following identities for $m\geq 1$; namely[1],

$$a_2(i,m)-a_1(i,m)=\big(2^{i-2}+(-1)^i\big)\frac{t_1(2m)+f_2(2m-1,m)}{2}\quad \text{for all } i \text{ s.t. } 2\leq i\leq 2m+1, \tag{20}$$

[1]Note: These bounds for i are sharp!

and

$$b_2(i,m) - b_1(i,m) = \big(2^{i-2} - (-1)^i\big)\frac{t_1(2m-1)}{2} \quad \text{for all } i \text{ s.t. } 2 \le i \le 2m, \tag{21}$$

and

$$b_2(2m+1,m) - b_1(2m+1,m) = -t_1(2m)t_1(2m-1) + \big(2^{2m-1}+1\big)\frac{t_1(2m-1)}{2}. \tag{22}$$

Ideally one would prove (20) to (22) directly with the Sigma package [Sch01, PS03, Sch04b, Sch04a]. However, it turns out that the computational complexity in this formulation of the problem is still too high. More precisely, the left hand sides of (20) and (21) are differences of two 4-fold sums, whereas the left hand side of (22) is a difference of two 3-fold sums. So, in order to be able to use Sigma, we transform the problem by an induction argument which is presented in the next subsection.

3.2. **Transforming the problem by induction.** The following transformation is based on an induction argument. It reduces the 4-fold sums on the left hand sides of (20) and (21) to 3-fold sums, and the 3-fold sums on the left hand side of (22) to 2-fold sums.

3.2.1. *Recurrences for the induction step.*

Definition 4. For $i, j \ge 1$ define

$$\alpha(i,j) := (-1)^{j-i-1} \qquad \text{and} \qquad \beta(i,j) := \sum_{s=i}^{j-1} \binom{i+j-2}{s}. \tag{23}$$

It is convenient to introduce the following lemma. Its proof is elementary.

Lemma 3. *For integers $i, j \ge 1$:*

$$\alpha(i+1,j) = -\alpha(i,j) \qquad \textit{and} \qquad \alpha(i,j+1) = -\alpha(i,j); \tag{24}$$

$$\beta(i+1,j) = 2\beta(i,j) - \binom{i+j-1}{i} \quad (1 \le i \le j-1); \tag{25}$$

$$\beta(i,j+1) = 2\beta(i,j) - \binom{i+j-1}{j} \quad (1 \le i \le j). \tag{26}$$

Lemma 3 implies the following recurrences for μ.

Proposition 7. *For integers[2] i, j:*

$$\mu(i+1,j) = 2\mu(i,j) + 3(-1)^{j-i} - \binom{i+j-1}{i} \qquad (1 \le i \le j-2); \tag{27}$$

$$\mu(i,j+1) = 2\mu(i,j) + 3(-1)^{j-i} + \binom{i+j-1}{j} \qquad (0 \le i \le j-1). \tag{28}$$

Proof: The definition of μ implies

$$\mu(i,j) = \alpha(i,j) + \beta(i,j) \qquad (1 \le i < j).$$

Hence Lemma 3 for $1 \le i \le j-2$ gives

$$\begin{aligned}\mu(i+1,j) &= \alpha(i+1,j) + \beta(i+1,j) = -\alpha(i,j) + 2\beta(i,j) - \binom{i+j-1}{i} \\ &= 2(\alpha(i,j) + \beta(i,j)) - 3\alpha(i,j) - \binom{i+j-1}{i},\end{aligned}$$

which proves (27). Recurrence (28) is proved analogously. □

[2]Note that the ranges for i and j in both recurrences are sharp.

With Proposition 7 in hand one finds the relations of Proposition 8 and Proposition 9 which will be used in the induction step and which involve the following sums.

Definition 5. For $2 \le i \le 2m+1$ define

$$A_1(i,m) := \sum_{k=1}^{i-3}(-1)^k f_2(2m-k,m), \qquad B_1(i,m) := \sum_{k=1}^{i-3}(-1)^k f_1(2m-k-1,m),$$
$$A_2(i,m) := \sum_{k=i}^{2m}(-1)^k f_2(2m-k,m), \qquad B_2(i,m) := \sum_{k=i}^{2m-1}(-1)^k f_1(2m-k-1,m),$$

and

$$A_0(i,m) := \sum_{k=0}^{2m}\binom{i+k-3}{i-2} f_2(2m-k,m), \quad B_0(i,m) := \sum_{k=0}^{2m-1}\binom{i+k-3}{i-2} f_1(2m-k-1,m).$$

Proposition 8. *For* $3 \le i \le 2m+1$,

$$a_2(i,m) - a_1(i,m) = 2(a_2(i-1,m) - a_1(i-1,m)$$
$$-\Big(f_2(2m-i+2,m) + 2f_2(2m-i+1,m)\Big) + 3(-1)^i\Big(A_2(i,m) - A_1(i,m)\Big) - A_0(i,m). \quad (29)$$

Proof: Applying (27) and (28), respectively, yields

$$\begin{aligned} a_2(i,m) - a_1(i,m) &= \sum_{k=i}^{2m}\mu(i-1,k) f_2(2m-k,m) \\ &\qquad - \Big(\sum_{k=1}^{i-3}\mu(k,i-1) f_2(2m-k,m) + \mu(i-2,i-1) f_2(2m-i+2,m)\Big) \\ &= 2\Big(a_2(i-1,m) - \mu(i-2,i-1) f_2(2m-i+1,m)\Big) \\ &\qquad + 3(-1)^i A_2(i,m) - \sum_{k=i}^{2m}\binom{i+k-3}{i-2} f_2(2m-k,m) - \Big[2a_1(i-1,m) + 3(-1)^i A_1(i,m) \\ &\qquad + \sum_{k=1}^{i-3}\binom{i+k-3}{i-2} f_2(2m-k,m) + \mu(i-2,i-1) f_2(2m-i+2,m)\Big] \\ &= 2\big(a_2(i-1,m) - a_1(i-1,m)\big) + 3(-1)^i\big(A_2(i,m) - A_1(i,m)\big) - A_0(i,m) \\ &\qquad + \Big(\binom{2i-5}{i-2} - \mu(i-2,i-1)\Big) f_2(2m-i+2,m) \\ &\qquad + \Big(\binom{2i-4}{i-2} - 2\mu(i-2,i-1)\Big) f_2(2m-i+1,m) \end{aligned}$$

which gives the right side of (29) by invoking the fact that

$$\mu(i-2,i-1) = 1 + \binom{2i-5}{i-2} \qquad (i \ge 3). \quad (30)$$

□

Proposition 9. *For* $3 \le i \le 2m+1$,

$$b_2(i,m) - b_1(i,m) = 2\big(b_2(i-1,m) - b_1(i-1,m)\big)$$
$$-\big(f_1(2m-i+1,m) + 2f_1(2m-i,m)\big) + 3(-1)^i\big(B_2(i,m) - B_1(i,m)\big) - B_0(i,m). \quad (31)$$

Proof: The proof is completely analogous to that of Proposition 8. □

3.2.2. *Identities for the induction step.* Suppose we have proved the case $i = 2$ of (20) and (21). Then proving (20) and (21) in their full generality can be done by induction on i for fixed $m \geq 1$. Namely, due to Proposition 8, identity (20) is equivalent to showing the case $i = 2$ and that for any i with[3] $3 \leq i \leq 2m + 1$ the equality

$$\begin{aligned}&\frac{3}{2}(-1)^i\big(t_1(2m) + f_2(2m-1,m)\big)\\ &= -\big(f_2(2m-i+2,m) + 2f_2(2m-i+1,m)\big) + 3(-1)^i\big(A_2(i,m) - A_1(i,m)\big) - A_0(i,m)\end{aligned} \tag{32}$$

holds. Analogously, because of Proposition 9, identity (21) is equivalent to showing the case $i = 2$ and that for any i with $3 \leq i \leq 2m$ the equality

$$\begin{aligned}&-\frac{3}{2}(-1)^i t_1(2m-1)\\ &\quad = -\big(f_1(2m-i+1,m) + 2f_1(2m-i,m)\big) + 3(-1)^i\big(B_2(i,m) - B_1(i,m)\big) - B_0(i,m)\end{aligned} \tag{33}$$

holds.

The base case $i = 2$ of (20) and (21), respectively, simplifies as explained in Subsection 3.2.3 below. Identities (32) and (33) are proved in Sections 7 and 6, respectively.

Finally, the proof of (19) is completed by showing (22). Since (21) will be proved independently from it, we can proceed as follows. Applying (31) and then (21) gives

$$b_2(2m+1,m) - b_1(2m+1,m) = \big(2^{2m-2} - 1\big)t_1(2m-1) - 1 + 3B_1(2m+1,m) - B_0(2m+1,m).$$

By (12) and Lemma 1 one has

$$B_1(2m+1,m) = -\sum_{k=1}^{2m-2}(-1)^k f_1(k,m) = 1 - \frac{t_1(2m-1)}{2}. \tag{34}$$

Consequently the proof of (22) is equivalent to showing that for $m \geq 1$,

$$B_0(2m+1,m) = t_1(2m)t_1(2m-1) - 3t_1(2m-1) + 2. \tag{35}$$

Proof: This will be done in Sections 5 and 8. □

3.2.3. *The base case for $i = 2$.* The case $i = 2$ and $m \geq 1$ of identity (20) reads as

$$\sum_{k=2}^{2m}\mu(1,k)f_2(2m-k,m) = t_1(2m) + f_2(2m-1,m). \tag{36}$$

Because of

$$\mu(1,k) = 2^{k-1} + (-1)^k - 1 \qquad (k \geq 2), \tag{37}$$

and then (11) and (16), the left side of (36) turns into $-\sum_{k=0}^{2m-2} f_2(k,m) + t_1(2m) - 2f_2(2m,m)$. Consequently, identity (36) is implied by the following lemma.

Lemma 4. *For $m \geq 1$,*

$$\sum_{k=0}^{2m} f_2(k,m) = -f_2(2m,m). \tag{38}$$

Proof: This follows by identity (118) and Lemma 2. □

The case $i = 2$ and $m \geq 1$ of identity (21) reads as

$$\sum_{k=2}^{2m-1}\mu(1,k)f_1(2m-k-1,m) = 0. \tag{39}$$

[3]Note that these bounds for identity (32) are sharp; similarly for identity (33).

The case $m=1$ is trivial. Because of (37) and then (12) and (16), the left side of (39) turns into $-\sum_{k=2}^{2m-1} f_1(2m-k-1,m)$. Consequently, identity (39) is implied by Lemma 2 and

Lemma 5. *For $m \geq 1$,*

$$\sum_{k=0}^{2m-1} f_1(2m-k-1,m) = f_1(2m-2,m). \tag{40}$$

Proof: See identity (97). □

3.3. **The even case: $j=2m$.** Analogously to Proposition 1, from all the definitions we can derive the following representation.

Proposition 10. *For $m \geq 2$ and $0 \leq i \leq 2m$,*

$$\begin{aligned} G(i,2m) &= \big(2\mu_1(i,0)+\mu_1(i,1)\big)\frac{t_1(2m-1)}{2} - b_1(i,m) + b_2(i,m) \\ &- x\frac{t_1(2m-1)}{t_1(2m-2)}\Big(\frac{\mu_1(i,1)}{2}\big(t_1(2m-2)+f_2(2m-3,m-1)\big) - a_1(i,m-1) + a_2(i,m-1)\Big). \end{aligned} \tag{41}$$

Suppose we have proved $G(i,2m+1)=0$ for $0 \leq i \leq 2m+1$. From Proposition 1 we know that the "x–free part" of the right side of (41) is 0 for $0 \leq i \leq 2m$, and also that the "x–part" of the right side of (41) is 0 for $0 \leq i \leq 2m-1$. Consequently, to complete the proof of

$$\forall m \geq 2: \qquad G(i,2m) = \begin{cases} 0, & \text{if } 0 \leq i \leq 2m-1, \\ x t_1(2m-1)^2, & \text{if } i = 2m, \end{cases} \tag{42}$$

it remains to show that

$$\begin{aligned} -x\frac{t_1(2m-1)}{t_1(2m-2)}\Big[\frac{\mu_1(2m,1)}{2}\big(t_1(2m-2)+f_2(2m-3,m-1)\big) \\ + a_2(2m,m-1) - a_1(2m,m-1)\Big] = x t_1(2m-1)^2 \end{aligned} \tag{43}$$

for all $m \geq 2$.

If we define

$$a(m) := a_2(2m,m-1) - a_1(2m,m-1) \qquad (m \geq 2) \tag{44}$$

then (43) is equivalent to

$$a(m) = \big(2^{2m-3}+\frac{1}{2}\big)\big(t_1(2m-2)+f_2(2m-3,m-1)\big) - t_1(2m-1)t_1(2m-2) \qquad (m \geq 2). \tag{45}$$

On the other hand, from (29) with $m \to m-1$ and $i=2m$ we obtain that

$$\begin{aligned} a(m) = 2\big(a_2(2m-1,m-1) - a_1(2m-1,m-1)\big) \\ - 1 - A_0(2m,m-1) - 3\sum_{k=1}^{2m-3}(-1)^k f_2(2m-k-2,m-1), \end{aligned}$$

which by (20) [with $m \to m-1$ and $i=2m-1$] and by (11) can be simplified further to

$$\begin{aligned} a(m) = 2 - A_0(2m,m-1) + 3f_2(2m-2,m-1) \\ - \frac{3}{2}t_1(2m-2) + \big(2^{2m-3}-1\big)\big(t_1(2m-2)+f_2(2m-3,m-1)\big). \end{aligned} \tag{46}$$

Comparing this to (45) and using the elementary property of f_2 described in Lemma 2, to prove identity (43) it suffices to show that for all $m \geq 2$,

$$t_1(2m-1)t_1(2m-2) - 3t_1(2m-2) + 2 = A_0(2m,m-1). \tag{47}$$

Proof: This will be done in Sections 5 and 8. □

4. The Proof Method

In this section we explain the method that we used for proving **all** of the identities stated in this article. The method consists in a combination of various algorithmic steps which are briefly described below. Remarkably most of the proving steps can be represented in the form of *proof certificates* which are identities that can be verified by elementary calculations independently from the way the algorithm has obtained them. This allows to produce compact descriptions of the proofs of the identities under consideration which can be found in the remaining sections.

Except the task of combining recurrences described in Subsection 4.1, all other algorithmic steps were executed by the computer algebra package Sigma [Sch01, PS03, Sch04b] developed by the third author. It should be noted that in order to handle the conjectured TSPP multiple sum identities, the Sigma tool-box was extended significantly. A detailed description of this work can be found in [Sch04a].

4.1. **Combining Recurrences.** Hypergeometric sequences are special instances of P-finite sequences where the latter are defined to be sequences that satisfy a linear recurrence with polynomial coefficients. Such recurrences are called P-finite recurrences. P-finite sequences enjoy various closure properties; see [Sta80].

For our purpose we need to combine P-finite recurrences additively and multiplicatively, namely:

Given P-finite sequences $\big(a_n\big)$ and $\big(b_n\big)$ satisfying the recurrences

$$p_\alpha(n)a_{n+\alpha} + \cdots + p_0(n)a_n = 0 \quad (n \geq 0)^4$$

and

$$q_\beta(n)b_{n+\beta} + \cdots + q_0(n)b_n = 0 \quad (n \geq 0)$$

respectively, **compute** a P-finite recurrence

$$r_\gamma(n)c_{n+\gamma} + \cdots + r_0(n)c_n = 0 \quad (n \geq 0) \tag{48}$$

which is satisfied by the sum sequence $\big(c_n\big)$ with $c_n := a_n + b_n$. — The analogous product problem is with $c_n := a_n\, b_n$.

Algorithms which compute the recurrence (48) for the sum (resp. product) sequence $\big(c_n\big)$ are described in [SZ94]. For our computations we have used Mallinger's Mathematica package GeneratingFunctions [Mal96].

4.2. **The General Proof Strategy.** All the identities we need to prove in this article are of the form

$$c_n^{(1)} + \cdots + c_n^{(k)} = 0 \quad (n \geq 0) \tag{49}$$

where k is a fixed positive integer and where each of the $c_n^{(i)}$ is a P-finite sequence. To prove that (49) is valid indeed for all $n \geq 0$ we proceed as follows. First we compute P-finite recurrences for all of the $c_n^{(i)}$, unless such recurrences are already given. Then, as described in Section 4.1, a P-finite recurrence for the sum sequence $s_n := c_n^{(1)} + \cdots + c_n^{(k)}$ is computed. Finally, we show that $s_n = 0$ for all $n \geq 0$ by checking sufficiently many initial values. — Note that the leading polynomial coefficient of the recurrence for s_n must not have any nonnegative integer root.

In our context the $c_n^{(i)}$ are given as hypergeometric sequences, as single, double and triple sums over hypergeometric sequences, or as the product of such sequences. So in view of Section 4.1 and of our general strategy described above, there remains the task to derive P-finite recurrences for such sums.

[4]For the sake of simplicity we have chosen $n \geq 0$; in concrete cases the initial values might be different from 0.

Concerning summation ranges the following remark is in place. In this section, for the sake of simplicity we restrict ourselves to (multiple) sums where all summations are taken over finite summand supports. With this restriction *homogeneous* sum recurrences are guaranteed.

4.3. **Single Sums.** Here the basic task is as follows.

Given a summand $F(r,s)$ which is hypergeometric[5] in r and s, **compute** a P-finite recurrence

$$p_\gamma(r)f(r+\gamma)+\cdots+p_0(r)f(r)=0 \quad (n\geq 0) \tag{50}$$

which is satisfied by the sum $f(r):=\sum_s F(r,s)$.

In case that $F(r,s)$ satisfies some mild side conditions this problem can be solved by applying Zeilberger's algorithm [PWZ96] which computes polynomials $p_i(r)$, free of s, and $G(r,s)$ such that

$$p_\gamma(r)F(r+\gamma,s)+p_{\gamma-1}(r)F(r+\gamma-1,s)+\cdots+p_0(r)F(r,s)=\Delta_s G(r,s). \tag{51}$$

Note that Δ_s denotes the difference operator defined as usual by $\Delta_s G(s)=G(s+1)-G(s)$.

One can show that $G(r,s)$ is a rational function multiple of $F(r,s)$. Hence recurrence (50) is obtained from (51) by summation over all s. — Consequently, all what is needed to prove the correctness of (50) is the knowledge of (51) which is called "certificate recurrence". Note that after dividing (51) by $F(r,s)$, its verification reduces to checking equality of rational functions, a simple check which is independent from the way the algorithm obtained (51).

Remark: It can be that for a fixed order γ there exists only the trivial solution, i.e., where all the $p_i(r)$ are 0. In this case one has to increase the order γ incrementally until a non-trivial solution is computed whose existence is guaranteed by the theory explained in [PWZ96].

4.3.1. *A slight but important variation.* Many TSPP identities involve summands in more than one independent variable. For instance, instead of the summand $F(r,s)$ take the summand $F(n,r,s)$, now hypergeometric in r,s and n. For the following it is important to note that completely analogously to (51) one can compute

$$p'_\gamma(n,r)F(n+1,r,s)+p'_{\gamma-1}(n,r)F(n,r+\gamma-1,s)+\cdots+p'_0(n,r)F(n,r,s)=\Delta_s G'(n,r,s) \tag{52}$$

if it exists. Also for such cases one can prove that $G'(m,r,s)$ is a rational function multiple of $F(n,r,s)$. — Recurrences like (52) are related to contiguous relations [Pau04]; see also [Sch04a]. For instance, summing (52) over all s (assuming finite support) yields

$$p'_\gamma(n,r)f(n+1,r)+p'_{\gamma-1}(n,r)f(n,r+\gamma-1)+\cdots+p'_0(n,r)f(n,r)=0 \quad (n\geq 0) \tag{53}$$

with $f(n,r)=\sum_s F(n,r,s)$ and where the $p'_i(n,r)$ are polynomials in n and r.

Remark: There remains the question whether relations like (53) or (52) do exist. However, in [Pau04] an existence theory is presented which closely relates to the situation of Zeilberger's algorithm; for multiple sums in [PS04] this question is analysed in further details.

4.4. **Double Sums.** Here the basic task is as follows.

Given a summand $F(n,r,s)$ which is hypergeometric in n,r and s, **compute** a P-finite recurrence

$$p_\gamma(n)S(n+\gamma)+\cdots+p_0(n)S(n)=0 \quad (n\geq 0) \tag{54}$$

which is satisfied by the sum $S(n):=\sum_r\sum_s F(n,r,s)$.

In principle, one could apply the WZ method which is based on ideas of Sister Celine Fasenmyer and which is described in [PWZ96] . However, it turns out that all available implementations of this approach or of variations of it (e.g., Wegschaider 's algorithm [Weg97]) meet serious problems of computational complexity when applied to the TSPP identities in question. As a consequence we will follow a different approach which can be viewed as a new, surprisingly simple variant of Chyzak's algorithm [Chy00]. The basic ideas of this method are as follows; a full account of the details and a comparison to [Chy00] is given in [Sch04a].

[5] $F(r)$ is hypergeometric in r iff $F(r+1)/F(r)=g(r)$ for some fixed rational function $g(r)$.

The overall goal of the method is to compute a certificate recurrence of type (51), i.e.,

$$p_\gamma(n)f(n+\gamma,r)+\cdots+p_0(n)f(n,r)=\Delta_r g(n,r) \tag{55}$$

where $f(n,r)$ is defined to be the inner sum, i.e.,

$$f(n,r):=\sum_s F(n,r,s),$$

and where $g(n,r)$ is suitably chosen. From (55) the desired recurrence (54) for $S(n)$ is obtained by summing over all r — as in Zeilberger's algorithm for single sums.

In order to find (55) we proceed as follows. First one computes recurrences of the following form,

$$f(n,r+\gamma)=\lambda_0(n,r)f(n,r)+\cdots+\lambda_{\gamma-1}(n,r)f(n,r+\gamma-1), \tag{56}$$

and

$$f(n+1,r)=\mu_0(n,r)f(n,r)+\cdots+\mu_{\gamma-1}(n,r)f(n,r+\gamma-1), \tag{57}$$

where the $\lambda_i(n,r)$ and $\mu_i(n,r)$ are rational functions in n and r. This can be accomplished by following the description to compute (50) and (53) via (51) and (52), respectively.

Second, for $g(n,r)$ one chooses an ansatz with undetermined coefficients of the following form,

$$g(n,r)=\phi_0(n,r)f(n,r)+\cdots+\phi_{\gamma-1}(n,r)f(n,r+\gamma-1). \tag{58}$$

In the third step, the unknown polynomials $p_i(n)$, free of r, and the unknown rational function coefficients $\phi_i(n,r)$ for $g(n,r)$ are computed such that the certificate recurrence (55) holds. In view of (56) and (57), the key observation is that any shift in n and r of $f(n,r)$ and also $g(n,r)$ can be represented as a linear combination of $f(n,r),\dots,f(n,r+\gamma-1)$ over rational functions in n and r. Then rewriting both sides of (55) in terms of these generators, allows — in all our applications — to compute the unknown data by comparing the coefficients of all the $f(n,r+i)$ involved.

The corresponding computational steps are carried out as follows. For the sake of simplicity we restrict to $\gamma=2$; the general case works completely analogously and is described in [Sch04a]. From the relations (56) and (57) one can find rational functions $\nu_i(n,r)$ such that

$$f(n+2,r)=\nu_0(n,r)f(n,r)+\nu_1(n,r)f(n,r+1). \tag{59}$$

This together with (57) implies for the hand side of (55) that

$$\begin{aligned}&p_2(n)f(n+2,r)+p_1(n)f(n+1,r)+p_0(n)f(n,r)\\&=\big(p_0(n)+p_1(n)\mu_0(n,r)+p_2(n)\nu_0(n,r)\big)f(n,r)+\big(p_1(n)\mu_1(n,r)+p_2(n)\nu_1(n,r)\big)f(n,r+1).\end{aligned} \tag{60}$$

To represent the right hand side of (55) in terms of the generators $f(n,r+i)$ one invokes (56) which gives that

$$\begin{aligned}\Delta_r g(n,r)=&\big(-\phi_0(n,r)+\phi_1(n,r+1)\lambda_0(n,r)\big)f(n,r)\\&+\big(\phi_0(n,r+1)-\phi_1(n,r)+\phi_1(n,r+1)\lambda_1(n,r)\big)f(n,r+1).\end{aligned} \tag{61}$$

Finally comparing the coefficients of $f(n,r)$ and $f(n,r+1)$ on the right hand sides of (60) and (61), respectively, after triangulation leads to the problem of solving the system

$$\begin{aligned}&\lambda_0(n,r+1)\phi_1(n,r+2)+\lambda_1(n,r)\phi_1(n,r+1)-\phi_1(n,r)\\&\qquad=p_0(n)+\big(\mu_0(n,r+1)+\mu_1(n,r)\big)p_1(n)+\big(\nu_0(n,r+1)+\nu_1(n,r)\big)p_2(n)\end{aligned} \tag{62}$$

and

$$\phi_0(n,r)=\phi_1(n,r+1)\lambda_0(n,r)-\big(p_0(n)+p_1(n)\mu_0(n,r)+p_2(n)\nu_0(n,r)\big). \tag{63}$$

Equation (62) is a parameterized difference equation which has to be solved for a rational function $\phi_1(n,r)$ and for the polynomials $p_i(n)$. This is done by the Sigma package by using a refinement of Abramov's algorithm [Abr89]. Finally $\phi_0(n,r)$ is computed from (63). It is important to note that not only for $\gamma=2$, but also for general γ the approach works entirely the same. In particular,

as pointed out in [Sch04a] triangularization of the system arising from this coefficient comparison can be avoided completely since the uncoupled system can be represented by a generic formula.

SUMMARY: The key identity for deriving a P-finite recurrence for the double sum $S(n) = \sum_r \sum_s F(n,r,s) = \sum_r f(n,r)$ is the certificate identity (55). Knowing (56) and (57) together with the $\phi_i(n,r)$ in (58), the reader can check the correctness of (55) independently from the steps of the method. Note that the correctness of (56) and (57) can be verified by standard creative telescoping. As a consequence, to certify that double sums satisfy certain P-finite recurrences, in the remaining sections we restrict ourselves to provide the data contained in (54), (55), (56), and (57).

4.5. **Triple Sums.** Based on what we said about single and double sums we are in the position to solve the following problem.

Given a summand $F(m,n,r,s)$ which is hypergeometric in m,n,r and s, **compute** a P-finite recurrence

$$p_\gamma(m)S(m+\gamma) + \cdots + p_0(m)S(m) = 0 \quad (m \geq 0) \tag{64}$$

which is satisfied by the sum $S(m) := \sum_n \sum_r \sum_s F(m,n,r,s)$.

As with double sums the overall goal of the method is to compute a certificate recurrence of the form

$$p_\gamma(m)h(m+\gamma,n) + \cdots + p_0(m)h(m,n) = \Delta_n g(m,n) \tag{65}$$

where we define $h(m,n)$ as

$$h(m,n) := \sum_r \sum_s F(m,n,r,s), \tag{66}$$

and where $g(m,n)$ is suitably chosen. Then from (65) the desired recurrence (64) for $S(m)$ is obtained by summation over all n.

To find (65) we proceed analogously to the double sum case. Namely, we first derive recurrences of the form

$$h(m,n+\gamma) = \lambda_0(m,n)h(m,n) + \cdots + \lambda_{\gamma-1}(m,n)h(m,n+\gamma-1), \tag{67}$$

and

$$h(m+1,n) = \mu_0(m,n)h(m,n) + \cdots + \mu_{\gamma-1}(m,n)h(m,n+\gamma-1), \tag{68}$$

and afterwards we apply the same method as in the double sum case in order to compute all the components for the certificate recurrence (65). In particular, due to our ansatz, $g(m,n)$ will be of the form

$$g(m,n) = \phi_0(m,n)h(m,n) + \cdots + \phi_{\gamma-1}(m,n)h(m,n+\gamma-1), \tag{69}$$

where the $\phi_i(m,n)$, $\lambda_i(m,m)$ and $\mu_i(m,n)$ are rational functions in m and n.

Obviously, from (67), (68) and (69) the correctness of (65) can be verified independently from the steps of our algorithm.

In order to apply the above strategy there remains the task to compute the recurrences (67) and (68). In principle, we could apply our techniques from above. Namely, with our description from Subsection 4.4 we can obtain a recurrence of the type (67) for the double sum (66). Similarly we can derive a recurrence of the form (68) by a slight variation of the same strategy which is described in [Sch04a]. Roughly spoken, this way we reduce triple summation first to double and then to single summation by recursion. Summarizing, we have a general method in hand to derive (67) and (68). But, by observing that the summand in the given TSPP triple sums are all of the type

$$h(m,n) = \sum_r \sum_{s=0}^{r} F(m,n,s)$$

where $F(m,n,s)$ is free of r, we can follow a more direct approach. Namely, as Zeilberger's algorithm can compute the required recurrences (56) and (57) for the double sum case, Sigma can

produce in an analogous fashion the recurrences (67) and (68) together with recurrence certificates for $h(m,n)$.

5. A first step to prove Identities (35) and (47)

First observe that

$$t_1(2m) = 2\prod_{i=1}^{m} \frac{3(6i-5)(6i-1)}{4(4i-3)(4i-1)} \qquad \text{and} \qquad t_1(2m-1) = -\frac{2}{3}\prod_{i=1}^{m} \frac{-3(3i-4)(3i-2)}{(4i-5)(4i-3)},$$

i.e., they satisfy the recurrences

$$-3(6m+1)(6m+5)t_1(2m) + 4(4m+1)(4m+3)t_1(2m+2) = 0 \tag{70}$$

and

$$-3(3m-1)(3m+1)t_1(2m-1) + (4m-1)(4m+1)t_1(2m+1) = 0 \tag{71}$$

for all $m \geq 1$.

Using Maple's gfun package [SZ94] or the Mathematica package GeneratingFunctions described in [Mal96], from (70) and (71) we can derive a recurrence which is satisfied by the right side of (35). More precisely, first we combine (70) and (71) into the recurrence

$$-9(3m-1)(3m+1)(6m+1)(6m+5)P(m) + 4(4m-1)(4m+1)^2(4m+3)P(1+m) = 0$$

which is satisfied by the sequence $P(m) := t_1(2m)t_1(2m-1)$ for all $m \geq 1$. Next, we combine this recurrence with (71) to obtain the recurrence

$$\begin{aligned}
&27(-1+3m)(1+3m)(2+3m)(4+3m)(1+6m)(5+6m)(91+132m+44m^2)S(m)\\
&-3(2+3m)(4+3m)(-1+4m)(1+4m)(1785+18544m+39544m^2+30272m^3+7568m^4)S(1+m)\\
&\quad+4(-1+4m)(1+4m)(3+4m)(5+4m)^2(7+4m)(3+44m+44m^2)S(2+m) = 0
\end{aligned} \tag{72}$$

which is satisfied by $S(m) := P(m) - 3t_1(2m-1)$. Finally, in order to add the constant sequence 2, we combine (72) with the recurrence $C(m) - C(m+1) = 0$ and obtain

$$\begin{aligned}
&27(-1+3m)(1+3m)(2+3m)(4+3m)(1+6m)(5+6m)(35582085+208705770m+502426266m^2\\
&+659838718m^3+522259397m^4+256674880m^5+76794344m^6+12820192m^7+915728m^8)b(m)\\
&\quad-3(2+3m)(4+3m)(-5091730875-64116259830m-268357073220m^2\\
&\quad-344918244168m^3+764136680690m^4+3625956026718m^5+6449149601689m^6\\
&\quad+6689203564428m^7+4423817173116m^8+1893138648192m^9\\
&\quad+508247453296m^{10}+77855194560m^{11}+5190346304m^{12})b(m+1)\\
&\quad+(3+4m)(5+4m)(-7125148800-88649425620m-376507014210m^2\\
&\quad-591098983182m^3+431261276465m^4+3340499936442m^5\\
&\quad+6290733211249m^6+6633648559692m^7+4412582090796m^8\\
&\quad+1892139974208m^9+508247453296m^{10}+77855194560m^{11}+5190346304m^{12})b(m+2)\\
&-4(3+4m)(5+4m)(7+4m)(9+4m)^2(11+4m)(-61740-644340m-1950794m^2-910998m^3\\
&\quad+6194397m^4+13852080m^5+12693384m^6+5494368m^7+915728m^8)b(m+3) = 0
\end{aligned} \tag{73}$$

that is satisfied by $b(m) := S(m) + 2 = t_1(2m)t_1(2m-1) - 3t_1(2m-1) + 2$ for $m \geq 1$.

In Section 8 we show that $B_0(2m+1,m)$ satisfies the same recurrence (73) for $m \geq 1$. Since the coefficient of $b(m+3)$ in (73) is nonzero for all $m \geq 1$ and since $b(m)$ and $B_0(2m+1,m)$ are equal for $m \in \{1,2,3\}$, this will prove identity (35).

In complete analogous fashion one derives the recurrence

$$\begin{aligned}
&27(2+3m)(4+3m)(1+6m)(5+6m)(7+6m)(11+6m)(389987325\\
&\quad+1563210054m+2677225618m^2+2567677266m^3+1512020037m^4\\
&\quad+560774016m^5+128075112m^6+16483104m^7+915728m^8)a(m)\\
&\quad-3(7+6m)(11+6m)(188256814200+3218702456970m\\
&\quad+17894686721409m^2+51759068834382m^3+91965480308606m^4\\
&\quad+108563491759548m^5+88575755915749m^6+50780656348884m^7\\
&\quad+20427064666140m^8+5647619363232m^9+1022091737392m^{10}\\
&\quad+108997272384m^{11}+5190346304m^{12})a(1+m)\\
&\quad+4(5+4m)(7+4m)(83466403020+2609224361478m+16370962889247m^2\\
&\quad+49594957358604m^3+90028764304859m^4+107424612589224m^5\\
&\quad+88133765355661m^6+50671172949012m^7+20411335550892m^8\\
&\quad+5646620689248m^9+1022091737392m^{10}+108997272384m^{11}+5190346304m^{12})a(2+m)\\
&\quad-16(5+4m)(7+4m)(9+4m)(11+4m)^2(13+4m)(79380+7185780m\\
&\quad+39195762m^2+91462910m^3+116468957m^4+87187760m^5\\
&\quad+38333768m^6+9157280m^7+915728m^8)a(3+m)=0 \qquad (74)
\end{aligned}$$

which is satisfied by $a(m) := t_1(2m+1)t_1(2m) - 3t_1(2m) + 2$ for all $m \geq 1$. In Section 8 we will show that this recurrence is also satisfied by $A_0(2m+2, m)$. Since the coefficient of $a(m+3)$ is nonzero for $m \geq 1$ and since $a(m)$ and $A_0(2m+2, m)$ are equal for the first three initial values $m \in \{1, 2, 3\}$, identity (47) must hold for all $m \geq 2$.

6. A Proof of Identity (33)

In this section we will use the abbreviation $h_1(k, m) := f_1(2m-k-1, m)$. By Lemma 2 we may write

$$B_1(i, m) = \sum_{k=0}^{i-3} (-1)^k h_1(k, m). \qquad (75)$$

Moreover the sum in Proposition 3 can be rewritten as

$$-\frac{1}{2}t_1(2m-1) = \sum_{k=0}^{2m-1} (-1)^k h_1(k, m) = B_1(i, m) - (-1)^i\, h_1(i-1, m) + (-1)^i\, h_1(i-2, m) + B_2(i, m).$$

Consequently to show identity (33) is equivalent to prove

$$\boxed{6(-1)^i B_2(i, m) - B_0(i, m) + 2h_1(i-2, m) - 5h_1(i-1, m) + 3(-1)^i t_1(2m-1) = 0, \qquad (76)}$$

or, equivalently,

$$\boxed{6(-1)^i B_1(i, m) + B_0(i, m) + 4h_1(i-2, m) - h_1(i-1, m) = 0 \qquad (77)}$$

for all $2 \leq i \leq 2m$.

Now it suffices to verify one of those two identities. To this end we will follow the same proof strategy as sketched in Section 5: derive recurrences for all the ingredients in (77), namely $h_1(i-2, m)$, $h_1(i-1, m)$, $B_0(i, m)$, and $B_1(i, m)$, or for all objects in (76), namely $h_1(i-2, m)$, $h_1(i-1, m)$, $B_0(i, m)$, and $B_2(i, m)$, and combine them into one homogeneous recurrence that contains the left side of (77), respectively (76) as solution. For the sake of simplicity we refrain from computing a recurrence for $B_1(i, m)$ (see Remark 2) and therefore compute a recurrence for the left side in (76).

However, as shown below, for a full proof of (76) we also need to validate that certain instances of (77) hold.

Using Sigma [Sch01] we first compute a recurrence for $h_1(k,m)$ that holds for all $m,k \geq 0$, namely

$$\begin{aligned}
&-2(2+k)^2(1+k-2m)(k+2m)h_1(k,m)\\
&+(29k^3+5k^4+k(46+20m-40m^2)+k^2(58+6m-12m^2)+12(1+m-2m^2))h_1(1+k,m)\\
&-(26k^3+4k^4+k(55+14m-28m^2)+k^2(59+6m-12m^2)+6(3+m-2m^2))h_1(2+k,m)\\
&+(1+k)^2(3+k-2m)(2+k+2m)h_1(3+k,m)=0 \quad (78)
\end{aligned}$$

Note that (78) can be computed with any package like [PS95] that implements Zeilberger's algorithm [Zei90].

Remark 1. In order to derive this recurrence, we first tried to apply Zeilberger's algorithm on the input of $h_1(k,m)=\sum_{s=0}^{\lfloor\frac{2m-k-1}{2}\rfloor} q(s,k,m)$ with

$$q(s,k,m)=\frac{(-1)^{s+k-1}\binom{m-s-1}{2m-2s-k-1}(1-3m)_s(m)_s}{(3/2-2m)_s\,s!\,4^s}\frac{3m-3s-1}{3m-1}. \quad (79)$$

More precisely, we computed constants $c_i(k,m)\in\mathbb{Q}[k,m]$, free of s, and $g(s,k,m):=r(s,k,m)q(s,k,m)$ given by a rational function $r(s,k,m)\in\mathbb{Q}(s,k,m)$ such that

$$g(s+1,k,m)-g(s,k,m)=c_0(k,m)q(s,k,m)+\cdots+c_3(k,m)q(s,k+3,m).$$

The usual strategy is now to sum this "creative telescoping equation" over s from 0 to $\lfloor\frac{2m-k-1}{2}\rfloor$, in order to obtain a recurrence for $h_1(k,m)$. But we failed, since the denominator of $q(s,k,m)$ contains the factors $(m-s-k-1)(m-s-l-2)$ which causes poles in the given summation range. By rewriting $q(s,k,m)$ to $q'(s,k,m)$ by replacing $\binom{m-s-1}{2m-2s-k-1}$ with $\frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k-1)(2m-2s-k-2)}\binom{m-s-1}{2m-2s-k-3}$ these poles can be eliminated in the result of Zeilberger's algorithm. But in this case the sum $h_1(k,m)$ can be only defined in the summation range $0\leq s\leq\lfloor\frac{2m-k-1}{2}\rfloor-1$. i.e., $h_1'(k,m):=\sum_{s=0}^{\lfloor\frac{2m-k-1}{2}\rfloor-1}q'(s,k,m)$. Applying this representation of the definite sum $h_1'(k,m)$ to Zeilberger's algorithm gives the creative telescoping equation

$$g(s+1,k,m)-g(s,k,m)=c_0(k,m)q'(s,k,m)+\cdots+c_3(k,m)q'(s,k+3,m). \quad (80)$$

with

$$\begin{aligned}
c_0(k,m)=&-2(2+k)^2(1+k-2m)(k+2m)(-1+3m),\\
c_1(k,m)=&(1-3m)(-29k^3-5k^4-12((1+m-2m^2)+2k^2(-29-3m+6m^2)\\
&+2k(-23-10m+20m^2)),\\
c_2(k,m)=&(-1+3m)(-26k^3-4k^4-6((3+m-2m^2)+k^2(-59-6m+12m^2)\\
&+k(-55-14m+28m^2)),\\
c_3(k,m)=&(1+k)^2(3+k-2m)(2+k+2m)((-1+3m),
\end{aligned}$$

and

$$\begin{aligned}
g(s,k,m)=&\frac{(-1)^{s+k-1}\binom{m-s-1}{2m-2s-k-3}(1-3m)_s(m)_s}{(3/2-2m)_s\,s!\,4^s}\\
&\times\frac{-2(3+k)(-1+4m-2s)(m-s)s(3k+2k^2+4m+6km+3k^2m-4s-6ks-3k^2s)}{(2m-2s-k-1)(2m-2s-k-2)}.
\end{aligned}$$

Now we have to check that this summand telescoping equation (80) holds within the summation range. For this denote $p(s,k,m)=(-1)^k\binom{m-s-1}{2m-2s-k-3}$. In a first attempt we tried to express $p(s+1,k,m)$, that occurs in $g(s+1,k,m)$, and $p(s,k+i,m)$, that occurs in $q(s,k+i,m)$, in terms of $p(s,k,m)$ times a rational function in $\mathbb{Q}(s,k,m)$. But we failed to show (80) in this representation, since we entered again into pole problems within our summation range $0\leq s\leq\lfloor\frac{2m-k-1}{2}\rfloor-1$:

for instance we have $p(s,k+1,m)=\frac{2m-2s-k-1}{m-s-k-1}p(s,k,m)$. In order to avoid this, we represent all ingredients in (80) not in terms of $p(s,k,m)$ but in terms of $p(s+1,k+3,m)$, i.e.,

$$p(s+1,k,m)=p(s+1,k+3,m)\prod_{j=0}^{2}\frac{m-s-k-j-4}{2m-2s-k-j-5}, \tag{81}$$

$$p(s,k+i,m)=-p(s+1,k+3,m)\frac{m-s-1}{2m-2s-k-7}\prod_{j=0}^{3-i}\frac{m-s-k+j-6}{2m-2s-k+j-6},\quad 0\le i\le 3. \tag{82}$$

With these representations in $p(s+1,k+3,m)$ we finally manage to verify that (80) holds for $0\le s\le\lfloor\frac{2m-k-1}{2}\rfloor-4$.

Hence, summing equation (80) over s from 0 to an arbitrary d with $0\le d\le\lfloor\frac{2m-k-1}{2}\rfloor-4$ gives the relation

$$g(d+1,k,m)-g(0,k,m)=c_0(k,m)\,y(k,m)+\cdots+c_3\,y(k+3,m) \tag{83}$$

for the sum $y(k,m)=\sum_{s=0}^{d}q(s,k,m)=\sum_{s=0}^{d}q'(s,k,m)$. Now we do a case distinction on k.

If k is even, we can choose $d=m-k/2-5$, and obtain

$$\begin{aligned}&c_0(k,m)\,h_1(k,m)+\cdots+c_3(k,m)\,h_1(k+3,m)\\ &\qquad=g(m-\frac{k}{2}-4,k,m)-g(0,k,m)+\sum_{i=0}^{3}c_i(k,m)\sum_{j=1}^{4}q(m-\frac{k}{2}-j,k+i,m).\end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m-k/2-5\ge 0$, k even, the relation (78) holds. Similarly, for odd k, one can choose $d=m-(k-1)/2-4$, and obtains

$$\begin{aligned}&c_0(k,m)\,h_1(k,m)+\cdots+c_3(k,m)\,h_1(k+3,m)\\ &\qquad=g(m-\frac{k-1}{2}-3,k,m)-g(0,k,m)+\sum_{i=0}^{3}c_i(k,m)\sum_{j=0}^{3}q(m-\frac{k-1}{2}-j,k+i,m).\end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (78) holds for $m-(k-1)/2-4\ge 0$, k odd. Summarizing, for all $m\ge 5$ and $0\le k\le 2m-10$ the recurrence (78) contains the solution $h_1(k,m)$. The special cases $0\le m\le 5$ and $k\ge 2m-9$ for $m\ge 5$ can be verified by simple evaluation. This shows that the recurrence holds for all $m,k\ge 0$. ◊

We want to emphasize that recurrence (78) enables us to describe the k-shifts $h_1(k+j,m)$, $j\ge 3$, by linear combinations in $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$. Given this information, in a next step we can obtain with Sigma a recurrence for B_0, namely,

$$\begin{aligned}&(-2-i-i^2)(-1+i-2m)(-2+i+2m)B_0(i,m)\\ &\qquad+(3+i)(-2+2i-i^2+i^3-2m+4m^2)B_0(i+1,m)\\ &\qquad+(-3+i)(2+2i+i^2+i^3+2m-4m^2)B_0(i+2,m)\\ &\qquad\qquad+(-2+i-i^2)(2+i-2m)(1+i+2m)B_0(i+3,m)=0\end{aligned} \tag{84}$$

which holds for $2\le i\le 2m$.

Remark 2. More precisely, Sigma is able to compute constants

$$\begin{aligned}c_0(i,m)&=(1-i)i(1+i)(2+i+i^2)(-1+i-2m)(-2+i+2m),\\ c_1(i,m)&=(-1+i)i(1+i)(3+i)(-2+2i-i^2+i^3-2m+4m^2),\\ c_2(i,m)&=(-3+i)(-1+i)i(1+i)(2+2i+i^2+i^3+2m-4m^2),\\ c_3(i,m)&=(1-i)i(1+i)(2-i+i^2)(2+i-2m)(1+i+2m),\end{aligned}$$

and

$$g(k,i,m)=\frac{p_0(i,m)h_1(k,i,m)+p_1(i,m)h_1(k+1,i,m)+p_2(i,m)h_1(k+2,i,m)}{(1+k)^2}(k-1)\binom{i+k-3}{i-2} \tag{85}$$

where

$$\begin{aligned}
p_0(i,m)=&-(12-6i-12i^2+6i^3+26k-82ik+7i^2k+16i^3k-3i^4k-24k^2\\
&-106ik^2+73i^2k^2-4i^3k^2+i^4k^2-60k^3+16ik^3+59i^2k^3-16i^3k^3\\
&+9i^4k^3-12k^4+60ik^4-i^2k^4+4i^3k^4+5i^4k^4+14k^5+12ik^5-4i^2k^5+6i^3k^5+4k^6\\
&-2ik^6+2i^2k^6+12m-90im+54i^2m+10i^3m-2i^4m-46km-20ikm\\
&+76i^2km-14i^3km+8i^4km-24k^2m+64ik^2m-4i^2k^2m+2i^3k^2m+6i^4k^2m\\
&+14k^3m+12ik^3m-4i^2k^3m+6i^3k^3m+4k^4m-2ik^4m+2i^2k^4m-24m^2+180im^2\\
&-108i^2m^2-20i^3m^2+4i^4m^2+92km^2+40ikm^2-152i^2km^2+28i^3km^2-16i^4km^2\\
&+48k^2m^2-128ik^2m^2+8i^2k^2m^2-4i^3k^2m^2-12i^4k^2m^2-28k^3m^2-24ik^3m^2+8i^2k^3m^2\\
&-12i^3k^3m^2-8k^4m^2+4ik^4m^2-4i^2k^4m^2),\\
p_1(i,m)=&(-2+i+k)(-10+10i^2-32k+39ik+22i^2k-i^3k-11k^2+85ik^2+5i^2k^2\\
&+5i^3k^2+34k^3+53ik^3-7i^2k^3+10i^3k^3+29k^4+4ik^4+3i^2k^4+4i^3k^4+6k^5-3ik^5+3i^2k^5\\
&-10m+18im+10i^2m-2i^3m-2km+40ikm+4i^2km+2i^3km+20k^2m+16ik^2m\\
&-6i^2k^2m+6i^3k^2m+8k^3m-4ik^3m+4i^2k^3m+20m^2-36im^2-20i^2m^2\\
&+4i^3m^2+4km^2-80ikm^2-8i^2km^2-4i^3km^2-40k^2m^2-32ik^2m^2\\
&+12i^2k^2m^2-12i^3k^2m^2-16k^3m^2+8ik^3m^2-8i^2k^3m^2),\\
p_2(i,m)=&(-2+i+k)(-1+i+k)(2+2i+5k+ik+2k^2\\
&-ik^2+i^2k^2)(2+k-2m)(1+k+2m).
\end{aligned}$$

This information is sufficient to produce (84). Namely, suppose that $m\geq 2$. Then for $q(k,i,m)=h_1(k,m)\binom{i+k-3}{i-2}$ we have that

$$g(k+1,i,m)-g(k,i,m)=c_0(i,m)\,q(k,i,m)+\cdots+c_3(i,m)\,q(k,i+3,m) \tag{86}$$

for $0\leq k<2m-3$ and $2\leq i\leq 2m$ which can be verified as follows. By applying the shift in k to the representation (85) of $g(k,i,m)$ the expression $g(k+1,i,m)$ depends on $\binom{i+k-2}{i-2}$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $h_1(k+3,m)$. Then the term $k\binom{i+k-2}{i-2}$ can be rewritten with $(i+k-2)\binom{i+k-3}{i-2}$, and $h_1(k+3,m)$ can be expressed by $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ by applying the relation given in (78) for any $0\leq k<2m-3$. Moreover, $q(k,i+j,m)=q_j\,q(k,i,m)$ with $q_1=\frac{i+k-2}{i-1}$, $q_2=\frac{(i+k-2)(i+k-1)}{(i-1)i}$, and $q_3=\frac{(i+k-2)(i+k-1)(i+k)}{(i-1)i(i+1)}$. In this representation the equation (86) can be verified by simple term rewriting for $i\geq 2$. Summing equation (86) over k from 0 to $2m-4$ gives

$$\begin{aligned}
&c_0(i,m)\,B_0(i,m)(k,m)+\cdots+c_3(i,m)\,B_0(i+3,m)\\
&\qquad=g(2m-3,i,m)-g(0,i,m)+\sum_{j=0}^{3}c_j(i,m)\sum_{r=2m-3}^{2m-1}\binom{i+r-3}{i-2}h_1(r,m).
\end{aligned}$$

Observe that under the assumption $2\leq i\leq 2m$ in $g(2m-3,i,m)$ the binomial $\binom{2m+i-7}{i-2}$ is well defined for $m>2$. In this case one can show that the right hand side vanishes to 0 which gives recurrence (84). Since $B_0(i,m)$ evaluates to a polynomial in $\mathbb{Q}[i]$ for any $m\in\mathbb{N}$, the special cases $m=1,2$ follow by plugging in those polynomials for the specific evaluations at $m=1,2$ in $\sum_{r=0}^{3}c_r(i,m)\,B_0(i+r,m)$ and showing that these expressions are the 0 polynomial. Summarizing, we have proven that $B_0(i,m)$ is a solution of recurrence (84) for all $2\leq i\leq 2m$. $\Diamond$

Next we compute a recurrence for $B_2(i,m)$, namely

$$\begin{aligned}
&2(2+i)(3+i)(1+i-2m)(i+2m)B_2(i,m)\\
&\quad+(3+i)(3i^2+i^3+8m(-1+2m)+i(2-2m+4m^2))B_2(i+1,m)\\
&\quad-2(1+i)(24+14i^2+2i^3+5m-10m^2+i(32+m-2m^2))B_2(i+2,m)\\
&\quad-2(2+i)(6+6i^2+i^3+i(11+2m-4m^2))B_2(i+3,m)\\
&\quad+2(1+i)(3+i)(8+6i+i^2+m-2m^2)B_2(i+4,m)\\
&\quad+(1+i)(2+i)(4+i-2m)(3+i+2m)B_2(i+5,m)=0 \quad (87)
\end{aligned}$$

for all $2 \le i \le 2m$.

Remark 3. In order to achieve this, we actually compute a creative telescoping equation for the sum

$$B_2^{(e)}(i,m,e) := \sum_{k=0}^{2m-4} q(k,i,m,e)$$

where

$$q(k,i,m,e) = (-1)^k \frac{(k-i+e)!}{(k-i)!} h_1(k,m).$$

Namely, with Sigma we can compute

$$\begin{aligned}
c_0(i,m,e) :=& 2(e^2-e(7+3i)+3(6+5i+i^2))(1+i-2m)(i+2m),\\
c_1(i,m,e) :=& 4e^3(1+i)-e^2(26+37i+11i^2+2m-4m^2)+e(58+56i^2+9i^3+14m-28m^2\\
&+i(105+6m-12m^2))+3(3+i)(3i^2+i^3+8m(-1+2m)+i(2-2m+4m^2)),\\
c_2(i,m,e) :=& -4e^3+2e^4-e^2(37+51i+16i^2+2m-4m^2)-6(1+i)(24+14i^2+2i^3+5m-10m^2\\
&+i(32+m-2m^2))+e(165+169i^2+30i^3+26m-52m^2+6i(50+m-2m^2)),\\
c_3(i,m,e) :=& 5e^4-e^3(35+17i)+e^2(82+82i+19i^2-4m+8m^2)\\
&-6(2+i)(6+6i^2+i^3+i(11+2m-4m^2))-2e(17+5i^2-5m+10m^2+i(19-6m+12m^2)),\\
c_4(i,m,e) :=& 4e^4-2e^3(19+9i)-e(7+3i)(33+37i+8i^2+2m-4m^2)+e^2(139+139i+32i^2+2m-4m^2)\\
&+6(3+4i+i^2)(8+6i+i^2+m-2m^2),\\
c_5(i,m,e) :=& (e^2-e(4+3i)+3(2+3i+i^2))(-3+e-i-2m)(-4+e-i+2m),
\end{aligned}$$

and

$$\begin{aligned}
g(k,i,m,e) = -(&p_0(k,i,m,e)h_1(k,i,m)+p_1(k,i,m,e)h_1(k+1,i,m)+p_2(k,i,m,e)h_1(k+2,i,m))\\
&\times \frac{q(k,i,e)}{d(k,i,e)}(-1)^k\frac{(k-i+e)!}{(k-i)!}
\end{aligned}$$

where

$$\begin{aligned}
q(k,i,e) :=& (-1+e)e(i-k),\\
d(k,i,e) :=& (1+k)^2(-4+e-i+k)(-3+e-i+k)(-2+e-i+k)(-1+e-i+k)(e-i+k),
\end{aligned}$$

and

$$p_0(k,i,m,e) = -((e^4(9k^3+5k^4+2m(-1+2m)+k(-3+8m-16m^2)+k^2(1+6m-12m^2))-e^3(3-40k+20k^2 \\ +126k^3+49k^4-14k^5-26m+122km+66k^2m-18k^3m+52m^2-244km^2-132k^2m^2+36k^3m^2+i(3-10k+10k^2 \\ +52k^3+29k^4-8m+48km+36k^2m+16m^2-96km^2-72k^2m^2))+e^2(33-202k+146k^2+688k^3+165k^4 \\ -132k^5+10k^6-108m+720km+258k^2m-176k^3m+14k^4m+216m^2-1440km^2-516k^2m^2+352k^3m^2-28k^4m^2 \\ +i^2(12-2k+45k^2+126k^3+67k^4+2m+124km+86k^2m-4m^2-248km^2-172k^2m^2)+i(45-87k+160k^2+579k^3 \\ +225k^4-62k^5-42m+590km+310k^2m-82k^3m+84m^2-1180km^2-620k^2m^2+164k^3m^2))-e(120-447k+459k^2 \\ +1681k^3+153k^4-442k^5+60k^6-150m+1894km+362k^2m-606k^3m+84k^4m+300m^2-3788km^2-724k^2m^2+1212k^3m^2 \\ -168k^4m^2+i^3(15+14k+69k^2+142k^3+72k^4+28m+152km+96k^2m-56m^2-304km^2-192k^2m^2)+i^2(102+13k+405k^2 \\ +933k^3+337k^4-102k^5+110m+1018km+482k^2m-138k^3m-220m^2-2036km^2-964k^2m^2+276k^3m^2)+i(207-238k \\ +765k^2+2116k^3+454k^4-422k^5+30k^6+40m+2358km+754k^2m-574k^3m+42k^4m-80m^2-4716km^2-1508k^2m^2 \\ +1148k^3m^2-84k^4m^2))+2(72-180k+262k^2+753k^3-65k^4-249k^5+55k^6+917km+7k^2m-353k^3m+77k^4m-1834km^2 \\ -14k^2m^2+706k^3m^2-154k^4m^2+i^4(3+4k+16k^2+30k^3+15k^4+11m+36km+21k^2m-22m^2-72km^2-42k^2m^2) \\ +i^3(30+28k+141k^2+260k^3+87k^4-30k^5+83m+310km+129k^2m-42k^3m-166m^2-620km^2-258k^2m^2 \\ +84k^3m^2)+i(150-100k+571k^2+1256k^3+17k^4-358k^5+60k^6+169m+205m+1017km+258k^2m-249k^3m+21k^4m \\ -410m^2-2034km^2-516k^2m^2+498k^3m^2-42k^4m^2))))),$$

$$p_1(k,i,m,e) = (-4+e-i+k)(-60-88k-257k^2-291k^3-7k^4+55k^5+18m-148km-174k^2m+88k^3m-36m^2 \\ +296km^2+348k^2m^2-176k^3m^2-i^3(10+32k+55k^2+48k^3+15k^4+10m+30km+24k^2m-20m^2-60km^2 \\ -48k^2m^2)-3i^2(20+56k+97k^2+83k^3+17k^4-5k^5+14m+52km+42k^2m-8k^3m-28m^2-104km^2-84k^2m^2 \\ 16k^3m^2)-i(110+256k+497k^2+450k^3+39k^4-60k^5+38m+270km+240k^2m-96k^3m-76m^2-540km^2 \\ -480k^2m^2+192k^3m^2)+e^3(10k^3+4k^4+2m(-1+2m)+k^2(5+6m-12m^2)+k(-1+2m-4m^2)) \\ -e^2(5-2k+50k^2+88k^3+26k^4-5k^5-16m+24km+54k^2m-8k^3m+32m^2-48km^2 \\ -108k^2m^2+16k^3m^2+i(5+9k+32k^2+45k^3+17k^4-4m+14km+26k^2m+8m^2-28km^2-52k^2m^2)) \\ +e(35+33k+197k^2+285k^3+56k^4-30k^5-38m+104km+174k^2m-48k^3m+76m^2-208km^2 \\ -348k^2m^2+96k^3m^2+i^2(15+41k+79k^2+80k^3+27k^4+6m+38km+42k^2m-12m^2-76km^2 \\ -84k^2m^2)+i(50+108k+262k^2+298k^3+79k^4-15k^5-4m+128km+166k^2m-24k^3m+8m^2-256km^2 \\ -332k^2m^2+48k^3m^2))),$$

$$p_2(k,i,m,e) = -((-4+e-i+k)(-3+e-i+k)(4+3k+11k^2+e^2k^2+i^2(2+3k+3k^2)+2i(3+4k+6k^2) \\ -e(1+6k^2+i(1+k+3k^2)))(2+k-2m)(1+k+2m)).$$

Note that for all $e \notin \mathbb{Z}$ and $k,i,m \in \mathbb{Z}$ we have that $d(k,i,e) \neq 0$. Now suppose that $2 \leq i \leq 2m-4$, $m \geq 3$. Then by using the relation (78) one can verify that the creative telescoping equation

$$g(k+1,i,m,e) - g(k,i,m,e) = c_0(i,m,e)\,q(k,i,m,e) + \cdots + c_5(i,m,e)\,q(k,i+5,m,e) \tag{88}$$

holds for $0 \leq k \leq 2m-4$ and $e \notin \mathbb{Z}$ as we have sketched it for the sum case $B_0(i,m)$ in Remark 2. Note that the factor $i-k$ in $q(k,i,e)$ becomes essential in order to avoid poles in this verification. Then summing this equation over k from 0 to $2m-4$ results the inhomogeneous recurrence

$$c_0(i,m,e)\,B_2^{(e)}(i,m,e) + \cdots + c_0(i+5,m,e)\,B_2^{(e)}(i+5,m,e) = g(2m-3,i,m,e) - g(0,i,m,e) \tag{89}$$

for all $e \notin \mathbb{Z}$ and $2 \leq i \leq 2m-4$.

But this means that also

$$B_2'(i,m) := \lim_{e \to 0} B_2^{(e)}(i,m) = \sum_{k=i}^{2m-4} (-1)^k\, h_1(k,m)$$

is a solution of the recurrence (89) in the limit $e \to 0$. Observe that $g(k,i,m,0) = 0$ which comes from the fact that $q(k,i,m,0) = 0$. Therefore $B_2'(i,m)$ is a solution of the recurrence

$$c_0(i,m,0)\,B_2'(i,m) + \cdots + c_5(i,m,0)\,B_2'(i+5,m) = 0 \tag{90}$$

for all $2 \leq i \leq 2m-4$ and $m \geq 3$. Note that this recurrence is nothing but recurrence (87) given from above. In the final step we observe that $\sum_{j=0}^{5} c_j(i,m,0) = 0$ which shows that any constant is a solution of (90) or (87). Therefore $B_2(i,m) = B_2'(i,m) + \sum_{j=2m-3}^{2m-1} (-1)^j h_1(j,m)$ is a solution

of (87) for all $2 \le i \le 2m$ and $m \ge 3$. Similarly, for the specific values $m = 1, 2$ with $2 \le r \le 2m$ the constant sum $B_2(r, m)$ is a solution of (87). This finally shows that $B_2(i, m)$ is a solution of (87) for all $2 \le i \le 2m$.

As a side remark we would like to mention that we also could have derived a recurrence for $B_1(i, m)$ that holds for all $2 \le i \le m$. In a naive attempt we might set up the definite summation problem with $B_1^{(e)}(i, m, e) := \sum_{k=0}^{2m-4} q'(k, i, m, e)$ where $q'(k, i, m, e) = (-1)^k \frac{(i-k-3+e)!}{(i-k-3)!} h_1(k, m)$ and derive a creative telescoping solution with Sigma. But now note that $\frac{(i-k-3+e)!}{(i-k-3)!}$ shifted in i equals $\frac{(i-k-3+e)!}{(i-k-3)!} \frac{i-k-2+e}{i-k-2}$. Hence, during the verification for the corresponding creative telescoping equation similarly to (88), we have a pole problem in the summation range $2 \le i \le 2m-4$. This could be avoided by choosing a different representation of $q'(k, i, m, e)$, namely in $q'(k, i-3, m, e)$, similarly to Remark 1. ◊

Let us denote by $L(i, m)$ the left hand side of (76), i.e.,

$$L(i, m) = 6(-1)^i B_2(i, m) - B_0(i, m) + 2h_1(i-2, m) - 5h_1(i-1, m) + 3(-1)^i t_1(2m-1). \quad (91)$$

Then we have all the ingredients in hand in order to compute a recurrence that contains $L(i, m)$ as solution for any $2 \le i \le 2m$. Verifying that sufficiently many initial values of the recurrence for $L(i, m)$ vanish, will prove that $L(i, m)$ indeed equals 0.

To this end, we first derive, for instance with the packages [SZ94] or [Mal96], a recurrence in i that contains the solutions

$$c_1 (-1)^i + c_2 (-1)^i B_2(i, m) + c_3 h_1(i-1, m) + c_4 h_2(i-2, m) + c_5 B_0(i, m) \quad (92)$$

for any constants $c_i \in \mathbb{Q}$ and any $2 \le i \le 2m$.

As already pointed out in Remark 3 the recurrence (87) contains besides $B_2(i, m)$ also the constant solution 1. Therefore changing the signs of the coefficients of $B_2(i, m)$, $B_2(i+2, m)$ and $B_2(i+4, m)$ in (87) gives a recurrence that contains the solutions $c_1 (-1)^i + c_2 (-1)^i B_2(i, m)$ for $2 \le i \le 2m$. Moreover one obtains immediately recurrences for $h_1(i-1, m)$ and $h_1(i-2, m)$, if one substitutes k by $i-1$ or $i-2$ in the recurrence (78); those recurrences are both valid in the range $2 \le i \le 2m$. Then combining these two recurrences with the recurrence (84) for $B_0(i, m)$ and the recurrence for $c_1 (-1)^i + c_2 (-1)^i B_2(i, m)$ by using procedures from the packages [SZ94] or [Mal96] enables us to compute a single homogeneous recurrence of order 9,

$$c_0(i, m) S(i, m) + c_1(i, m) S(i+1, m) + \cdots + c_9(i, m) S(i+9, m) = 0, \quad (93)$$

which contains the solution for (92) and therefore also (91) for all $2 \le i \le 2m$. The explicit expressions for the $c_j(i, m)$ can be easily recomputed with the Maple package [SZ94] gfun or the Mathematica package [Mal96] GeneratingFunctions.

Moreover the following remark shows that in the range $2 \le i \le 2m - 9$ the leading coefficient $c_9(i, m)$ in (93) does not vanish.

Remark 4. The leading coefficient $c_9(i, m)$ in (93) is

$$(i-1)i(6+i-2m)(7+i-2m)(8+i-2m)(5+i+2m)(6+i+2m)(7+i+2m)\, p(i, m)$$

with the polynomial $p(i, m) \in \mathbb{Z}[i, m]$ defined as

```
p(i, m) := 286934400 i + 1022116320 i^2 + 1518851088 i^3 + 1277708580 i^4 + 764090658 i^5 + 449853750 i^6 + 295140024 i^7 + 169048296 i^8 +
  70449102 i^9 + 20137410 i^10 + 3830652 i^11 + 462672 i^12 + 32076 i^13 + 972 i^14 + 1012435200 m + 5482935360 i m +
  14278564656 i^2 m + 23213538792 i^3 m + 26151610626 i^4 m + 21597662355 i^5 m + 13483008435 i^6 m + 6458953002 i^7 m +
  2388711936 i^8 m + 682573911 i^9 m + 150177099 i^10 m + 25219524 i^11 m + 3183792 i^12 m + 295824 i^13 m + 19752 i^14 m + 912 i^15 m +
  24 i^16 m + 2024870400 m^2 + 6988032000 i m^2 + 10391171184 i^2 m^2 + 7815524436 i^3 m^2 + 1403744502 i^4 m^2 - 3433378715 i^5 m^2 -
  4306205557 i^6 m^2 - 2795766398 i^7 m^2 - 1193680594 i^8 m^2 - 353952095 i^9 m^2 - 72888049 i^10 m^2 - 9860680 i^11 m^2 -
  699794 i^12 m^2 + 13148 i^13 m^2 + 7588 i^14 m^2 + 608 i^15 m^2 + 16 i^16 m^2 - 13619181888 m^3 - 62360574624 i m^3 - 138500395704 i^2 m^3 -
  196534634088 i^3 m^3 - 197750758622 i^4 m^3 - 148547525347 i^5 m^3 - 85793698617 i^6 m^3 - 38793616174 i^7 m^3 -
  13884128774 i^8 m^3 - 3953893039 i^9 m^3 - 896125845 i^10 m^3 - 160785656 i^11 m^3 - 22549686 i^12 m^3 - 2412656 i^13 m^3 -
  188176 i^14 m^3 - 9728 i^15 m^3 - 256 i^16 m^3 + 1472884992 m^4 + 17277933456 i m^4 + 55255553160 i^2 m^4 + 97387495728 i^3 m^4 +
  114340534314 i^4 m^4 + 96961318441 i^5 m^4 + 61869893729 i^6 m^4 + 30390885826 i^7 m^4 + 11641806632 i^8 m^4 + 3500762485 i^9 m^4 +
  827525561 i^10 m^4 + 153150896 i^11 m^4 + 21942460 i^12 m^4 + 2380016 i^13 m^4 + 187216 i^14 m^4 + 9728 i^15 m^4 + 256 i^16 m^4 +
  25007101632 m^5 + 96066457488 i m^5 + 181925812632 i^2 m^5 + 221098537536 i^3 m^5 + 189184506516 i^4 m^5 + 118702812708 i^5 m^5 +
  55719392004 i^6 m^5 + 19763963784 i^7 m^5 + 5314961340 i^8 m^5 + 1080128100 i^9 m^5 + 164117244 i^10 m^5 + 18299232 i^11 m^5 +
  1456536 i^12 m^5 + 78336 i^13 m^5 + 2304 i^14 m^5 - 3283424064 m^6 - 24487119216 i m^6 - 62740182552 i^2 m^6 - 91955201268 i^3 m^6 -
  89825565356 i^4 m^6 - 62243885388 i^5 m^6 - 31512673972 i^6 m^6 - 11837036352 i^7 m^6 - 3321422232 i^8 m^6 - 695467968 i^9 m^6 -
  107666704 i^10 m^6 - 12118848 i^11 m^6 - 968336 i^12 m^6 - 52224 i^13 m^6 - 1536 i^14 m^6 - 21238328448 m^7 - 64255523184 i m^7 -
  96366768720 i^2 m^7 - 92096792976 i^3 m^7 - 60749842800 i^4 m^7 - 28468056528 i^5 m^7 - 9555442224 i^6 m^7 - 2283228864 i^7 m^7 -
  379758816 i^8 m^7 - 42201312 i^9 m^7 - 2991072 i^10 m^7 - 138240 i^11 m^7 - 4608 i^12 m^7 + 6290303040 m^8 + 23115529440 i m^8 +
  38148908736 i^2 m^8 + 38686906320 i^3 m^8 + 26702560752 i^4 m^8 + 13015763424 i^5 m^8 + 4522579392 i^6 m^8 + 1111374432 i^7 m^8 +
  188339328 i^8 m^8 + 21100656 i^9 m^8 + 1495536 i^10 m^8 + 69120 i^11 m^8 + 2304 i^12 m^8 + 5912676864 m^9 + 12381602112 i m^9 +
  13532698560 i^2 m^9 + 9775140480 i^3 m^9 + 4854527040 i^4 m^9 + 1615086720 i^5 m^9 + 339571200 i^6 m^9 + 40320000 i^7 m^9 +
  2053440 i^8 m^9 - 2620173312 m^10 - 5651547264 i m^10 - 5713697280 i^2 m^10 - 3829696704 i^3 m^10 - 1857903168 i^4 m^10 -
  627501888 i^5 m^10 - 134592960 i^6 m^10 - 16128000 i^7 m^10 - 821376 i^8 m^10 + 209811456 m^11 + 681806592 i m^11 +
  312221952 i^2 m^11 - 87664896 i^3 m^11 - 91535616 i^4 m^11 - 20217600 i^5 m^11 - 1347840 i^6 m^11 + 46356480 m^12 - 92874240 i m^12 -
  77865984 i^2 m^12 + 29221632 i^3 m^12 + 30511872 i^4 m^12 + 6739200 i^5 m^12 + 449280 i^6 m^12 - 111992832 m^13 -
  124056576 i m^13 - 24192000 i^2 m^13 + 38633472 m^14 + 35444736 i m^14 + 6912000 i^2 m^14 - 5308416 m^15 + 1327104 m^16.
```

By inspection the first 8 factors in $c_9(i,m)$ do not vanish in the range $2 \le i \le 2m-9$. Moreover by the cylindric algebraic decomposition method [CH91] one can decide that there does not exist any solution to the problem

$$\exists i \in \mathbb{R} \exists m \in \mathbb{R} \big[p(i,m) = 0 \wedge 3 \le i \le 2m \big]$$

where $\mathbb{R}$ stands for the real numbers. More precisely, in our instance the algorithm returns the equivalent formula *FALSE* which means that there does not exist any solution $i, m \in \mathbb{R}$ with $3 \le i \le 2m$ and $p(i,m) = 0$. In particular it follows that for any integer i, m with $3 \le i \le 2m$ there does not exist any diophantine solution $p(i,m) = 0$. We want to mention that we verified this result with the cylindric algebraic decomposition implementations of Mathematica 5.0 and with the package QEPCAD B version 1.30 of Hoon Hong. Note that for $2 \le i < 3$ there are solutions $m \in \mathbb{R}$ with $p(i,m)$. As a side remark note that the implementations mentioned above failed to compute a cylindric algebraic decomposition for the larger range $2 \le i \le 2m$ in a reasonable amount of time. But by checking that $p(2,m)$ does not have any integer root (by factoring the polynomial $p(2,m) \in \mathbb{Z}[m]$ over $\mathbb{Z}$), we complete our verification that $p(i,m) \neq 0$ for any integer with $2 \le i \le 2m$. $\Diamond$

In order to prove Identity (76) for $2 \le i \le 2m$ it suffices to show that the first nine initial values of $L(i,m)$ are 0, i.e., we will verify that $L(r,m) = 0$ for all $2 \le r \le 10$ and all $2m \ge r$. Then together with the fact that $L(i,m)$ is a solution of (93) for all $2 \le i \le 2m$ and $c_9(i,m) \neq 0$ for all $2 \le i \le 2m-9$, it follows that $L(i,m) = 0$ for all $2 \le i \le 2m$.

To accomplish this task we will use the equivalent relation (77). Then showing that $L(r,m) = 0$ for $2 \le r \le 10$ and all m with $2m \ge r$ is equivalent to proving

$$6(-1)^r B_1(r,m) + B_0(r,m) + 4h_1(r-2,m) - h_1(r-1,m) = 0 \tag{94}$$

for the same range of values.

For $r = 2$ we can prove this last identity for instance as follows. With Sigma we compute the expression

$$g(k,m) := \frac{1}{2\,(1+k)\,m\,(-1+2\,m)} \big(-2\,\big(3\,k^2+k^3+3\,(1-2\,m)\,m+k\,\big(2+m-2\,m^2\big)\big)\,h_1(k,m) \\ + \big(9\,k^2+3\,k^3+2\,(1-2\,m)\,m+k\,\big(6+4\,m-8\,m^2\big)\big)\,h_1(k+1,m)$$

$$- k\,\left(2+3\,k+k^2+2\,m-4\,m^2\right)\,h_1(k+2,m)\big) \quad (95)$$

such that

$$g(k+1,m)-g(k,m)=h_1(k,m) \quad (96)$$

holds for all $0 \le k \le 2m-4$. We want to emphasize that this equation can be verified by using the relation (78). Now summing this equation from 0 to $2m-4$ gives $\sum_{k=0}^{2m-4} h_1(k,m) = g(2m-3,m)-g(0,m)$. Adding $h_1(2m-3,m)+h_1(2m-2,m)+h_1(2m-1,m)$ to both sides together with Lemma 2 and

$$h_1(2m-3,m)=\frac{-6+21m-18m^2+4m^3}{2(-3+4m)}, \quad h_1(2m-2,m)=1-m, \quad h_1(2m-1,m)=1$$

finally gives

$$B_0(2,m)=h_1(1,m), \quad m\ge 1. \quad (97)$$

Hence for $r=2$ we have that

$$\begin{aligned}6(-1)^i B_1(r,m)+B_0(r,m)+4h_1(r-2,m)-h_1(r-1,m)&\\ = h_1(1,m)+4h_1(0,m)-h_1(1,m)=4h_1(0,m)=0, \quad \forall m\ge 1,&\end{aligned}$$

which verifies the identity (94) for the case $r=2$.

Similar to the derivation of (97) one can obtain the following representations[6] of $B_0(r,m)$ that hold for all m with $2m \ge r$, $2 \le r \le 10$.

$$\begin{aligned}
B_0(2,m) =& h_1(1,m),\\
B_0(3,m) =& -4h_1(1,m)+h_1(2,m),\\
B_0(4,m) =& \frac{-12h_1(1,m)+(3+m-2m^2)h_1(2,m)}{(1+m)(-3+2m)},\\
B_0(5,m) =& \frac{3(-24-3m+7m^2-4m^3+4m^4)h_1(1,m)+(18+3m-7m^2+4m^3-4m^4)h_1(2,m)}{(-2+m)(1+m)(-3+2m)(3+2m)},\\
B_0(6,m) =& \frac{(120+19m-31m^2-28m^3+28m^4)h_1(1,m)+(-30-13m+25m^2+4m^3-4m^4)h_1(2,m)}{(1+m)(2+m)(-5+2m)(-3+2m)},\\
B_0(7,m) =& (-10800-3060m+6339m^2-838m^3+647m^4+464m^5-328m^6+32m^7-16m^8)h_1(1,m)\\
&+2(1350+495m-954m^2-134m^3+85m^4+112m^5-56m^6-32m^7+16m^8)h_1(2,m)\Big/\\
&((-3+m)(-2+m)(1+m)(2+m)(-5+2m)(-3+2m)(3+2m)(5+2m)),\\
B_0(8,m) =& (-226800-98820m+172035m^2+100077m^3-88313m^4-28501m^5+19850m^6-1288m^7+224m^8\\
&+560m^9-224m^{10})h_1(1,m)+2(28350+19305m-34983m^2-14460m^3+14195m^4+775m^5-974m^6\\
&+760m^7-320m^8-80m^9+32m^{10})h_1(2,m)\Big/((-3+m)(-2+m)(1+m)(2+m)(3+m)(-7+2m)\\
&(-5+2m)(-3+2m)(3+2m)(5+2m)),\\
B_0(9,m) =& -10(-211680-8442m+15159m^2+5996m^3-1538m^4-10353m^5+5754m^6+1944m^7-912m^8\\
&-80m^9+32m^{10})h_1(1,m)+(-529200-76860m+125976m^2+107021m^3-87398m^4\\
&-46245m^5+28002m^6+4824m^7-2352m^8-80m^9+32m^{10})h_1(2,m))\Big/\\
&((-4+m)(-3+m)(-2+m)(2+m)(3+m)(-7+2m)(-5+2m)(3+2m)(5+2m)(7+2m)),\\
B_0(10,m) =& 2(-114307200-56269080m+98214930m^2+56493117m^3-52362987m^4-10634788m^5+9427703m^6\\
&-3727349m^7+1168354m^8+909516m^9-328536m^{10}-38768m^{11}+13408m^{12}-448m^{13}\\
&+128m^{14})h_1(1,m)-3(3+m-2m^2)^2(-2116800-114240m+247244m^2-74384m^3+70991m^4\\
&+8327m^5-6158m^6+1016m^7-448m^8-80m^9+32m^{10})h_1(2,m)\Big/((-4+m)(-3+m)(-2+m)\\
&(1+m)(2+m)(3+m)(4+m)(-9+2m)(-7+2m)(-5+2m)(-3+2m)(3+2m)(5+2m)(7+2m)).
\end{aligned}$$

[6]Note that identity $B_0(2,m)=h_1(1,m)$ for $m\ge 1$ proves Lemma 5.

Remark 5. As for the case $r=2$, for $B_0(r,m)=\sum_{k=0}^{2m}\binom{r+k-3}{r-2}h_1(k,m)$ with $r\in\{3,\ldots,10\}$ we can compute with Sigma expressions $g_r(k,m)$, listed in Appendix A, such that

$$g_r(k+1,m)-g_r(k,m)=\binom{r+k-3}{r-2}h_1(k,m)$$

holds for all $0\leq k\leq 2m-4$. Similarly to the case $r=2$, the corresponding equations can be verified by using the relation (78). Finally with telescoping, as for the case $r=2$, one derives the above closed form evaluations of the $B_0(r,m)$. ◊

Hence we see that for $r\in\{2,\ldots,10\}$ each $B_0(r,m)$ can be expressed in terms of $h_1(1,m)$ and $h_1(2,m)$. Moreover, by (75) each corresponding $B_1(r,m)$ can be written as a linear combination of the $h_1(1,m),\ldots,h_1(r-3,m)$. In particular by applying relation (78) each of the $h_1(j,m)$ for $3\leq j\leq r-3$ can be reduced to $h_1(1,m)$ and $h_1(2,m)$ where $m>r/2$. In other words, the left side of identity (94) can be expressed in terms of $h_1(1,m)$ and $h_1(2,m)$ that occur only linearly. Since this expression collapses to 0, identity (94) holds for $m>r/2$. The special case $2m=r$ of identity (94) can be shown separately by simple evaluation.

Applying this proof strategy for all $r\in\{2,\ldots,10\}$ shows that $L(r,m)=0$ for all r,m with $2m\geq r$ and $2\leq r\leq 10$ which finally proves that $L(i,m)=0$ for all $2\leq i\leq 2m$.

7. A proof of identity (32)

In this section we will use the abbreviation $h_2(k,m):=f_2(2m-k,m)$. By redefining $A_1(i,m)$ from above with

$$A_1(i,m)=\sum_{k=0}^{i-3}(-1)^k h_2(k,m) \tag{98}$$

and Lemma 2 it is immediate that showing (32) is equivalent to proving

$$-\frac{3}{2}(-1)^i t_1(2m)=h_2(i-2,m)+2h_2(i-1,m)+3(-1)^i(A_1(i,m)-A_2(i,m))+A_0(i,m) \tag{99}$$

for $3\leq i\leq 2m+1$. Moreover the sum in Proposition 3 can be rewritten as

$$\frac{1}{2}t_1(2m)=\sum_{k=0}^{2m}(-1)^k h_2(k,m)=A_1(i,m)+A_2(i,m)+(-1)^i h_2(i-2,m)-(-1)^i h_2(i-1,m)$$

for $i\geq 3$. Consequently to show identity (99) is equivalent to prove[7]

$$6(-1)^i A_2(i,m)-A_0(i,m)+2h_2(i-2,m)-5h_2(i-1,m)-3(-1)^i t_1(2m-1)=0 \tag{100}$$

or, equivalently,

$$6(-1)^i A_1(i,m)+A_0(i,m)+4h_2(i-2,m)-h_2(i-1,m)=0 \tag{101}$$

for all $3\leq i\leq 2m+1$. To verify these relations we proceed analogously to the proof of Identity (33). Using Sigma we first compute a recurrence of $h_2(k,m)$ for all $m,k\geq 0$.

$$\begin{aligned}&2(2+k)^2(k-2m)(1+k+2m)h_2(k,m)\\ &-(29k^3+5k^4+k(46-20m-40m^2)-12(-1+m+2m^2)-2k^2(-29+3m+6m^2))h_2(1+k,m)\\ &+(26k^3+4k^4+k(55-14m-28m^2)+k^2(59-6m-12m^2)-6(-3+m+2m^2))h_2(2+k,m)\\ &-(1+k)^2(2+k-2m)(3+k+2m)h_2(3+k,m)=0\end{aligned} \tag{102}$$

[7]Note that equations (101) and (100) are analogous to (77) and (76), except a sign change of $3(-1)^i t_1(2m-1)$ in (76) and (100).

Remark 6. Denote by $q(s,k,m)$ the summand of $h_2(k,m)$, i.e., $h_2(k,m)=\sum_{s=0}^{\lfloor\frac{2m-k-1}{2}\rfloor} q(s,k,m)$. First note that $\binom{m-s-1}{2m-2s-k}=\frac{s-m+k}{m-s}\binom{m-s}{2m-2s-k}$ for all $0\le s<m$. Therefore with

$$h_2'(k,m):=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}\frac{k}{m-s}\binom{m-s}{2m-2s-k}\frac{(-1)^{s+k}}{2m\,4^s}\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac12-2m)_r} \tag{103}$$

we have that $h_2(k,m)=h_2'(k,m)+q(\lfloor\frac{2m-k}{2}\rfloor,k,m)$ for $0\le k\le 2m$. Now similarly as in Remark 1 we transform the sum $h_2'(s,k,m)=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}q'(s,k,m)$ to $h_2''(s,k,m)=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}q''(s,k,m)$ by replacing $\binom{m-s}{2m-2s-k}$ with $\frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k)(2m-2s-k-1)}\binom{m-s}{2m-2s-k-2}$ in $q'(s,k,m)$.

Afterwards, we compute with Sigma the constants $c_i(k,m)$ and $g(s,k,m)$ for the creative telescoping equation

$$g(s+1,k,m)-g(s,k,m)=c_0(k,m)q''(s,k,m)+\cdots+c_3q''(s,k+3,m). \tag{104}$$

More precisely, those constants turn out to be

$$\begin{aligned}
c_0(k,m)&=2m(2+k)^2(k-2m)(1+k+2m),\\
c_1(k,m)&=-m(29k^3+5k^4+k(46-20m-40m^2)-12(-1+m+2m^2)-2k^2(-29+3m+6m^2)),\\
c_2(k,m)&=m(26k^3+4k^4+k(55-14m-28m^2)+k^2(59-6m-12m^2)-6(-3+m+2m^2)),\\
c_3(k,m)&=-m(1+k)^2(2+k-2m)(3+k+2m)h_2(3+k)=0,
\end{aligned}$$

and

$$\begin{aligned}
g(s,k,m)=&\binom{m-s}{2m-2s-k-2}\frac{-2(-1)^{s+k}}{4^s(2m-2s-k-2)(2m-2s-k-1)}\\
&\Big[p_1(s,k,m)\,\frac{(m)_s(-3m-1)_s}{s!(\frac12-2m)_s}+p_2(s,k,m)\,m\sum_{r=0}^{s}\frac{(m-r)(m)_r(-3m-1)_r}{r!(\frac12-2m)_r}\Big]
\end{aligned}$$

where

$$\begin{aligned}
p_1(s,k,m):=&(k(-1+2m-2s)+2(-1+m-s)+k^2(m-s))\\
&\times(1+3m-s)(m+s)(k-2m+2s)(1+k-2m+2s)
\end{aligned}$$

and

$$\begin{aligned}
p_2(s,k,m):=&\,k^4(m+3m^2+s+2ms-s^2)+k^3(-1-12m^3+10s+24ms-9s^2+m^2(5+12s))\\
&+4(6m^4-3(-1+s)s+2m^2s(2+3s)-m^3(7+12s)+m(1+11s+3s^2))\\
&+2k(-1+12m^4+16s-15s^2-4m^3(5+6s)+2m^2(5+7s+6s^2)+m(3+43s+6s^2))\\
&+k^2(-3+12m^4+29s-26s^2-2m^3(13+12s)+2m^2(7+10s+6s^2)+m(-1+68s+6s^2)).
\end{aligned}$$

In order to show the correctness of the creative telescoping equation (104) within the summation range we proceed as in Remark (1). If we denote $p(s,k,m)=(-1)^k\binom{m-s-1}{2m-2s-k-2}$, we represent all ingredients in (104) in terms of $p(s+1,k+3,m)$, i.e.,

$$p(s+1,k,m)=p(s+1,k+3,m)\prod_{j=0}^{2}\frac{m-s-k-j-4}{2m-2s-k-j-4}, \tag{105}$$

$$p(s,k+i,m)=-p(s+1,k+3,m)\frac{m-s}{2m-2s-k-6}\prod_{j=0}^{3-i}\frac{m-s-k+j-6}{2m-2s-k+j-5},\quad 0\le i\le 3. \tag{106}$$

With these representations in $p(s+1,k+3,m)$ we finally manage to verify that (80) holds for $0\le s\le\lfloor\frac{2m-k}{2}\rfloor-4$. Hence, summing equation (104) over s from 0 to an arbitrary d with $0\le$

$d \le \lfloor \frac{2m-k}{2} \rfloor - 4$ gives the relation (83) for the sum $b(k,m) = \sum_{s=0}^{d} q''(s,k,m) = \sum_{s=0}^{d} q'(s,k,m)$. Now we do a case distinction on k. If k is even, we can choose $d = m - k/2 - 4$, and obtain

$$\begin{aligned} &c_0(k,m)\,h_2(k,m) + \cdots + c_3(k,m)\,h_2(k+3,m) \\ &\qquad = g(m-\frac{k}{2}-3,k,m) - g(0,k,m) + \sum_{i=0}^{3} c_i(k,m) \sum_{j=0}^{3} h_2(m-\frac{k}{2}-j,k+i,m). \end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m-k/2-4 \ge 0$, k even, the relation (102) holds. Similarly, for odd k, one can choose $d = m-(k-1)/2-4$, and obtains

$$\begin{aligned} &c_0(k,m)\,h_2(k,m) + \cdots + c_3(k,m)\,h_2(k+3,m) \\ &\qquad = g(m-\frac{k-1}{2}-3,k,m) - g(0,k,m) + \sum_{i=0}^{3} c_i(k,m) \sum_{j=0}^{3} h_2(m-\frac{k-1}{2}-j,k+i,m). \end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (102) holds for $m-(k-1)/2-4 \ge 0$, k odd.

Summarizing, for all $m \ge 5$ and $0 \le k \le 2m-9$ the recurrence (102) contains the solution $h_2(k,m)$. The special cases $0 \le m \le 4$ or $k \ge 2m-8$ for $m \ge 4$ can be easily verified by evaluation. This shows that the recurrence holds for all $m, k \ge 0$. ◊

In the next step we compute with Sigma a recurrence for A_0, namely,

$$\begin{aligned} &(-2-i-i^2)(2-3i+i^2-2m-4m^2)A_0(i,m) \\ &\qquad + (3+i)(-2+2i-i^2+i^3+2m+4m^2)A_0(i+1,m) \\ &\qquad + (-3+i)(2+2i+i^2+i^3-2m-4m^2)A_0(i+2,m) \\ &\qquad - (2-i+i^2)(2+3i+i^2-2m-4m^2)A_0(i+3,m) = 0 \end{aligned} \tag{107}$$

which holds for $3 \le i \le 2m+1$.

Remark 7. More precisely, Sigma is able to compute constants for the creative telescoping summand recurrence (109) below, namely,

$$\begin{aligned} c_0(i,m) &:= (1-i)i(1+i)(2+i+i^2)(-2+i-2m)(-1+i+2m), \\ c_1(i,m) &:= (-1+i)i(1+i)(3+i)(-2+2i-i^2+i^3+2m+4m^2), \\ c_2(i,m) &:= (-3+i)(-1+i)i(1+i)(2+2i+i^2+i^3-2m-4m^2), \\ c_3(i,m) &:= (1-i)i(1+i)(2-i+i^2)(1+i-2m)(2+i+2m), \end{aligned}$$

and

$$g(k,i,m) = -\Big[p_0(i,m)h_2(i,m) + p_1(i,m)h_2(i+1,m) + p_2(i,m)h_2(i+2,m)\Big]\frac{k-1}{(k+1)^2}\binom{i+k-3}{i-2} \tag{108}$$

where

$$\begin{aligned} p_0(i,m) &:= 2(-6-k+2k^2)(-1+4k^3+k^4+m+2m^2-k^2(-2+m+2m^2)-2k(1+2m+4m^2)) \\ &\quad + 2i^3(3+2k^4+3k^5-5m-10m^2-k^2(2+m+2m^2)-k^3(8+3m+6m^2)+k(8+7m+14m^2)) \\ &\quad + i^4(9k^3+5k^4+2m(1+2m)+k^2(1-6m-12m^2)-k(3+8m+16m^2)) \\ &\quad + i^2(-4k^5+2k^6+k(7-76m-152m^2)-k^4(1+2m+4m^2)+k^3(59+4m+8m^2) \\ &\quad + k^2(73+4m+8m^2)-6(2+9m+18m^2))-2i(3-6k^5+k^6-45m-90m^2+k(41-10m-20m^2) \\ &\quad - k^4(30+m+2m^2)+2k^3(-4+3m+6m^2)+k^2(53+32m+64m^2)), \\ p_1(i,m) &:= (2-i-k)(29k^4+6k^5+2k(-16+m+2m^2)+10(-1+m+2m^2)-2k^3(-17+4m+8m^2) \\ &\quad - k^2(11+20m+40m^2)+i^3(10k^3+4k^4+2m(1+2m)+k^2(5-6m-12m^2) \end{aligned}$$

$$
\begin{aligned}
&- k(1+2m+4m^2)) + i(4k^4 - 3k^5 - 18m(1+2m) + k(39 - 40m - 80m^2)\\
&+ k^2(85 - 16m - 32m^2) + k^3(53 + 4m + 8m^2)) + i^2(3k^4 + 3k^5 - 10(-1 + m + 2m^2)\\
&- 2k(-11 + 2m + 4m^2) - k^3(7 + 4m + 8m^2) + k^2(5 + 6m + 12m^2))),\\
&(-2+i+k)(-1+i+k)(2+5k+2k^2+i^2k^2+i(2+k-k^2))(1+k-2m)(2+k+2m).
\end{aligned}
$$

Then for the summand $q(k,i,m) := h_2(k,m)\binom{i+k-3}{i-2}$ of $A_0(i,m)$ we have that

$$g(k+1,i,m) - g(k,i,m) = c_0(i,m)\,q(k,i,m) + \cdots + c_3(i,m)\,q(k,i+3,m) \tag{109}$$

for $m \geq 2$, $0 \leq k \leq 2m-3$ and $3 \leq i \leq 2m+1$. The verification of this creative telescoping equation and correctness of the recurrence (107) for $3 \leq i \leq 2m+1$ is now completely analogous to Remark 2. ◊

Next we compute a recurrence for $A_2(i,m)$, namely

$$
\begin{aligned}
&2(6+5i+i^2)(i+i^2-2m(1+2m))A_2(i,m)\\
&\quad + (3+i)(3i^2+i^3+8m(1+2m)+2i(1+m+2m^2))A_2(1+i,m)\\
&\quad - 2(1+i)(24+14i^2+2i^3-5m-10m^2-i(-32+m+2m^2))A_2(2+i,m)\\
&\quad - 2(2+i)(6+6i^2+i^3+i(11-2m-4m^2))A_2(3+i,m)\\
&\quad + 2(3+4i+i^2)(8+6i+i^2-m-2m^2)A_2(4+i)\\
&\quad + (2+3i+i^2)(7i+i^2-2(-6+m+2m^2))A_2(5+i,m) = 0
\end{aligned}
\tag{110}
$$

for all $3 \leq i \leq 2m+1$.

Remark 8. In order to achieve this, we actually derive the creative telescoping equation (111) below for the sum

$$A_2^{(e)}(i,m,e) := \sum_{k=0}^{2m-3} q(k,i,k,e)$$

where

$$q(k,i,m,e) = (-1)^k \frac{(k-i+e)!}{(k-i)!} h_2(k,m).$$

More precisely, with Sigma we can compute

$$
\begin{aligned}
c_0(i,m,e) :=& 2(e^2 - e(7+3i) + 3(6+5i+i^2))(i-2m)(1+i+2m),\\
c_1(i,m,e) :=& 4e^3(1+i) + e^2(-26-37i-11i^2+2m+4m^2) + 3(3+i)(3i^2+i^3+8m(1+2m)\\
&+ 2i(1+m+2m^2)) + e(58+56i^2+9i^3-14m-28m^2-3i(-35+2m+4m^2))\\
&- 4e^2m^2 - 36im^2 - 12eim^2 - 12i^2m^2,\\
c_2(i,m,e) :=& -4e^3 + 2e^4 + e^2(-37-51i-16i^2+2m+4m^2) + e(165+169i^2+30i^3\\
&- 26m - 52m^2 - 6i(-50+m+2m^2)) - 6(1+i)(24+14i^2+2i^3-5m-10m^2\\
&- i(-32+m+2m^2)),\\
c_3(i,m,e) :=& 5e^4 - e^3(35+17i) + e^2(82+82i+19i^2+4m+8m^2) - 6(2+i)(6+6i^2+i^3\\
&+ i(11-2m-4m^2)) - 2e(17+5i^2+5m+10m^2+i(19+6m+12m^2)),\\
c_4(i,m,e) :=& 4e^4 - 2e^3(19+9i) - e(7+3i)(33+37i+8i^2-2m-4m^2)\\
&+ e^2(139+139i+32i^2-2m-4m^2) + 6(3+4i+i^2)(8+6i+i^2-m-2m^2),\\
c_5(i,m,e) :=& (e^2 - e(4+3i) + 3(2+3i+i^2))(-4+e-i-2m)(-3+e-i+2m),
\end{aligned}
$$

and

$$g(k,i,m) = -(p_0(k,i,m,e)h_2(k,i,m)+p_1(k,i,m,e)h_2(k+1,i,m)+p_2(k,i,m,e)h_2(k+2,i,m)) \times \frac{q(k,i,e)}{d(k,i,e)}(-1)^k\frac{(k-i+e)!}{(k-i)!}$$

where

$$q(k,i,e) := (-1+e)e(i-k),$$

$$d(k,i,e) := (1+k)^2(-4+e-i+k)(-3+e-i+k)(-2+e-i+k)(-1+e-i+k)(e-i+k),$$

and

$$\begin{aligned}
p_0 :=& -(e^4(9k^3+5k^4+2m(1+2m)+k^2(1-6m-12m^2)-k(3+8m+16m^2)))+e^3(3-40k+20k^2+126k^3+49k^4\\
&-14k^5+26m-122km-66k^2m+18k^3m+52m^2-244km^2-132k^2m^2+36k^3m^2+i(3-10k+10k^2+52k^3\\
&+29k^4+8m-48km-36k^2m+16m^2-96km^2-72k^2m^2))-e^2(33-202k+146k^2+688k^3+165k^4\\
&-132k^5+10k^6+108m-720km-258k^2m+176k^3m-14k^4m+216m^2-1440km^2-516k^2m^2\\
&+352k^3m^2-28k^4m^2+i^2(12-2k+45k^2+126k^3+67k^4-2m-124km-86k^2m-4m^2\\
&-248km^2-172k^2m^2)+i(45-87k+160k^2+579k^3+225k^4-62k^5+42m-590km-310k^2m+82k^3m+84m^2\\
&-1180km^2-620k^2m^2+164k^3m^2))-2(72-180k+262k^2+753k^3-65k^4-249k^5+55k^6-917km\\
&-7k^2m+353k^3m-77k^4m-1834km^2-14k^2m^2+706k^3m^2-154k^4m^2\\
&+i^4(3+4k+16k^2+30k^3+15k^4-11m-36km-21k^2m-22m^2-72km^2-42k^2m^2)\\
&+i^3(30+28k+141k^2+260k^3+87k^4-30k^5-83m-310km-129k^2m+42k^3m-166m^2\\
&-620km^2-258k^2m^2+84k^3m^2)+i(150-100k+571k^2+1256k^3+17k^4\\
&-358k^5+60k^6-169m-1534km-159k^2m+506k^3m-84k^4m-338m^2-3068km^2-318k^2m^2\\
&+1012k^3m^2-168k^4m^2)+i^2(105+32k+434k^2+843k^3+144k^4-177k^5+15k^6-205m\\
&-1017km-258k^2m+249k^3m-21k^4m-410m^2-2034km^2-516k^2m^2\\
&+498k^3m^2-42k^4m^2))-e(-120+447k-459k^2-1681k^3-153k^4\\
&+442k^5-60k^6-150m+1894km+362k^2m-606k^3m+84k^4m-300m^2+3788km^2\\
&+724k^2m^2-1212k^3m^2+168k^4m^2+i^3(-15-14k-69k^2-142k^3-72k^4+28m+152km+96k^2m\\
&+56m^2+304km^2+192k^2m^2)+i^2(-102-13k-405k^2-933k^3-337k^4+102k^5+110m+1018km\\
&+482k^2m-138k^3m+220m^2+2036km^2+964k^2m^2-276k^3m^2)+i(-207+238k-765k^2\\
&-2116k^3-454k^4+422k^5-30k^6+40m+2358km+754k^2m-574k^3m+42k^4m+80m^2+4716km^2\\
&+1508k^2m^2-1148k^3m^2+84k^4m^2)),\\
p_1 :=&(-4+e-i+k)(-60-88k-257k^2-291k^3-7k^4+55k^5-18m+148km+174k^2m-88k^3m-36m^2\\
&+296km^2+348k^2m^2-176k^3m^2+i^3(-10-32k-55k^2-48k^3-15k^4+10m\\
&+30km+24k^2m+20m^2+60km^2+48k^2m^2)+3i^2(-20-56k-97k^2-83k^3-17k^4\\
&+5k^5+14m+52km+42k^2m-8k^3m+28m^2+104km^2+84k^2m^2-16k^3m^2)\\
&+e^3(10k^3+4k^4+2m(1+2m)+k^2(5-6m-12m^2)-k(1+2m+4m^2))+i(-39k^4+60k^5\\
&-6k^3(75+16m+32m^2)+2(-55+19m+38m^2)+2k(-128+135m+270m^2)+k^2(-497+240m+480m^2))\\
&-e^2(5-2k+50k^2+88k^3+26k^4-5k^5+16m-24km-54k^2m+8k^3m+32m^2\\
&-48km^2-108k^2m^2+16k^3m^2+i(5+9k+32k^2+45k^3+17k^4+4m-14km-26k^2m+8m^2\\
&-28km^2-52k^2m^2))+e(35+33k+197k^2+285k^3+56k^4-30k^5+38m-104km-174k^2m+48k^3m\\
&+76m^2-208km^2-348k^2m^2+96k^3m^2+i^2(15+41k+79k^2+80k^3+27k^4\\
&-6m-38km-42k^2m-12m^2-76km^2-84k^2m^2)+i(50+108k+262k^2+298k^3\\
&+79k^4-15k^5+4m-128km-166k^2m+24k^3m+8m^2-256km^2-332k^2m^2+48k^3m^2))),\\
p_2 :=&(4-e+i-k)(-3+e-i+k)(4+3k+11k^2+e^2k^2+i^2(2+3k+3k^2)+2i(3+4k+6k^2)-e(1+6k^2\\
&+i(1+k+3k^2)))(1+k-2m)(2+k+2m).
\end{aligned}$$

The verification of recurrence (110) is analogous to Remark (8). For completeness, we summarize the steps. Note that for all $e \notin \mathbb{Z}$ and $k,i,m \in \mathbb{Z}$ we have that $d(k,i,e) \neq 0$. Now suppose that

$3 \le i \le 2m-3$, $m \ge 2$. Then by using the relation (78) one can verify that the creative telescoping equation

$$g(k+1,i,m,e) - g(k,i,m,e) = c_0(i,m,e)\, q(k,i,m,e) + \cdots + c_5(i,m,e)\, q(k,i+5,m,e) \quad (111)$$

holds for $0 \le k \le 2m-3$ and $e \notin \mathbb{Z}$ as we have sketched it for the sum case $B_0(i,m)$ in Remark 2. Then summing this equation over k from 0 to $2m-3$ results the inhomogeneous recurrence

$$c_0(i,m,e)\, A_2^{(e)}(i,m,e) + \cdots + c_5(i,m,e)\, A_2^{(e)}(i+5,m,e) = g(2m-2,i,m,e) - g(0,i,m,e) \quad (112)$$

for all $e \notin \mathbb{Z}$ and $3 \le i \le 2m-3$.

But this means that also

$$A_2'(i,m) := \lim_{e\to 0} A_2^{(e)}(i,m) = \sum_{k=i}^{2m-3} (-1)^k\, h_2(k,m)$$

is a solution of the recurrence (112) in the limit $e \to 0$. Observe that $g(k,i,m,0) = 0$ which comes from the fact that $q(k,i,m,0) = 0$. Therefore $A_2'(i,m)$ is a solution of the recurrence

$$c_0(i,m,0)\, A_2'(i,m) + \cdots + c_0(i+5,m,0)\, A_2'(i+5,m) = 0 \quad (113)$$

for all $3 \le i \le 2m-3$ and $m \ge 2$. Note that this recurrence is nothing but recurrence (110) from above. In the next step observe that $\sum_{j=0}^{5} c_j(i,m,0) = 0$ which shows that any constant is a solution of (113) or (110). Therefore $A_2(i,m) = A_2'(i,m) + \sum_{j=2m-2}^{2m} (-1)^j h_2(j,m)$ is a solution of (110) for all $3 \le i \le 2m+1$ and $m \ge 2$.

$\Diamond$

Also for the final part we follow the strategy of the proof for identity (76) and compute a recurrence that contains the left hand side

$$L(i,m) := 6(-1)^i A_2(i,m) - A_0(i,m) + 2h_2(i-2,m) - 5h_2(i-1,m) - 3(-1)^i t_1(2m-1) \quad (114)$$

of (100) as solution for any $3 \le i \le 2m+1$. More precisely, we derive a recurrence in i that is satisfied by all solutions of the form

$$c_1\,(-1)^i + c_2\,(-1)^i\, A_2(i,m) + c_3\, h_2(i-1) + c_4\, h_2(i-2) + c_5\, A_0(i,m) \quad (115)$$

where the constants $c_i \in \mathbb{Q}$ and $3 \le i \le 2m+1$.

By Remark 8 the recurrence (110) contains besides $A_2(i,m)$ also the constant solution 1. Therefore changing the signs of the coefficients of $A_2(i,m)$, $A_2(i+2,m)$ and $A_2(i+4,m)$ in (110) gives a recurrence that contains the solutions $c_1\,(-1)^i + c_2\,(-1)^i\, A_2(i,m)$ for $3 \le i \le 2m+1$. Moreover one obtains immediately recurrences for $h_2(i-1)$ and $h_2(i-2)$, if one substitutes k by $i-1$ or $i-2$ in the recurrence (102); those recurrences are both valid in the range $3 \le i \le 2m$. Then adding these two recurrences with the recurrence (107) for $A_0(i,m)$ and the recurrence for $c_1\,(-1)^i + c_2\,(-1)^i\, A_2(i,m)$ for instance with the packages [SZ94] or [Mal96] gives a homogeneous recurrence of order 9,

$$c_0(i,m)\, S(i,m) + c_1(i,m)\, S(i+1,m) + \cdots + c_9(i,m)\, S(i+9,m) = 0, \quad (116)$$

which contains the solution for (115) and therefore also (114) for all $3 \le i \le 2m+1$. The explicit expressions for the $c_j(i,m)$ can be easily recomputed with the Maple package [SZ94] gfun or the Mathematica package [Mal96] GeneratingFunctions.

The following remark shows that in the range $3 \le i \le 2m-8$ the leading coefficient $c_9(i,m)$ in (116) does not vanish.

Remark 9. The leading coefficient $c_9(i,m)$ in (116) is

$$(-1+i)i(5+i-2m)(6+i-2m)(7+i-2m)(6+i+2m)(7+i+2m)(8+i+2m)\, p(i,m)$$

with the polynomial

$$
\begin{aligned}
p(i,m) = {} & 286934400\,i + 1022116320\,i^2 + 1518851088\,i^3 + 1277708580\,i^4 + 764090658\,i^5 + 449853750\,i^6 + 295140024\,i^7 + 169048296\,i^8 + \\
& 70449102\,i^9 + 20137410\,i^{10} + 3830652\,i^{11} + 462672\,i^{12} + 32076\,i^{13} + 972\,i^{14} - 1012435200\,m - 5482935360\,i\,m - \\
& 14278564656\,i^2\,m - 23213538792\,i^3\,m - 26151610626\,i^4\,m - 21597662355\,i^5\,m - 13483008435\,i^6\,m - 6458953002\,i^7\,m - \\
& 2388711936\,i^8\,m - 682573911\,i^9\,m - 150177099\,i^{10}\,m - 25219524\,i^{11}\,m - 3183792\,i^{12}\,m - 295824\,i^{13}\,m - 19752\,i^{14}\,m - 912\,i^{15}\,m - \\
& 24\,i^{16}\,m + 2024870400\,m^2 + 6988032000\,i\,m^2 + 10391171184\,i^2\,m^2 + 7815524436\,i^3\,m^2 + 1403744502\,i^4\,m^2 - 3433378715\,i^5\,m^2 - \\
& 4306205557\,i^6\,m^2 - 2795766398\,i^7\,m^2 - 1193680594\,i^8\,m^2 - 353952095\,i^9\,m^2 - 72888049\,i^{10}\,m^2 - 9860680\,i^{11}\,m^2 - \\
& 699794\,i^{12}\,m^2 + 13148\,i^{13}\,m^2 + 7588\,i^{14}\,m^2 + 608\,i^{15}\,m^2 + 16\,i^{16}\,m^2 + 13619181888\,m^3 + 62360574624\,i\,m^3 + \\
& 138500395704\,i^2\,m^3 + 196534634088\,i^3\,m^3 + 197750758622\,i^4\,m^3 + 148547525347\,i^5\,m^3 + 85793698617\,i^6\,m^3 + \\
& 38793616174\,i^7\,m^3 + 13884128774\,i^8\,m^3 + 3953893039\,i^9\,m^3 + 896125845\,i^{10}\,m^3 + 160785656\,i^{11}\,m^3 + 22549686\,i^{12}\,m^3 + \\
& 2412656\,i^{13}\,m^3 + 188176\,i^{14}\,m^3 + 9728\,i^{15}\,m^3 + 256\,i^{16}\,m^3 + 1472884992\,m^4 + 17277933456\,i\,m^4 + 55255553160\,i^2\,m^4 + \\
& 97387495728\,i^3\,m^4 + 114340534314\,i^4\,m^4 + 96961318441\,i^5\,m^4 + 61869893729\,i^6\,m^4 + 30390885826\,i^7\,m^4 + 11641806632\,i^8\,m^4 + \\
& 3500762485\,i^9\,m^4 + 827525561\,i^{10}\,m^4 + 153150896\,i^{11}\,m^4 + 21942460\,i^{12}\,m^4 + 2380016\,i^{13}\,m^4 + 187216\,i^{14}\,m^4 + 9728\,i^{15}\,m^4 + \\
& 256\,i^{16}\,m^4 - 25007101632\,m^5 - 96066457488\,i\,m^5 - 181925812632\,i^2\,m^5 - 221098537536\,i^3\,m^5 - 189184506516\,i^4\,m^5 - \\
& 118702812708\,i^5\,m^5 - 55719392004\,i^6\,m^5 - 19763963784\,i^7\,m^5 - 5314961340\,i^8\,m^5 - 1080128100\,i^9\,m^5 - 164117244\,i^{10}\,m^5 - \\
& 18299232\,i^{11}\,m^5 - 1456536\,i^{12}\,m^5 - 78336\,i^{13}\,m^5 - 2304\,i^{14}\,m^5 - 3283424064\,m^6 - 24487119216\,i\,m^6 - 62740182552\,i^2\,m^6 - \\
& 91955201268\,i^3\,m^6 - 89825565356\,i^4\,m^6 - 62243885388\,i^5\,m^6 - 31512673972\,i^6\,m^6 - 11837036352\,i^7\,m^6 - 3321422232\,i^8\,m^6 - \\
& 695467968\,i^9\,m^6 - 107666704\,i^{10}\,m^6 - 12118848\,i^{11}\,m^6 - 968336\,i^{12}\,m^6 - 52224\,i^{13}\,m^6 - 1536\,i^{14}\,m^6 + 21238328448\,m^7 + \\
& 64255523184\,i\,m^7 + 96366768720\,i^2\,m^7 + 92096792976\,i^3\,m^7 + 60749842800\,i^4\,m^7 + 28468056528\,i^5\,m^7 + 9555442224\,i^6\,m^7 + \\
& 2283228864\,i^7\,m^7 + 379758816\,i^8\,m^7 + 42201312\,i^9\,m^7 + 2991072\,i^{10}\,m^7 + 138240\,i^{11}\,m^7 + 4608\,i^{12}\,m^7 + 6290303040\,m^8 + \\
& 23115529440\,i\,m^8 + 38148908736\,i^2\,m^8 + 38686906320\,i^3\,m^8 + 26702560752\,i^4\,m^8 + 13015763424\,i^5\,m^8 + 4522579392\,i^6\,m^8 + \\
& 1111374432\,i^7\,m^8 + 188339328\,i^8\,m^8 + 21100656\,i^9\,m^8 + 1495536\,i^{10}\,m^8 + 69120\,i^{11}\,m^8 + 2304\,i^{12}\,m^8 - 5912676864\,m^9 - \\
& 12381602112\,i\,m^9 - 13532698560\,i^2\,m^9 - 9775140480\,i^3\,m^9 - 4854527040\,i^4\,m^9 - 1615086720\,i^5\,m^9 - 339571200\,i^6\,m^9 - \\
& 40320000\,i^7\,m^9 - 2053440\,i^8\,m^9 - 2620173312\,m^{10} - 5651547264\,i\,m^{10} - 5713697280\,i^2\,m^{10} - 3829696704\,i^3\,m^{10} - \\
& 1857903168\,i^4\,m^{10} - 627501888\,i^5\,m^{10} - 134592960\,i^6\,m^{10} - 16128000\,i^7\,m^{10} - 821376\,i^8\,m^{10} - 209811456\,m^{11} - \\
& 681806592\,i\,m^{11} - 312221952\,i^2\,m^{11} + 87664896\,i^3\,m^{11} + 91535616\,i^4\,m^{11} + 20217600\,i^5\,m^{11} + 1347840\,i^6\,m^{11} + 46356480\,m^{12} - \\
& 92874240\,i\,m^{12} - 77865984\,i^2\,m^{12} + 29221632\,i^3\,m^{12} + 30511872\,i^4\,m^{12} + 6739200\,i^5\,m^{12} + 449280\,i^6\,m^{12} + 111992832\,m^{13} + \\
& 124056576\,i\,m^{13} + 24192000\,i^2\,m^{13} + 38633472\,m^{14} + 35444736\,i\,m^{14} + 6912000\,i^2\,m^{14} + 5308416\,m^{15} + 1327104\,m^{16}.
\end{aligned}
$$

By inspection the first 8 linear factors in $c_9(i,m)$ do not vanish in the range $3 \le i \le 2m-8$. Moreover by the cylindric algebraic decomposition method [CH91] (QEPCAD B version 1.30 of Hoon Hong) there does not exist any solution $i \in \mathbb{R}$ and $m \in \mathbb{R}$ such that $p(i,m) = 0$ and $3 \le i \le 2m$. Hence there also does not exist any positive integer i, m with $3 \le i \le 2m-8$ and $c_9(i,m) \neq 0$. ◊

Now we show that the first nine initial values of $L(i,m)$ are 0, i.e., we will verify that $L(r,m) = 0$ for all $3 \le r \le 11$ and all $2m+1 \ge r$. Then together with the fact that $L(i,m)$ is a solution of (116) for all $3 \le i \le 2m+1$ and $c_9(i,m) \neq 0$ for all $3 \le i \le 2m-8$, it follows that $L(i,m) = 0$ for all $3 \le i \le 2m+1$. The crucial step is that $L(r,m) = 0$ for $3 \le r \le 11$ and all m with $2m+1 \ge r$ if and only if

$$6(-1)^r A_1(r,m) + A_0(r,m) + 4h_2(r-2,m) - h_2(r-1,m) = 0. \tag{117}$$

Similarly as in the case of identity (94), we derive representations[8] for $A_0(r,m)$ that hold for all m with $2m+1 \ge r$, namely

$$A_0(2,m) = -3h_2(0,m) + h_2(1,m), \quad m \ge 1, \tag{118}$$

and

$$
\begin{aligned}
A_0(3,m) &= 6\,h_2(0,m) - 4\,h_2(1,m) + h_2(2,m), \\
A_0(4,m) &= \frac{2\,(9+m+2\,m^2)\,h_2(0,m) - 12\,h_2(1,m) - (-3+m+2\,m^2)\,h_2(2,m)}{(-1+m)(3+2m)}, \\
A_0(5,m) &= \frac{-12\,(-9+2\,m+4\,m^2)\,h_2(0,m) + 3\,(-24+3\,m+7\,m^2+4\,m^3+4\,m^4)\,h_2(1,m) - (-18+3\,m+7\,m^2+4\,m^3+4\,m^4)\,h_2(2,m)}{(-1+m)\,(2+m)\,(-3+2\,m)\,(3+2\,m)}, \\
A_0(6,m) &= (-12\,(15-m-m^2+4\,m^3+4\,m^4)\,h_2(0,m) + (120-19\,m-31\,m^2+28\,m^3+28\,m^4)\,h_2(1,m) + \\
&\qquad (-30+13\,m+25\,m^2-4\,m^3-4\,m^4)\,h_2(2,m))/(-2+m)\,(-1+m)\,(3+2\,m)\,(5+2\,m), \\
A_0(7,m) &= (-4\,(-4050+1440\,m+2865\,m^2-82\,m^3-191\,m^4-256\,m^5-152\,m^6+32\,m^7+16\,m^8)\,h_2(0,m) + \\
&\qquad (-10800+3060\,m+6339\,m^2+838\,m^3+647\,m^4-464\,m^5-328\,m^6-32\,m^7-16\,m^8)\,h_2(1,m) + \\
&\qquad 2\,(1350-495\,m-954\,m^2+134\,m^3+85\,m^4-112\,m^5-56\,m^6+32\,m^7+16\,m^8)\,h_2(2,m))/ \\
&\quad ((-2+m)\,(-1+m)\,(2+m)\,(3+m)\,(-5+2\,m)\,(-3+2\,m)\,(3+2\,m)\,(5+2\,m)), \\
A_0(8,m) &= (4\,(85050-31050\,m-51525\,m^2+40671\,m^3+32579\,m^4-19123\,m^5-11750\,m^6+1736\,m^7+928\,m^8+80\,m^9+32\,m^{10})\,h_2(0,m) + \\
&\qquad (-226800+98820\,m+172035\,m^2-100077\,m^3-88313\,m^4+28501\,m^5+19850\,m^6+1288\,m^7+224\,m^8-560\,m^9-224\,m^{10})\,h_2(1,m) + \\
&\qquad 2\,(28350-19305\,m-34983\,m^2+14460\,m^3+14195\,m^4-775\,m^5-974\,m^6-760\,m^7-320\,m^8+80\,m^9+32\,m^{10})\,h_2(2,m))/ \\
&\quad ((-3+m)\,(-2+m)\,(-1+m)\,(2+m)\,(3+m)\,(-5+2\,m)\,(-3+2\,m)\,(3+2\,m)\,(5+2\,m)\,(7+2\,m)),
\end{aligned}
$$

[8]Note that identity (118) implies Lemma 4.

```
A0(9,m) = (18 ( - 176400 + 16800 m + 34090 m^2 + 2353 m^3 + 4274 m^4 + 4365 m^5 + 2098 m^6 - 1368 m^7 - 624 m^8 + 80 m^9 + 32 m^10) h2(0,m)-
          10 ( - 211680 + 8442 m + 15159 m^2 - 5996 m^3 - 1538 m^4 + 10353 m^5 + 5754 m^6 - 1944 m^7 - 912 m^8 + 80 m^9 + 32 m^10) h2(1,m)+
          ( - 529200 + 76860 m + 125976 m^2 - 107021 m^3 - 87398 m^4 + 46245 m^5 + 28002 m^6 - 4824 m^7 - 2352 m^8 + 80 m^9 + 32 m^10) h2(2,m))/
      ((-3 + m) (-2 + m) (2 + m) (3 + m) (4 + m) (-7 + 2 m) (-5 + 2 m) (-3 + 2 m) (5 + 2 m) (7 + 2 m)),

A0(10,m) = (6 (57153600 - 24872400 m - 41202000 m^2 + 33029298 m^3 + 27322823 m^4 - 13673507 m^5 - 9012177 m^6+
                 296549 m^7 + 441274 m^8 + 375684 m^9 + 120424 m^10 - 32272 m^11 - 10272 m^12 + 448 m^13 + 128 m^14) h2(0,m)+
           2 ( - 114307200 + 56269080 m + 98214930 m^2 - 56493117 m^3 - 52362987 m^4 + 10634788 m^5 + 9427703 m^6 + 3727349 m^7+
                 1168354 m^8 - 909516 m^9 - 328536 m^10 + 38768 m^11 + 13408 m^12 + 448 m^13 + 128 m^14) h2(1,m) - 3 ( - 3 + m + 2 m^2)^2
             ( - 2116800 + 114240 m + 247244 m^2 + 74384 m^3 + 70991 m^4 - 8327 m^5 - 6158 m^6 - 1016 m^7 - 448 m^8 + 80 m^9 + 32 m^10) h2(2,m))/
      ((-4 + m) (-3 + m) (-2 + m) (-1 + m) (2 + m) (3 + m) (4 + m) (-7 + 2 m) (-5 + 2 m) (-3 + 2 m) (3 + 2 m) (5 + 2 m) (7 + 2 m) (9 + 2 m)),

A0(11,m) = ( - 4 ( - 3857868000 + 1828915200 m + 3455037180 m^2-
                 769680063 m^3 - 573285609 m^4 + 410292357 m^5 + 75600887 m^6 - 317122186 m^7 - 103854870 m^8 + 70617976 m^9+
                 23616736 m^10 - 5008224 m^11 - 1596608 m^12 + 68992 m^13 + 22272 m^14 + 2048 m^15 + 512 m^16) h2(0,m)+
           ( - 10287648000 + 4421239200 m + 8271754020 m^2 - 2100636018 m^3-
                 1223321355 m^4 + 1917190986 m^5 + 662194436 m^6 - 1008261760 m^7 - 385832709 m^8 + 155348368 m^9+
                 57307408 m^10 - 5439936 m^11 - 2089952 m^12 - 245504 m^13 - 56064 m^14 + 11264 m^15 + 2816 m^16) h2(1,m)-
           3 ( - 857304000 + 470156400 m + 809515080 m^2-
                 491016438 m^3 - 333442341 m^4 + 359794142 m^5 + 179100812 m^6 - 102079240 m^7 - 45758047 m^8+
                 7260176 m^9 + 3326576 m^10 + 426304 m^11 + 84832 m^12 - 51968 m^13 - 13568 m^14 + 1024 m^15 + 256 m^16) h2(2,m))/
      ((-4 + m) (-3 + m) (-2 + m) (-1 + m)
           (2 + m) (3 + m) (4 + m) (5 + m) (-9 + 2 m) (-7 + 2 m) (-5 + 2 m) (-3 + 2 m) (3 + 2 m) (5 + 2 m) (7 + 2 m) (9 + 2 m)).
```

Details about the way how these representations have been derived and proved can be found in

Remark 10. For $A_0(r,m) = \sum_{k=0}^{2m} \binom{r+k-3}{r-2} h_2(k,m)$ with $r \in \{3,\dots,11\}$ we can compute with Sigma expressions $g_r(k,m)$ such that

$$g_r(k+1,m) - g_r(k,m) = \binom{r+k-3}{r-2} h_2(k,m)$$

holds for all $0 \le k \le 2m-3$. The g_r are listed explicitly in Appendix B. Note that the telescoping equations for each case $2 \le r \le 11$ can be verified by applying the relation (102). Moreover, the identities for $A_0(r,m)$ can be derived by telescoping and using additionally the representations

$$h_2(2m-2,m) = \frac{1+12\,m-24\,m^2+8\,m^3}{-4+16\,m}, \qquad h_2(2m-1,m) = \frac{1-2\,m}{2}, \qquad h_2(2m,m) = 1$$

that hold for $m \ge 1$. ◊

Hence we see that for $r \in \{2,\dots,12\}$ each $A_0(r,m)$ can be expressed in terms of $h_2(0,m)$, $h_2(1,m)$ and $h_2(2,m)$. Moreover, by (98) each corresponding $A_1(r,m)$ can be written as a linear combination in terms of $h_2(0,m),\dots,h_2(r-3,m)$.

In particular by applying relation (102) each of the $h_2(j,m)$ for $3 \le j \le r-3$ can be reduced to $h_2(0,m)$, $h_2(1,m)$ and $h_2(2,m)$ where $m > (r-1)/2$. In other words, the left side of identity (94) can be expressed in terms of $h_2(0,m)$, $h_2(1,m)$ and $h_2(2,m)$ that occur only linearly. Since this expression collapses to 0, identity (94) holds for $m > (r-1)/2$. The special case $2m+1 = r$ of identity (94) can be shown separately by simple evaluation.

Applying this proof strategy for all $r \in \{3,\dots,11\}$ shows that $L(r,m) = 0$ for all m,r with $2m+1 \ge r$ and $3 \le r \le 11$ which finally proves that $L(i,m) = 0$ for all $3 \le i \le 2m+1$.

8. Completing missing proof steps

What remains to show are Propositions 3 and 5 and that

$$b(m) := B_0(2m+1,m) = \sum_{k=0}^{2m-1} \binom{2m+k-2}{2m-1} h_1(k,m),$$

$$a(m) := A_0(2m+2,m) = \sum_{k=0}^{2m} \binom{2m+k-2}{2m-1} h_2(k,m) \tag{119}$$

are solutions of (73) and (74) respectively for all $m \geq 1$. In order to accomplish this, we first compute

$$\begin{aligned}
&2(k+2m)(-6k-13k^2-3k^3+11k^4+9k^5+2k^6+12m-6km-72k^2m-72k^3m-2k^4m\\
&\quad+24k^5m+8k^6m+66m^2+138km^2-20k^2m^2-192k^3m^2-160k^4m^2-48k^5m^2+42m^3+342km^3\\
&+306k^2m^3+186k^3m^3+96k^4m^3-240m^4+108km^4+72k^2m^4-108k^3m^4-312m^5-144km^5+72k^2m^5)\,h_1(k,m)\\
&\quad(1+4m)(18k^2+39k^3+9k^4-33k^5-27k^6-6k^7-6km+46k^2m+112k^3m+100k^4m\\
&\quad+56k^5m+16k^6m-60m^2-162km^2+14k^2m^2+252k^3m^2+172k^4m^2+24k^5m^2-180m^3-528km^3\\
&-504k^2m^3-192k^3m^3-24k^4m^3-240km^4-432k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5)\,h_1(k+1,m)\\
&\quad(2+k-2m)(1+4m)(-3k^2-5k^3+k^4+5k^5+2k^6-14k^2m-18k^3m-4k^4m+12m^2\\
&+24km^2-42k^2m^2-54k^3m^2-12k^4m^2+48m^3+120km^3-36k^3m^3+48m^4+144km^4+72k^2m^4)\,h_1(k+2,m)\\
&\quad+4(1+k)^2(-1+k-2m)(k-2m)m(1+2m)(-1+4m)(1+4m)\,h_1(k,m+1)=0, \qquad (120)
\end{aligned}$$

which holds for all $m, k \geq 0$, with Sigma [Sch00, Sch01] or an extended version of [PS95].

Remark 11. As in Remark 1 we consider the sum $h_1'(k,m) := \sum_{s=0}^{\lfloor \frac{2m-k-1}{2} \rfloor - 1} q'(s,k,m)$. Recall that $q'(s,k,m) = q(s,k,m)$ for $0 \leq s \leq \lfloor \frac{2m-k-1}{2} \rfloor - 1$ where the summand $q(s,k,m)$ of $h_1(k,m) = \sum_{s=0}^{\lfloor \frac{2m-k-1}{2} \rfloor} q(s,k,m)$ is given in (79).

In the sequel denote

$$p(s,k,m) = (-1)^k \binom{m-s-1}{2m-2s-k-3} \qquad \text{and} \qquad u(s,m) = \frac{(-1)^s (1-3m)_s (m)_s}{(3/2-2m)_s \, s! \, 4^s},$$

i.e.,

$$q'(s,k,m) = -p(s,k,m)\, u(s,m) \frac{3m-3s-1}{3m-1} \frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k-1)(2m-2s-k-2)}.$$

Now applying this representation of the definite sum $h_1'(k,m)$ to Sigma gives the creative telescoping equation

$$\begin{aligned}
&g(s+1,k,m) - g(s,k,m)\\
&\qquad = c_0(k,m) q'(s,k,m) + \cdots + c_2(k,m) q'(s,k+2,m) + \kappa(k,m)\, q'(s,k,m+1). \qquad (121)
\end{aligned}$$

with

$$\begin{aligned}
c_0(k,m) &= 2(k+2m)(-1+3m)(-6k-13k^2-3k^3+11k^4+9k^5+2k^6+12m-6km-72k^2m-72k^3m-2k^4m\\
&\quad+24k^5m+8k^6m+66m^2+138km^2-20k^2m^2-192k^3m^2-160k^4m^2-48k^5m^2+42m^3+342km^3\\
&\quad+306k^2m^3+186k^3m^3+96k^4m^3-240m^4+108km^4+72k^2m^4-108k^3m^4-312m^5-144km^5+72k^2m^5),\\
c_1(k,m) &= (-1+3m)(1+4m)(18k^2+39k^3+9k^4-33k^5-27k^6-6k^7-6km+46k^2m+112k^3m+100k^4m\\
&\quad+56k^5m+16k^6m-60m^2-162km^2+14k^2m^2+252k^3m^2+172k^4m^2+24k^5m^2-180m^3-528km^3\\
&\quad-504k^2m^3-192k^3m^3-24k^4m^3-240km^4-432k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5),\\
c_2(k,m) &= (2+k-2m)(-1+3m)(1+4m)(-3k^2-5k^3+k^4+5k^5+2k^6-14k^2m-18k^3m-4k^4m+12m^2\\
&\quad+24km^2-42k^2m^2-54k^3m^2-12k^4m^2+48m^3+120km^3-36k^3m^3+48m^4+144km^4+72k^2m^4),\\
\kappa(k,m) &= 4(1+k)^2(-1+k-2m)(k-2m)m(1+2m)(-1+3m)(-1+4m)(1+4m)
\end{aligned}$$

and

$$g(s,k,m) = -p(s,k,m)\, u(k,m)\, \frac{a(s,k,m)}{b(s,k,m)}$$

where

$$\begin{aligned}
b(s,k,m) &= (3m-s)(3m-s+1)(3m-s+2)\\
&\qquad (2m-2s-k+1)(2m-2s-k)(2m-2s-k-1)(2m-2s-k-2)
\end{aligned}$$

and

```
a(s, k, m) = 2 (−1 + 4 m − 2 s) (m − s) s (2 + k − m + s) (12 k^2 + 20 k^3 − 16 k^4 − 40 k^5 − 4 k^6 + 20 k^7 + 8 k^8 − 48 k m + 38 k^2 m + 368 k^3 m+
  120 k^4 m − 452 k^5 m − 238 k^6 m + 132 k^7 m + 80 k^8 m + 48 m^2 − 396 k m^2 − 786 k^2 m^2 + 1937 k^3 m^2 + 2200 k^4 m^2 − 1476 k^5 m^2−
  1720 k^6 m^2 + 103 k^7 m^2 + 282 k^8 m^2 + 504 m^3 − 762 k m^3 − 6194 k^2 m^3 + 4457 k^3 m^3 + 9396 k^4 m^3 − 848 k^5 m^3−
  4204 k^6 m^3 − 843 k^7 m^3 + 414 k^8 m^3 + 2280 m^4 + 1080 k m^4 − 22806 k^2 m^4 + 10392 k^3 m^4 + 22080 k^4 m^4 + 1308 k^5 m^4−
  3462 k^6 m^4 − 2016 k^7 m^4 + 216 k^8 m^4 + 7764 m^5 + 4926 k m^5 − 72276 k^2 m^5 + 22866 k^3 m^5 + 49980 k^4 m^5 − 492 k^5 m^5+
  864 k^6 m^5 − 1296 k^7 m^5 + 26400 m^6 + 19260 k m^6 − 183792 k^2 m^6 − 16608 k^3 m^6 + 84012 k^4 m^6 − 1152 k^5 m^6 + 1944 k^6 m^6+
  68940 m^7 + 89892 k m^7 − 247536 k^2 m^7 − 132444 k^3 m^7 + 59004 k^4 m^7 + 432 k^5 m^7 + 106608 m^8 + 192264 k m^8 − 104424 k^2 m^8−
  137592 k^3 m^8 + 8424 k^4 m^8 + 84672 m^9 + 171936 k m^9 + 43200 k^2 m^9 − 33696 k^3 m^9 + 26784 m^10 + 53568 k m^10 + 26784 k^2 m^10+
  24 k s + 10 k^2 s − 148 k^3 s − 120 k^4 s + 140 k^5 s + 134 k^6 s − 16 k^7 s − 24 k^8 s − 48 m s + 288 k m s + 886 k^2 m s − 1176 k^3 m s−
  2324 k^4 m s + 508 k^5 m s + 1414 k^6 m s + 68 k^7 m s − 192 k^8 m s − 624 m^2 s + 468 k m^2 s + 8320 k^2 m^2 s − 1403 k^3 m^2 s − 12894 k^4 m^2 s−
  2314 k^5 m^2 s + 5000 k^6 m^2 s + 1233 k^7 m^2 s − 474 k^8 m^2 s − 3456 m^3 s − 4494 k m^3 s + 35338 k^2 m^3 s + 5388 k^3 m^3 s − 32542 k^4 m^3 s−
  12234 k^5 m^3 s + 5508 k^6 m^3 s + 3540 k^7 m^3 s − 360 k^8 m^3 s − 14076 m^4 s − 21870 k m^4 s + 113340 k^2 m^4 s + 11634 k^3 m^4 s−
  64344 k^4 m^4 s − 14436 k^5 m^4 s − 2952 k^6 m^4 s + 2880 k^7 m^4 s − 56196 m^5 s − 68496 k m^5 s + 313104 k^2 m^5 s + 68664 k^3 m^5 s−
  123732 k^4 m^5 s − 240 k^5 m^5 s − 6048 k^6 m^5 s − 171144 m^6 s − 238824 k m^6 s + 508944 k^2 m^6 s + 280704 k^3 m^6 s − 126096 k^4 m^6 s+
  3312 k^5 m^6 s − 308280 m^7 s − 535920 k m^7 s + 279240 k^2 m^7 s + 362448 k^3 m^7 s − 31248 k^4 m^7 s − 284976 m^8 s − 561312 k m^8 s−
  127728 k^2 m^8 s + 117504 k^3 m^8 s − 103680 m^9 s − 207360 k m^9 s − 103680 k^2 m^9 s − 60 k s^2 − 232 k^2 s^2 + 79 k^3 s^2 + 468 k^4 s^2+
  40 k^5 s^2 − 258 k^6 s^2 − 59 k^7 s^2 + 22 k^8 s^2 + 120 m s^2 − 114 k m s^2 − 3266 k^2 m s^2 − 965 k^3 m s^2 + 4696 k^4 m s^2 + 1832 k^5 m s^2−
  1788 k^6 m s^2 − 645 k^7 m s^2 + 130 k^8 m s^2 + 1176 m^2 s^2 + 2952 k m^2 s^2 − 17924 k^2 m^2 s^2 − 11450 k^3 m^2 s^2 + 16234 k^4 m^2 s^2+
  11382 k^5 m^2 s^2 − 2690 k^6 m^2 s^2 − 2200 k^7 m^2 s^2 + 168 k^8 m^2 s^2 + 6756 m^3 s^2 + 17646 k m^3 s^2 − 63760 k^2 m^3 s^2 − 37310 k^3 m^3 s^2+
  29100 k^4 m^3 s^2 + 21416 k^5 m^3 s^2 + 3160 k^6 m^3 s^2 − 2256 k^7 m^3 s^2 + 37284 m^4 s^2 + 59616 k m^4 s^2 − 196536 k^2 m^4 s^2−
  86712 k^3 m^4 s^2 + 55860 k^4 m^4 s^2 + 7776 k^5 m^4 s^2 + 6792 k^6 m^4 s^2 + 147432 m^5 s^2 + 211320 k m^5 s^2 − 393120 k^2 m^5 s^2−
  237096 k^3 m^5 s^2 + 84600 k^4 m^5 s^2 − 6768 k^5 m^5 s^2 + 327072 m^6 s^2 + 544704 k m^6 s^2 − 284064 k^2 m^6 s^2 − 356256 k^3 m^6 s^2+
  38880 k^4 m^6 s^2 + 362304 m^7 s^2 + 688320 k m^7 s^2 + 134208 k^2 m^7 s^2 − 157248 k^3 m^7 s^2 + 155520 m^8 s^2 + 311040 k m^8 s^2+
  155520 k^2 m^8 s^2 + 348 k^2 s^3 + 271 k^3 s^3 − 454 k^4 s^3 − 350 k^5 s^3 + 112 k^6 s^3 + 79 k^7 s^3 − 6 k^8 s^3 − 534 k m s^3 + 3486 k^2 m s^3+
  3936 k^3 m s^3 − 2786 k^4 m s^3 − 3286 k^5 m s^3 + 284 k^6 m s^3 + 508 k^7 m s^3 − 24 k^8 m s^3 − 324 m^2 s^3 − 4410 k m^2 s^3 + 15680 k^2 m^2 s^3+
  18170 k^3 m^2 s^3 − 4716 k^4 m^2 s^3 − 9632 k^5 m^2 s^3 − 1520 k^6 m^2 s^3 + 768 k^7 m^2 s^3 − 7212 m^3 s^3 − 17760 k m^3 s^3 + 54968 k^2 m^3 s^3+
  43056 k^3 m^3 s^3 − 6188 k^4 m^3 s^3 − 7200 k^5 m^3 s^3 − 3456 k^6 m^3 s^3 − 50160 m^4 s^3 − 73776 k m^4 s^3 + 140256 k^2 m^4 s^3+
  97248 k^3 m^4 s^3 − 18672 k^4 m^4 s^3 + 3600 k^5 m^4 s^3 − 152784 m^5 s^3 − 241488 k m^5 s^3 + 140928 k^2 m^5 s^3 + 164304 k^3 m^5 s^3−
  19440 k^4 m^5 s^3 − 213600 m^6 s^3 − 388800 k m^6 s^3 − 57312 k^2 m^6 s^3 + 100224 k^3 m^6 s^3 − 112320 m^7 s^3 − 224640 k m^7 s^3−
  112320 k^2 m^7 s^3 + 60 k s^4 − 158 k^2 s^4 − 306 k^3 s^4 + 118 k^4 s^4 + 270 k^5 s^4 + 40 k^6 s^4 − 24 k^7 s^4 − 120 m s^4 + 468 k m s^4−
  1356 k^2 m s^4 − 2580 k^3 m s^4 + 344 k^4 m s^4 + 1668 k^5 m s^4 + 376 k^6 m s^4 − 96 k^7 m s^4 − 324 m^2 s^4 + 1428 k m^2 s^4 − 6520 k^2 m^2 s^4−
  8376 k^3 m^2 s^4 − 800 k^4 m^2 s^4 + 2208 k^5 m^2 s^4 + 864 k^6 m^2 s^4 + 4812 m^3 s^4 + 7524 k m^3 s^4 − 23088 k^2 m^3 s^4 − 18684 k^3 m^3 s^4−
  180 k^4 m^3 s^4 − 576 k^5 m^3 s^4 + 28656 m^4 s^4 + 41784 k m^4 s^4 − 35448 k^2 m^4 s^4 − 36072 k^3 m^4 s^4 + 3384 k^4 m^4 s^4 + 56832 m^5 s^4+
  98208 k m^5 s^4 + 6912 k^2 m^5 s^4 − 30240 k^3 m^5 s^4 + 38880 m^6 s^4 + 77760 k m^6 s^4 + 38880 k^2 m^6 s^4 − 24 k s^5 + 20 k^2 s^5 + 84 k^3 s^5+
  4 k^4 s^5 − 60 k^5 s^5 − 24 k^6 s^5 + 48 m s^5 − 96 k m s^5 + 152 k^2 m s^5 + 456 k^3 m s^5 + 64 k^4 m s^5 − 240 k^5 m s^5 − 96 k^6 m s^5 + 120 m^2 s^5+
  24 k m^2 s^5 + 1104 k^2 m^2 s^5 + 1056 k^3 m^2 s^5 + 192 k^4 m^2 s^5 − 1272 m^3 s^5 − 1344 k m^3 s^5 + 3768 k^2 m^3 s^5 + 3168 k^3 m^3 s^5−
  5232 m^4 s^5 − 8352 k m^4 s^5 + 720 k^2 m^4 s^5 + 3456 k^3 m^4 s^5 − 5184 m^5 s^5 − 10368 k m^5 s^5 − 5184 k^2 m^5 s^5).
```

In the next step we have to check that this summand telescoping equation (121) holds within the summation range. In order to avoid poles in the evaluation, similarly to Remark 1, we represent all ingredients in (121) not in terms of $p(s,k,m)$ but in terms of $p(s+1,k+2,m)$, i.e.,

$$p(s+1,k,m) = p(s+1,k+2,m)\frac{(m-s-k-4)(m-s-k-5)}{(2m-2s-k-5)(2m-2s-k-6)}$$

and

$$p(s,k+i,m) = p(s+1,k+2,m)\,p_i(s,k,m), \qquad 0 \le i \le 2,$$

where

$$p_i(s,k,m) = -\frac{m-s-1}{2m-2s-k-6}\prod_{j=0}^{2-i}\frac{m-s-k+j-5}{2m-2s-k+j-5}.$$

Moreover, the shifted version of $p(s,k,m)$ in m can be expressed by

$$p(s,k,m+1) = p(s+1,k+2,m)\,p_0(s,k,m)\,\frac{-(m-s)(m-s-k-2)}{(2m-2s-k-2)(2m-2s-k-1)}.$$

Note that the representation of $u(s,k,m)$ will not be changed. Hence, with

$$u(s,m+1) = u(s,m)\,\frac{3(3m+1)(3m+2)(4m-2s-1)(4m-2s+1)(m+s)}{(4m-1)(4m+1)(3m-s)(3m-s+1)(3m-s+2)}$$

we finally manage to verify that (121) holds for $0 \le s \le \lfloor\frac{2m-k-1}{2}\rfloor - 3$ in terms of $u(s,k,m)$ and $p(s+1,k+2,m)$.

Therefore, summing equation (80) over s from 0 to an arbitrary d with $0 \le d \le \lfloor \frac{2m-k-1}{2} \rfloor - 3$ gives the relation

$$g(d+1,k,m)-g(0,k,m) = c_0(k,m)\,y(k,m)+\cdots+c_2(k,m)\,y(k+2,m)+\kappa(k,m)\,y(k,m+1) \quad (122)$$

for the sum $y(k,m) = \sum_{s=0}^{d} q(s,k,m) = \sum_{s=0}^{d} q'(s,k,m)$. Now we do a case distinction on k.

If k is even, we can choose $d = m - k/2 - 4$, and obtain

$$\begin{aligned} &c_0(k,m)\,h_1(k,m) + \cdots + c_2(k,m)\,h_1(k+2,m) + \kappa(k,m+1)\,h_1(k,m+1) \\ &\quad = g(m-\frac{k}{2}-3,k,m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=1}^{3} q(m-\frac{k}{2}-j,k+i,m) \\ &\qquad + \kappa(k,m) \sum_{j=1}^{4} q(m+1-\frac{k}{2}-j,k,m+1). \end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m-k/2-4 \ge 0$, k even, the relation (120) holds. Similarly, for odd k, one can choose $d = m-(k-1)/2-3$, and obtains

$$\begin{aligned} &c_0(k,m)\,h_1(k,m) + \cdots + c_2(k,m)\,h_1(k+2,m) \\ &\quad = g(m-\frac{k-1}{2}-2,k,m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=0}^{2} q(m-\frac{k-1}{2}-j,k+i,m) \\ &\qquad + \kappa(k,m) \sum_{j=0}^{3} q(m+1-\frac{k-1}{2}-j,k,m+1). \end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (120) holds for $m-(k-1)/2-3 \ge 0$, k odd. Summarizing, for all $m \ge 4$ and $0 \le k \le 2m-8$ the recurrence (120) contains the solution $h_1(k,m)$. The special cases $0 \le m \le 4$ and $k \ge 2m-7$ for $m \ge 4$ can be verified by simple evaluation. This shows that the recurrence holds for all $m,k \ge 0$. ◊

Now let us turn to the identities (12), (17) and the sum expression $b(m)$.

Given the recurrences (78) and (120) we are able to compute with Sigma the recurrence

$$3(3m-1)(3m+1)S(m) - (4m-1)(4m+1)S(m+1) = 0 \quad (123)$$

that is satisfied by the left sides of (12) and (17) for $m \ge 1$. The verification is explained in

Remark 12. Denote $f(k,m) = (-1)^k h_1(k,m)$. With Sigma we obtain polynomials

$$c_0(m) = 12m(1+2m)(-1+3m)(1+3m), \qquad c_1(m) = -4m(1+2m)(-1+4m)(1+4m)$$

and the expression

```
g(k,m) = 1/((1+k)^2 (-1+k-2m) (k-2m))
  ( -2 (-1)^k k (1+4m) ( -2k-5k^2-2k^3+4k^4+4k^5+k^6+8m+6km-11k^2m-19k^3m-15k^4m-5k^5m+8m^2+32km^2+6k^2m^2-
        2k^3m^2+4k^4m^2-64m^3-36km^3+52k^2m^3+36k^3m^3-192km^4-144k^2m^4+192m^5+144km^5) h1(k,m)+
    (-1)^k k (1+4m) ( -6k-15k^2-6k^3+12k^4+12k^5+3k^6+16m+2km-37k^2m-45k^3m-33k^4m-11k^5m+40m^2+16km^2-
        54k^2m^2-46k^3m^2-4k^4m^2-8m^3+12km^3+92k^2m^3+60k^3m^3-96km^4-144k^2m^4+96m^5+144km^5) h1(1+k,m)-
    (-1)^k k (2+k-2m) (1+k+2m) (1+4m) ( -k-k^2+k^3+k^4+2m-3km-2k^2m-3k^3m+6m^2-12km^2+12m^3) h1(2+k,m))
```

such that

$$c_0(m)f(k,m) + c_1(m)f(k,m+1) = g(k+1,m) - g(k,m) \quad (124)$$

holds for all $0 \le k \le 2m-4$. The verification can be achieved as follows. By applying the shift in k to $g(k,m)$ the expression $g(k+1,m)$ depends on $(-1)^{k+1}$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $h_1(k+3,m)$. Then the term $(-1)^{k+1}$ can be rewritten with $-(-1)^k$, and $h_1(k+3,m)$ can be expressed by $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ by applying the relation given in (78) for any $0 \le k < 2m-3$. Moreover, the expression $f(m+1,k)$ depends on $(-1)^k$ and $h_1(k,m+1)$, $h_1(k+1,m+1)$ and $h_1(k+2,m+1)$. Now the crucial step is to apply rule (120) and represent

$h_1(k,m+1)$ in terms of $h_1(k,m)$ and $h_1(k+1,m)$. This rewriting holds for all $0 \le k < 2m$. In this representation the equation (124) can be verified by simple term rewriting for $0 \le k \le 2m-4$. Summing equation (124) over k from 0 to $2m-4$ gives

$$c_0(m)S(m) + c_1(m)S(m+1)$$
$$= g(2m-3,i,m) - g(0,i,m) + c_0(m)\sum_{r=2m-3}^{2m-1} f(r,m) + c_1(m)\sum_{r=2m-3}^{2m+1} f(r,m+1) \quad (125)$$

where $S(m)$ denotes the left hand side of (12). In this case one can show that the right hand side of (125) vanishes to 0 which gives recurrence (123). The special cases $m = 1, 2$ follow by simple evaluation.

Completely analogously, by choosing $f(k,m) = 2^k h_1(k,m)$,

$$c_0(m) = -12m(1+2m)(-1+3m)(1+3m), \qquad c_1(m) = 4m(1+2m)(-1+4m)(1+4m)$$

and

```
g(k, m) = 1/((1 + k)^2 (-1 + k - 2 m) (k - 2 m))( - 2^(1+k) (1 + 4 m) (2 k^2 - k^3 - 7 k^4 - k^5 + 5 k^6 + 2 k^7 - 2 k m -
      18 k^2 m + 8 k^3 m + 16 k^4 m - 24 k^5 m - 16 k^6 m + 12 m^2 + 10 k m^2 - 38 k^2 m^2 - 60 k^3 m^2 + 68 k^4 m^2 + 56 k^5 m^2 +
      36 m^3 - 8 k m^3 - 16 k^3 m^3 - 24 k^4 m^3 + 48 k m^4 - 240 k^2 m^4 - 144 k^3 m^4 - 48 m^5 + 240 k m^5 + 144 k^2 m^5) h1(k, m) +
  2^k (1 + 4 m) (6 k^2 - 3 k^3 - 21 k^4 - 3 k^5 + 15 k^6 + 6 k^7 - 34 k m - 62 k^2 m + 40 k^3 m +
      48 k^4 m - 60 k^5 m - 40 k^6 m + 60 m^2 + 2 k m^2 - 262 k^2 m^2 - 120 k^3 m^2 + 160 k^4 m^2 + 112 k^5 m^2 + 180 m^3 +
      248 k m^3 - 120 k^2 m^3 - 152 k^3 m^3 + 240 k m^4 - 48 k^2 m^4 - 432 k^3 m^4 - 240 m^5 + 48 k m^5 + 432 k^2 m^5) h1(1 + k, m) -
  2^k (2 + k - 2 m) (1 + k + 2 m) (1 + 4 m)
    (k^2 - 2 k^3 - k^4 + 2 k^5 - 8 k m + 14 k^3 m - 12 k^4 m + 12 m^2 - 12 k m^2 - 42 k^2 m^2 + 36 k^3 m^2 + 24 m^3 + 24 k m^3 - 36 k^2 m^3) h1(2 + k, m))
```

one can verify that the left hand side of (17) is a solution of recurrence (123). ◊

Since $t_1(2m-1)$ is also a solution of (71), identities (12) and (17) follow by checking their equalities at $m = 1$.

Similarly, we are able to derive the recurrence (73) for $b(m)$. The correctness of this result can be found in

Remark 13. Denote $f(k,m) = \binom{2m+k-2}{2m-1} h_1(k,m)$. With Sigma we obtain polynomials

$$\begin{aligned}
c_0(m) &= -3456m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(-1+3m)(1+3m)(2+3m)(4+3m)\\
&\quad (-1+4m)(1+4m)(1+6m)(5+6m)(35582085 + 208705770m + 502426266m^2 + 659838718m^3 + 522259397m^4\\
&\quad + 256674880m^5 + 76794344m^6 + 12820192m^7 + 915728m^8),\\
c_1(m) &= 384m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(2+3m)(4+3m)(-1+4m)(1+4m)\\
&\quad (-5091730875 - 64116259830m - 268357073220m^2 - 344918244168m^3 + 764136680690m^4 + 3625956026718m^5\\
&\quad + 6449149601689m^6 + 6689203564428m^7 + 4423817173116m^8 + 1893138648192m^9 + 508247453296m^{10}\\
&\quad + 77855194560m^{11} + 5190346304m^{12}),\\
c_2(m) &= -128m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(-1+4m)(1+4m)(3+4m)(5+4m)\\
&\quad (-7125148800 - 88649425620m - 376507014210m^2 - 591098983182m^3 + 431261276465m^4 + 3340499936442m^5\\
&\quad + 6290733211249m^6 + 6633648559692m^7 + 4412582090796m^8 + 1892139974208m^9 + 508247453296m^{10}\\
&\quad + 77855194560m^{11} + 5190346304m^{12}),\\
c_3(m) &= 512m^2(1+m)^2(2+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(-1+4m)(1+4m)(3+4m)(5+4m)(7+4m)\\
&\quad (9+4m)^2(11+4m)(-61740 - 644340m - 1950794m^2 - 910998m^3 + 6194397m^4 + 13852080m^5\\
&\quad + 12693384m^6 + 5494368m^7 + 915728m^8),
\end{aligned}$$

and the expression

$$g(k,m) = \frac{-(k-1)\big[p_0(k,m)h(k,m) + p_1(k,m)h(k+1,m) + p_2(k,m)h(k+2,m)\big]}{(k+1)^2\prod_{i=0}^{5}(2m-k+i)\prod_{i=2}^{4}(2m+k+i)} \times \binom{2m+k-2}{2m-1}, \quad (126)$$

where the polynomials $p_i(k,m)$ in k and m are given in Appendix D, such that

$$c_0(m)f(k,m)+\cdots+c_3(m)f(k,m+3)=g(k+1,m)-g(k,m) \tag{127}$$

holds for all $0\le k\le 2m-4$. The verification can be achieved as follows. By applying the shift in k to $g(k,m)$ the expression $g(k+1,m)$ depends on $\binom{2m+k-1}{2m-1}$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $h_1(k+3,m)$. Then the term $\binom{2m+k-1}{2m-1}$ can be rewritten with $\frac{2m+k-1}{k}\binom{2m+k-1}{2m-1}$, and $h_1(k+3,m)$ can be expressed by $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ by applying the relation given in (78) for any $0\le k<2m-3$. Hence $g(k+1,m)$ can be expressed in terms of $h_1(k,m)$, $h_1(k+1,m)$, $h_1(k+2,m)$, and $\binom{2m+k-1}{2m-1}$ for all $0\le k\le 2m-4$; for the correctness at $k=0$ the factor $k-1$ in the numerator of (126) needed. Moreover, the expressions $f(m+i,k)$ depend on $\binom{2m+2i+k-1}{2m+2i-1}$ and $h_1(m+i,k)$, $h_1(m+i,k+1)$ and $h_1(m+i,k+2)$. $\binom{2m+2i+k-1}{2m+2i-1}$ can be represented in the form $\frac{q_1(k,m)}{q_2(k,m)}\binom{2m+k-1}{2m-1}$ with $q_i(k,m)\in\mathbb{Z}[k,m]$ where $q_2(k,m)\neq 0$ for $0\le k\le 2m-4$. Now the crucial step is to apply rule (120) and represent $h_1(k,m+1)$ in terms of $h_1(k,m)$ and $h_1(k+1,m)$. Similarly, the shifted version of (120) in m enables one to reduce $h_1(k,m+2)$ first to expressions in $h_1(k,m+1)$, $h_1(k+1,m+1)$, and $h_1(k+2,m+1)$ for $0\le k<2m-1$. Second, by applying the rules (120) and (78) those expressions can be again represented in terms of $h_1(k,m),\dots,h_1(k+2,m)$ for $0\le k\le 2m-4$. Following this rewriting process recursively, all expressions in $f(k+j,m+i)$, $0\le i\le 2$ amd $0\le j\le 3$ can be represented in terms of $h_1(k,m)$, $h_1(k+1,m)$ and $h_1(k+2,m)$ for $0\le k\le 2m-4$. In this representation the equation (126) can be verified by simple term rewriting for $0\le k\le 2m-4$. Summing equation (127) over k from 0 to $2m-4$ gives

$$\begin{aligned}&c_0(m)b(m)+\cdots+c_3(m)b(m+3)\\ &\qquad=g(2m-3,i,m)-g(0,i,m)+\sum_{j=0}^{3}c_j(m)\sum_{r=2m-3}^{2(m+j)-1}f(m+j,r).\end{aligned}$$

In this case one can show that the right hand side vanishes to 0 which gives recurrence (74) for $b(m)$. The special cases $m=1,2$ follow by simple evaluation. ◊

Together with Section 5 this result shows identity (35).

Finally, we show the remaining identities in Propositions 3 and 5 and we prove that $a(m)$ is a solution of (74) for all $m\ge 1$. In order to achieve this, we compute the following recurrence of $h_2(k,m)$ for all $m,k\ge 0$ with Sigma.

$$\begin{aligned}&(1+k+2m)(6+141k-18k^2-141k^3-36k^4+18k^5+12k^6-216m+795km\\
&\quad+392k^2m-357k^3m-180k^4m-48k^5m+16k^6m-1242m^2+1266km^2+1274k^2m^2\\
&\quad-150k^3m^2-32k^4m^2-96k^5m^2-2436m^3+396km^3+1260k^2m^3-60k^3m^3\\
&\quad+192k^4m^3-2040m^4-504km^4+504k^2m^4-216k^3m^4-624m^5-288km^5+144k^2m^5)\,h_2(k,m)\\
&\quad+(3+4m)(-30-111k-41k^2+125k^3+99k^4+k^5-19k^6-6k^7-120m\\
&\quad-549km-489k^2m+148k^3m+254k^4m+80k^5m+16k^6m-30m^2-774km^2\\
&\quad-1210k^2m^2-252k^3m^2+136k^4m^2+24k^5m^2+420m^3+72km^3-1008k^2m^3-480k^3m^3\\
&\quad-24k^4m^3+600m^4+840km^4-72k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5)\,h_2(k+1,m)\\
&\quad-(-1-k+2m)(3+4m)(12+30k-16k^2-32k^3-4k^4+5k^5+2k^6+72m\\
&\quad+186km-20k^2m-99k^3m-16k^4m+156m^2+420km^2+66k^2m^2-108k^3m^2\\
&\quad-12k^4m^2+144m^3+408km^3+144k^2m^3-36k^3m^3+48m^4+144km^4+72k^2m^4)\,h_2(k+2,m)\\
&\quad+4(1+k)^2(-2+k-2m)(-1+k-2m)(1+m)(1+2m)(1+4m)(3+4m)\,h_2(k,m+1)=0\end{aligned} \tag{128}$$

Remark 14. As in Remark 6 we consider the sum

$$h_2''(s,k,m)=\sum_{s=0}^{\lfloor\frac{2m-k}{2}\rfloor-1}q''(s,k,m)$$

Recall that $q''(s,k,m) = q(s,k,m)$ for $0 \le s \le \lfloor \frac{2m-k}{2} \rfloor - 1$ where $q(s,k,m)$ is the summand representation of $f_2(k,2m-k) = h_2(k,m)$. In the sequel denote

$$p(s,k,m) = (-1)^k \binom{m-s}{2m-2s-k-2},$$

and

$$u_1(r,m) = \frac{(m)_r(-3m-1)_r}{r!(\frac{1}{2}-2m)_r}, \qquad u_2(s,k,m) = \sum_{r=0}^{s}(m-r)u_1(r,m),$$

i.e.,

$$q''(s,k,m) = \frac{k}{2m(m-s)}\frac{(m-s-k-1)(m-s-k-2)}{(2m-2s-k)(2m-2s-k-1)}\frac{(-1)^s}{4^s}\,p(s,k,m)\,u_2(s,k,m).$$

Now applying this representation of the definite sum $h_2''(k,m)$ to Sigma gives the creative telescoping equation

$$g(s+1,k,m)-g(s,k,m) = c_0(k,m)q''(s,k,m)+\cdots+c_2(k,m)q''(s,k+2,m)+\kappa(k,m)\,q''(s,k,m+1). \tag{129}$$

with

$$\begin{aligned}
c_0(k,m) &= m(1+k+2m)(6+141k-18k^2-141k^3-36k^4+18k^5+12k^6-216m+795km+392k^2m-357k^3m\\
&\quad -180k^4m-48k^5m+16k^6m-1242m^2+1266km^2+1274k^2m^2-150k^3m^2-32k^4m^2-96k^5m^2-2436m^3\\
&\quad +396km^3+1260k^2m^3-60k^3m^3+192k^4m^3-2040m^4-504km^4+504k^2m^4-216k^3m^4-624m^5\\
&\quad -288km^5+144k^2m^5),\\
c_1(k,m) &= m(3+4m)(-30-111k-41k^2+125k^3+99k^4+k^5-19k^6-6k^7-120m-549km-489k^2m+148k^3m\\
&\quad +254k^4m+80k^5m+16k^6m-30m^2-774km^2-1210k^2m^2-252k^3m^2+136k^4m^2+24k^5m^2+420m^3+72km^3\\
&\quad -1008k^2m^3-480k^3m^3-24k^4m^3+600m^4+840km^4-72k^2m^4-144k^3m^4+240m^5+432km^5+144k^2m^5),\\
c_2(k,m) &= -(m(-1-k+2m)(3+4m)(12+30k-16k^2-32k^3-4k^4+5k^5+2k^6+72m+186km-20k^2m-99k^3m\\
&\quad -16k^4m+156m^2+420km^2+66k^2m^2-108k^3m^2-12k^4m^2+144m^3+408km^3\\
&\quad +144k^2m^3-36k^3m^3+48m^4+144km^4+72k^2m^4)),\\
\kappa(k,m) &= 4(1+k)^2(-2+k-2m)(-1+k-2m)m(1+m)(1+2m)(1+4m)(3+4m),
\end{aligned}$$

and

$$g(s,k,m) = p(s,k,m)\frac{(-1)^s}{4^s}\,\frac{u_1(s,m)\,a_1(s,k,m)+u_2(s,k,m)\,a_2(s,k,m)}{b(s,k,m)}$$

where r is substituted with s in $u(r,m)$,

$$\begin{aligned}
b(s,k,m) &= (3m-s+2)(3m-s+3)\\
&\qquad (2m-2s-k+2)(2m-2s-k+1)(2m-2s-k)(2m-2s-k-1),
\end{aligned}$$

and

$$
\begin{aligned}
a_1(s,k,m) = {} & (-2-k+m-s)\,(m+s)\,\big(-216-432\,k+1278\,k^2+864\,k^3-1476\,k^4-468\,k^5+432\,k^6+36\,k^7-18\,k^8-2988\,m-8100\,k\,m+\\
& 15333\,k^2\,m+17766\,k^3\,m-17406\,k^4\,m-8136\,k^5\,m+5076\,k^6\,m+234\,k^7\,m-159\,k^8\,m+36\,k^9\,m-17100\,m^2-64818\,k\,m^2+64500\,k^2\,m^2+\\
& 146391\,k^3\,m^2-76314\,k^4\,m^2-59949\,k^5\,m^2+25098\,k^6\,m^2+1170\,k^7\,m^2-792\,k^8\,m^2+138\,k^9\,m^2-49752\,m^3-283356\,k\,m^3+\\
& 54240\,k^2\,m^3+631371\,k^3\,m^3-123441\,k^4\,m^3-237219\,k^5\,m^3+65184\,k^6\,m^3+4266\,k^7\,m^3-1851\,k^8\,m^3+174\,k^9\,m^3-61776\,m^4-\\
& 724608\,k\,m^4-477300\,k^2\,m^4+1561185\,k^3\,m^4+129030\,k^4\,m^4-550083\,k^5\,m^4+94842\,k^6\,m^4+8310\,k^7\,m^4-1920\,k^8\,m^4+72\,k^9\,m^4+\\
& 57348\,m^5-1037808\,k\,m^5-2013969\,k^2\,m^5+2211387\,k^3\,m^5+887745\,k^4\,m^5-771255\,k^5\,m^5+76608\,k^6\,m^5+7680\,k^7\,m^5-720\,k^8\,m^5+\\
& 358452\,m^6-531342\,k\,m^6-3824454\,k^2\,m^6+1512048\,k^3\,m^6+1620966\,k^4\,m^6-644538\,k^5\,m^6+31248\,k^6\,m^6+2664\,k^7\,m^6+\\
& 654336\,m^7+780048\,k\,m^7-4066884\,k^2\,m^7-139788\,k^3\,m^7+1511748\,k^4\,m^7-296100\,k^5\,m^7+4752\,k^6\,m^7+669312\,m^8+\\
& 1715184\,k\,m^8-2344680\,k^2\,m^8-1036296\,k^3\,m^8+734184\,k^4\,m^8-57672\,k^5\,m^8+408960\,m^9+1443456\,k\,m^9-507168\,k^2\,m^9-\\
& 694656\,k^3\,m^9+147744\,k^4\,m^9+139968\,m^{10}+603936\,k\,m^{10}+124416\,k^2\,m^{10}-158112\,k^3\,m^{10}+20736\,m^{11}+103680\,k\,m^{11}+\\
& 62208\,k^2\,m^{11}+828\,s+3564\,k\,s-4209\,k^2\,s-8064\,k^3\,s+6642\,k^4\,s+3636\,k^5\,s-2340\,k^6\,s-36\,k^7\,s+87\,k^8\,s-36\,k^9\,s+8412\,m\,s+\\
& 53580\,k\,m\,s-20785\,k^2\,m\,s-125706\,k^3\,m\,s+52019\,k^4\,m\,s+52960\,k^5\,m\,s-22914\,k^6\,m\,s-1138\,k^7\,m\,s+1028\,k^8\,m\,s-168\,k^9\,m\,s+\\
& 26160\,m^2\,s+324470\,k\,m^2\,s+132903\,k^2\,m^2\,s-771404\,k^3\,m^2\,s+81512\,k^4\,m^2\,s+310086\,k^5\,m^2\,s-85148\,k^6\,m^2\,s-7990\,k^7\,m^2\,s+\\
& 3289\,k^8\,m^2\,s-250\,k^9\,m^2\,s-26496\,m^3\,s+988470\,k\,m^3\,s+1441421\,k^2\,m^3\,s-2410212\,k^3\,m^3\,s-439263\,k^4\,m^3\,s+940994\,k^5\,m^3\,s-\\
& 153414\,k^6\,m^3\,s-20540\,k^7\,m^3\,s+4060\,k^8\,m^3\,s-120\,k^9\,m^3\,s-420588\,m^4\,s+1382778\,k\,m^4\,s+5393078\,k^2\,m^4\,s-3966280\,k^3\,m^4\,s-\\
& 2232326\,k^4\,m^4\,s+1613374\,k^5\,m^4\,s-138960\,k^6\,m^4\,s-22248\,k^7\,m^4\,s+1728\,k^8\,m^4\,s-1372572\,m^5\,s-303546\,k\,m^5\,s+\\
& 10829192\,k^2\,m^5\,s-2658282\,k^3\,m^5\,s-4422872\,k^4\,m^5\,s+1579620\,k^5\,m^5\,s-56736\,k^6\,m^5\,s-8640\,k^7\,m^5\,s-2431584\,m^6\,s-\\
& 4434180\,k\,m^6\,s+12497240\,k^2\,m^6\,s+1549124\,k^3\,m^6\,s-4598480\,k^4\,m^6\,s+825160\,k^5\,m^6\,s-6576\,k^6\,m^6\,s-2627808\,m^7\,s-\\
& 7556208\,k\,m^7\,s+7743264\,k^2\,m^7\,s+4133472\,k^3\,m^7\,s-2484864\,k^4\,m^7\,s+178704\,k^5\,m^7\,s-1734144\,m^8\,s-6400256\,k\,m^8\,s+\\
& 1625216\,k^2\,m^8\,s+2824576\,k^3\,m^8\,s-552384\,k^4\,m^8\,s-644544\,m^9\,s-2824320\,k\,m^9\,s-659904\,k^2\,m^9\,s+688704\,k^3\,m^9\,s-\\
& 103680\,m^{10}\,s-518400\,k\,m^{10}\,s-311040\,k^2\,m^{10}\,s-360\,s^2-8802\,k\,s^2-3105\,k^2\,s^2+24399\,k^3\,s^2-6195\,k^4\,s^2-10671\,k^5\,s^2+\\
& 4608\,k^6\,s^2+360\,k^7\,s^2-324\,k^8\,s^2+30\,k^9\,s^2+5496\,m\,s^2-94460\,k\,m\,s^2-130204\,k^2\,m\,s^2+283891\,k^3\,m\,s^2+9557\,k^4\,m\,s^2-\\
& 124737\,k^5\,m\,s^2+30964\,k^6\,m\,s^2+5060\,k^7\,m\,s^2-1713\,k^8\,m\,s^2+82\,k^9\,m\,s^2+98832\,m^2\,s^2-342106\,k\,m^2\,s^2-1187447\,k^2\,m^2\,s^2+\\
& 1246970\,k^3\,m^2\,s^2+429561\,k^4\,m^2\,s^2-550642\,k^5\,m^2\,s^2+72370\,k^6\,m^2\,s^2+17762\,k^7\,m^2\,s^2-2800\,k^8\,m^2\,s^2+56\,k^9\,m^2\,s^2+\\
& 580296\,m^3\,s^2-167196\,k\,m^3\,s^2-4962936\,k^2\,m^3\,s^2+2454572\,k^3\,m^3\,s^2+2092520\,k^4\,m^3\,s^2-1209448\,k^5\,m^3\,s^2+67904\,k^6\,m^3\,s^2+\\
& 23480\,k^7\,m^3\,s^2-1456\,k^8\,m^3\,s^2+1819992\,m^4\,s^2+2383628\,k\,m^4\,s^2-11351580\,k^2\,m^4\,s^2+1292292\,k^3\,m^4\,s^2+4653772\,k^4\,m^4\,s^2-\\
& 1418296\,k^5\,m^4\,s^2+16512\,k^6\,m^4\,s^2+10584\,k^7\,m^4\,s^2+3450000\,m^5\,s^2+8175768\,k\,m^5\,s^2-14731512\,k^2\,m^5\,s^2-3272216\,k^3\,m^5\,s^2+\\
& 5478328\,k^4\,m^5\,s^2-851488\,k^5\,m^5\,s^2-5584\,k^6\,m^5\,s^2+4089936\,m^6\,s^2+13010288\,k\,m^6\,s^2-9950560\,k^2\,m^6\,s^2-6520608\,k^3\,m^6\,s^2+\\
& 3325712\,k^4\,m^6\,s^2-206176\,k^5\,m^6\,s^2+2974272\,m^7\,s^2+11472896\,k\,m^7\,s^2-1897088\,k^2\,m^7\,s^2-4647040\,k^3\,m^7\,s^2+\\
& 822912\,k^4\,m^7\,s^2+1215936\,m^8\,s^2+5415360\,k\,m^8\,s^2+1413696\,k^2\,m^8\,s^2-1224192\,k^3\,m^8\,s^2+214272\,m^9\,s^2+1071360\,k\,m^9\,s^2+\\
& 642816\,k^2\,m^9\,s^2-2448\,s^3+5154\,k\,s^3+23109\,k^2\,s^3-31650\,k^3\,s^3-6972\,k^4\,s^3+15942\,k^5\,s^3-2712\,k^6\,s^3-1008\,k^7\,s^3+\\
& 243\,k^8\,s^3-6\,k^9\,s^3-42432\,m\,s^3+3778\,k\,m\,s^3+347291\,k^2\,m\,s^3-251692\,k^3\,m\,s^3-151185\,k^4\,m\,s^3+130030\,k^5\,m\,s^3-8322\,k^6\,m\,s^3-\\
& 6156\,k^7\,m\,s^3+708\,k^8\,m\,s^3-8\,k^9\,m\,s^3-291768\,m^2\,s^3-346540\,k\,m^2\,s^3+1961590\,k^2\,m^2\,s^3-610590\,k^3\,m^2\,s^3-886438\,k^4\,m^2\,s^3+\\
& 396250\,k^5\,m^2\,s^3+1064\,k^6\,m^2\,s^3-11024\,k^7\,m^2\,s^3+512\,k^8\,m^2\,s^3-1076040\,m^3\,s^3-2279388\,k\,m^3\,s^3+5565216\,k^2\,m^3\,s^3+\\
& 83480\,k^3\,m^3\,s^3-2358368\,k^4\,m^3\,s^3+572884\,k^5\,m^3\,s^3+20672\,k^6\,m^3\,s^3-6144\,k^7\,m^3\,s^3-2367504\,m^4\,s^3-6764632\,k\,m^4\,s^3+\\
& 8484056\,k^2\,m^4\,s^3+2863472\,k^3\,m^4\,s^3-3235208\,k^4\,m^4\,s^3+396840\,k^5\,m^4\,s^3+14512\,k^6\,m^4\,s^3-3212496\,m^5\,s^3-\\
& 11179144\,k\,m^5\,s^3+6351984\,k^2\,m^5\,s^3+5222808\,k^3\,m^5\,s^3-2241392\,k^4\,m^5\,s^3+106160\,k^5\,m^5\,s^3-2640960\,m^6\,s^3-\\
& 10639024\,k\,m^6\,s^3+932896\,k^2\,m^6\,s^3+3976208\,k^3\,m^6\,s^3-623616\,k^4\,m^6\,s^3-1207488\,m^7\,s^3-5469120\,k\,m^7\,s^3-\\
& 1569600\,k^2\,m^7\,s^3+1144320\,k^3\,m^7\,s^3-235776\,m^8\,s^3-1178880\,k\,m^8\,s^3-707328\,k^2\,m^8\,s^3+4608\,s^4+6498\,k\,s^4-32829\,k^2\,s^4+\\
& 15045\,k^3\,s^4+16785\,k^4\,s^4-9963\,k^5\,s^4-516\,k^6\,s^4+720\,k^7\,s^4-48\,k^8\,s^4+57012\,m\,s^4+122932\,k\,m\,s^4-338727\,k^2\,m\,s^4+\\
& 42905\,k^3\,m\,s^4+165223\,k^4\,m\,s^4-53185\,k^5\,m\,s^4-6352\,k^6\,m\,s^4+2256\,k^7\,m\,s^4-64\,k^8\,m\,s^4+293940\,m^2\,s^4+815490\,k\,m^2\,s^4-\\
& 1340342\,k^2\,m^2\,s^4-218164\,k^3\,m^2\,s^4+588342\,k^4\,m^2\,s^4-99182\,k^5\,m^2\,s^4-14608\,k^6\,m^2\,s^4+1728\,k^7\,m^2\,s^4+820896\,m^3\,s^4+\\
& 2712600\,k\,m^3\,s^4-2542124\,k^2\,m^3\,s^4-1224428\,k^3\,m^3\,s^4+988652\,k^4\,m^3\,s^4-76060\,k^5\,m^3\,s^4-9408\,k^6\,m^3\,s^4+1343712\,m^4\,s^4+\\
& 5071136\,k\,m^4\,s^4-2157368\,k^2\,m^4\,s^4-2273256\,k^3\,m^4\,s^4+801144\,k^4\,m^4\,s^4-19672\,k^5\,m^4\,s^4+1291008\,m^5\,s^4+5426400\,k\,m^5\,s^4-\\
& 129888\,k^2\,m^5\,s^4-1897248\,k^3\,m^5\,s^4+253728\,k^4\,m^5\,s^4+674880\,m^6\,s^4+3109728\,k\,m^6\,s^4+963072\,k^2\,m^6\,s^4-604896\,k^3\,m^6\,s^4+\\
& 148224\,m^7\,s^4+741120\,k\,m^7\,s^4+444672\,k^2\,m^7\,s^4-3420\,s^5-10230\,k\,s^5+20196\,k^2\,s^5+1734\,k^3\,s^5-10692\,k^4\,s^5+1896\,k^5\,s^5+\\
& 888\,k^6\,s^5-144\,k^7\,s^5-33756\,m\,s^5-114602\,k\,m\,s^5+147528\,k^2\,m\,s^5+51026\,k^3\,m\,s^5-66360\,k^4\,m\,s^5+4232\,k^5\,m\,s^5+3104\,k^6\,m\,s^5-\\
& 192\,k^7\,m\,s^5-136704\,m^2\,s^5-512892\,k\,m^2\,s^5+387648\,k^2\,m^2\,s^5+257732\,k^3\,m^2\,s^5-149272\,k^4\,m^2\,s^5+304\,k^5\,m^2\,s^5+\\
& 2560\,k^6\,m^2\,s^5-290496\,m^3\,s^5-1184432\,k\,m^3\,s^5+391120\,k^2\,m^3\,s^5+532560\,k^3\,m^3\,s^5-146000\,k^4\,m^3\,s^5-\\
& 2624\,k^5\,m^3\,s^5-341376\,m^4\,s^5-1497152\,k\,m^4\,s^5-34432\,k^2\,m^4\,s^5+502720\,k^3\,m^4\,s^5-52800\,k^4\,m^4\,s^5-210240\,m^5\,s^5-\\
& 985920\,k\,m^5\,s^5-322752\,k^2\,m^5\,s^5+180480\,k^3\,m^5\,s^5-52992\,m^6\,s^5-264960\,k\,m^6\,s^5-158976\,k^2\,m^6\,s^5+\\
& 1152\,s^6+4800\,k\,s^6-5496\,k^2\,s^6-3264\,k^3\,s^6+2472\,k^4\,s^6+288\,k^5\,s^6-192\,k^6\,s^6+9168\,m\,s^6+39424\,k\,m\,s^6-\\
& 26264\,k^2\,m\,s^6-23792\,k^3\,m\,s^6+9368\,k^4\,m\,s^6+1440\,k^5\,m\,s^6-256\,k^6\,m\,s^6+28752\,m^2\,s^6+127744\,k\,m^2\,s^6-\\
& 34944\,k^2\,m^2\,s^6-61296\,k^3\,m^2\,s^6+11408\,k^4\,m^2\,s^6+1408\,k^5\,m^2\,s^6+44352\,m^3\,s^6+204064\,k\,m^3\,s^6+11264\,k^2\,m^3\,s^6-\\
& 68192\,k^3\,m^3\,s^6+4416\,k^4\,m^3\,s^6+33600\,m^4\,s^6+160704\,k\,m^4\,s^6+54720\,k^2\,m^4\,s^6-28032\,k^3\,m^4\,s^6+9984\,m^5\,s^6+\\
& 49920\,k\,m^5\,s^6+29952\,k^2\,m^5\,s^6-144\,s^7-744\,k\,s^7+528\,k^2\,s^7+648\,k^3\,s^7-144\,k^4\,s^7-96\,k^5\,s^7-912\,m\,s^7-\\
& 4568\,k\,m\,s^7+1184\,k^2\,m\,s^7+2616\,k^3\,m\,s^7-192\,k^4\,m\,s^7-128\,k^5\,m\,s^7-2112\,m^2\,s^7-10384\,k\,m^2\,s^7-800\,k^2\,m^2\,s^7+\\
& 3632\,k^3\,m^2\,s^7-2112\,m^3\,s^7-10368\,k\,m^3\,s^7-3648\,k^2\,m^3\,s^7+1728\,k^3\,m^3\,s^7-768\,m^4\,s^7-3840\,k\,m^4\,s^7-2304\,k^2\,m^4\,s^7\big),
\end{aligned}
$$

```
a2(s,k,m) = -m (-2-k+m-s) (2+3 m-s) (3+3 m-s) ( -36-72 k+213 k^2+144 k^3-246 k^4-78 k^5+72 k^6+6 k^7-3 k^8-
  408 m-1170 k m+2023 k^2 m+2601 k^3 m-2286 k^4 m-1161 k^5 m+666 k^6 m+24 k^7 m-19 k^8 m+6 k^9 m-1776 m^2-7770 k m^2+
  5373 k^2 m^2+17680 k^3 m^2-6635 k^4 m^2-6972 k^5 m^2+2410 k^6 m^2+126 k^7 m^2-80 k^8 m^2+8 k^9 m^2-3240 m^3-26046 k m^3-7427 k^2 m^3+
  57127 k^3 m^3-557 k^4 m^3-20365 k^5 m^3+3840 k^6 m^3+360 k^7 m^3-80 k^8 m^3+468 m^4-43998 k m^4-69042 k^2 m^4+90860 k^3 m^4+
  32850 k^4 m^4-30310 k^5 m^4+2592 k^6 m^4+296 k^7 m^4+13248 m^5-23904 k m^5-151916 k^2 m^5+55724 k^3 m^5+66668 k^4 m^5-
  22220 k^5 m^5+528 k^6 m^5+25920 m^6+37200 k m^6-154056 k^2 m^6-23592 k^3 m^6+54216 k^4 m^6-6408 k^5 m^6+24384 m^7+
  72864 k m^7-64800 k^2 m^7-47904 k^3 m^7+16416 k^4 m^7+11712 m^8+47904 k m^8+2304 k^2 m^8-17568 k^3 m^8+2304 m^9+11520 k m^9+
  6912 k^2 m^9+216 s+516 k s-582 k^2 s-897 k^3 s+489 k^4 s+423 k^5 s-150 k^6 s-66 k^7 s+9 k^8 s+6 k^9 s+1944 m s+5952 k m s-
  1554 k^2 m s-10101 k^3 m s+1242 k^4 m s+3869 k^5 m s-660 k^6 m s-256 k^7 m s+12 k^8 m s+8 k^9 m s+6240 m^2 s+26400 k m^2 s+
  16028 k^2 m^2 s-42078 k^3 m^2 s-8466 k^4 m^2 s+14552 k^5 m^2 s-1136 k^6 m^2 s-464 k^7 m^2 s+5736 m^3 s+51396 k m^3 s+
  101424 k^2 m^3 s-76068 k^3 m^3 s-48944 k^4 m^3 s+26552 k^5 m^3 s-448 k^6 m^3 s-320 k^7 m^3 s-14280 m^4 s+18096 k m^4 s+
  238792 k^2 m^4 s-33712 k^3 m^4 s-97720 k^4 m^4 s+22816 k^5 m^4 s+336 k^6 m^4 s-46320 m^5 s-101040 k m^5 s+266448 k^2 m^5 s+
  77328 k^3 m^5 s-87264 k^4 m^5 s+7248 k^5 m^5 s-54816 m^6 s-185088 k m^6 s+117216 k^2 m^6 s+109728 k^3 m^6 s-29376 k^4 m^6 s-
  30912 m^7 s-130944 k m^7 s-15552 k^2 m^7 s+42048 k^3 m^7 s-6912 m^8 s-34560 k m^8 s-20736 k^2 m^8 s-396 s^2-1032 k s^2+
  453 k^2 s^2+1701 k^3 s^2+87 k^4 s^2-687 k^5 s^2-114 k^6 s^2+72 k^7 s^2+24 k^8 s^2-2760 m s^2-8130 k m s^2-2797 k^2 m s^2+
  10899 k^3 m s^2+4087 k^4 m s^2-3087 k^5 m s^2-644 k^6 m s^2+72 k^7 m s^2+32 k^8 m s^2-5964 m^2 s^2-21186 k m^2 s^2-33402 k^2 m^2 s^2+
  22208 k^3 m^2 s^2+21206 k^4 m^2 s^2-5442 k^5 m^2 s^2-1088 k^6 m^2 s^2-32 k^7 m^2 s^2+912 m^3 s^2-5832 k m^3 s^2-103748 k^2 m^3 s^2-
  1228 k^3 m^3 s^2+44108 k^4 m^3 s^2-3956 k^5 m^3 s^2-576 k^6 m^3 s^2+23520 m^4 s^2+74016 k m^4 s^2-136008 k^2 m^4 s^2-63048 k^3 m^4 s^2+
  41112 k^4 m^4 s^2-744 k^5 m^4 s^2+38784 m^5 s^2+149376 k m^5 s^2-58848 k^2 m^5 s^2-80064 k^3 m^5 s^2+14112 k^4 m^5 s^2+
  26688 m^6 s^2+118176 k m^6 s^2+24192 k^2 m^6 s^2-31392 k^3 m^6 s^2+6912 m^7 s^2+34560 k m^7 s^2+20736 k^2 m^7 s^2+
  216 s^3+684 k s^3+180 k^2 s^3-804 k^3 s^3-540 k^4 s^3+60 k^5 s^3+108 k^6 s^3+24 k^7 s^3+1152 m s^3+3828 k m s^3+
  3876 k^2 m s^3-1936 k^3 m s^3-2808 k^4 m s^3-208 k^5 m s^3+144 k^6 m s^3+32 k^7 m s^3+1368 m^2 s^3+4392 k m^2 s^3+
  15576 k^2 m^2 s^3+3312 k^3 m^2 s^3-5376 k^4 m^2 s^3-672 k^5 m^2 s^3-2736 m^3 s^3-12288 k m^3 s^3+22368 k^2 m^3 s^3+
  15744 k^3 m^3 s^3-4320 k^4 m^3 s^3-384 k^5 m^3 s^3-8352 m^4 s^3-37152 k m^4 s^3+6432 k^2 m^4 s^3+18240 k^3 m^4 s^3-
  1152 k^4 m^4 s^3-7488 m^5 s^3-35136 k m^5 s^3-10944 k^2 m^5 s^3+6912 k^3 m^5 s^3-2304 m^6 s^3-11520 k m^6 s^3-6912 k^2 m^6 s^3).
```

In the next step we have to check that this summand telescoping equation (129) holds within the summation range. In order to avoid poles in the evaluation, similarly to Remark 6, we represent all ingredients in (129) not in terms of $p(s,k,m)$ but in terms of $p(s+1,k+2,m)$, i.e.,

$$p(s+1,k,m) = p(s+1,k+2,m)\frac{(m-s-k-4)(m-s-k-5)}{(2m-2s-k-4)(2m-2s-k-5)} \tag{130}$$

and

$$p(s,k+i,m) = p(s+1,k+2,m)\, p_i(s,k,m), \qquad 0 \le i \le 2, \tag{131}$$

where

$$p_i(s,k,m) = -\frac{m-s}{2m-2s-k-5}\prod_{j=0}^{2-i}\frac{m-s-k+j-5}{2m-2s-k+j-4}.$$

Moreover, the shifted version of $p(s,k,m)$ in m can be expressed by

$$p(s,k,m+1) = p(s,k,m)\, p_0(s,k,m)\,\frac{-(m-s+1)(m-s-k-2)}{(2m-2s-k-1)(2m-2s-k)}.$$

Note that we keep the representations in $u_1(s,m)$ and $u_2(s,m)$. In addition we need the nontrivial relation

$$\begin{aligned} u_2(s,m+1) = \Big[&-3m(1+6m)(5+6m)(2+3m-s)(3+3m-s)\, u_2(s,m) \\ &+6(m+s)(18+169m+461m^2+496m^3+186m^4-97s-471ms-715m^2s \\ &-348m^3s+98s^2+278ms^2+198m^2s^2-27s^3-36ms^3)\, u_1(s,k,m)\Big] \Big/ \\ &\big(m(1+4m)(3+4m)(2+3m-s)(3+3m-s)\big); \end{aligned} \tag{132}$$

this identity can be verified by checking the telescoping equation

$$g'(s,k,m) - g'(s-1,k,m) = u_1(s,m+1)$$

where $g'(s,k,m)$ denotes the right side of (132).

With these relations we finally manage to verify that (129) holds for $0 \le s \le \lfloor \frac{2m-k}{2} \rfloor - 4$ in terms of $p(s+1,k+2,m)$, $u_1(s,m)$ and $u_2(s,m)$.

Hence, summing equation (129) over s from 0 to an arbitrary d with $0 \le d \le \lfloor \frac{2m-k}{2} \rfloor - 4$ gives the relation (122) for the sum $y(k,m) = \sum_{s=0}^{d} q(s,k,m) = \sum_{s=0}^{d} q''(s,k,m)$. Now we do a case distinction on k.

If k is even, we can choose $d = m - k/2 - 4$, and obtain

$$\begin{aligned} &c_0(k,m)\,h_2(k,m) + \cdots + c_2(k,m)\,h_2(k+2,m) + \kappa(k,m+1)\,h_2(k,m+1) \\ &\quad = g(m-\frac{k}{2}-3,k,m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=0}^{3} q(m-\frac{k}{2}-j, k+i, m) \\ &\qquad\qquad + \kappa(k,m) \sum_{j=0}^{4} q(m+1-\frac{k}{2}-j, k, m+1). \end{aligned}$$

By term rewriting one can now show that the right hand side is equal to 0. Hence for $m-k/2-4 \ge 0$, k even, the relation (78) holds. Similarly, for odd k, one can choose $d = m-(k-1)/2-4$, and obtains

$$\begin{aligned} &c_0(k,m)\,h_2(k,m) + \cdots + c_2(k,m)\,h_2(k+2,m) \\ &\quad = g(m-\frac{k-1}{2}-3,k,m) - g(0,k,m) + \sum_{i=0}^{2} c_i(k,m) \sum_{j=0}^{3} q(m-\frac{k-1}{2}-j, k+i, m) \\ &\qquad\qquad + \kappa(k,m) \sum_{j=0}^{4} q(m+1-\frac{k-1}{2}-j, k, m+1). \end{aligned}$$

Again one can prove that the right hand side vanishes to 0, and hence the relation (128) holds for $m-(k-1)/2-4 \ge 0$, k odd. Summarizing, for all $m \ge 5$ and $0 \le k \le 2m-9$ the recurrence (128) contains the solution $h_2(k,m)$. The special cases $0 \le m \le 5$ and $k \ge 2m-8$ for $m \ge 4$ can be verified by simple evaluation. This shows that the recurrence holds for all $m, k \ge 0$. ◊

With this recurrence relation one can prove similarly as above that $T_1(m) = \sum_{k=0}^{2m} (-1)^k h_2(k,m)$ fulfills

$$3(6m+1)(6m+5)T_1(m) - 4(4m+1)(4m+3)T_1(m+1) = 0,$$

$T_2(m) = \sum_{k=0}^{2m} 2^k h_2(k,m)$ fulfills

$$\begin{aligned} &3(6m+1)(6m+5)T_2(m) - 4(4m+1)(4m+3)T_2(m+1) \\ &\qquad = 3(2m-1)(4m+3)(2h_2(0,m) - 5h_2(1,m) + 2h_2(2,m)), \end{aligned}$$

and $a(m)$ fulfills (74) for all $m \ge 1$.

Remark 15. The verification of those recurrences is completely analogously to Remarks 12 and 13. For completeness we present the proof certificates only.

Denote $f(k,m) = (-1)^k h_2(k,m)$. Sigma provides the recurrence certificate (124) with

$$c_0(m) = 6(1+m)(1+2m)(1+6m)(5+6m), \quad c_1(m) = -8(1+m)(1+2m)(1+4m)(3+4m)$$

and

```
g(k,m) = 1/((1+k)^2 (-2+k-2m) (-1+k-2m)) ( - 2 (-1)^k k (3+4m)
    (8 - 6 k - 23 k^2 - 15 k^3 - 5 k^4 + 3 k^5 + 2 k^6 + 56 m - 80 k m - 76 k^2 m + 12 k^3 m - 22 k^4 m - 10 k^5 m + 304 m^2 - 260 k m^2 - 264 k^2 m^2 +
      104 k^3 m^2 + 8 k^4 m^2 + 832 m^3 - 120 k m^3 - 472 k^2 m^3 + 72 k^3 m^3 + 960 m^4 + 336 k m^4 - 288 k^2 m^4 + 384 m^5 + 288 k m^5) h_2(k,m) +
  (-1)^k k (3+4m) (40 - 2 k - 89 k^2 - 65 k^3 - 11 k^4 + 13 k^5 + 6 k^6 +
      160 m + 48 k m - 188 k^2 m - 92 k^3 m - 74 k^4 m - 22 k^5 m + 296 m^2 + 140 k m^2 - 264 k^2 m^2 + 88 k^3 m^2 - 8 k^4 m^2 +
      464 m^3 + 360 k m^3 - 392 k^2 m^3 + 120 k^3 m^3 + 480 m^4 + 528 k m^4 - 288 k^2 m^4 + 192 m^5 + 288 k m^5) h_2(1+k,m) -
  (-1)^k k (1+k-2m) (2+k+2m) (3+4m) (8 - 11 k - 4 k^2 - k^3 + 2 k^4 + 34 m - 30 k m - 4 k^2 m - 6 k^3 m + 48 m^2 - 24 k m^2 + 24 m^3)
    h_2(2+k,m)).
```

Similarly, denote $f(k,m) = 2^k h_2(k,m)$. Sigma provides the recurrence certificate (124) with

$$c_0(m) = -6(1+m)(1+2m)(1+6m)(5+6m), \quad c_1(m) = 8(1+m)(1+2m)(1+4m)(3+4m)$$

and

$$\mathtt{g(k,m)} = \frac{1}{(1+\mathtt{k})^2\,(-2+\mathtt{k}-2\,\mathtt{m})\,(-1+\mathtt{k}-2\,\mathtt{m})}$$
$$\big(-2^{1+\mathtt{k}}\,(3+4\,\mathtt{m})\,(6+11\,\mathtt{k}-27\,\mathtt{k}^2-23\,\mathtt{k}^3+15\,\mathtt{k}^4+\mathtt{k}^5-3\,\mathtt{k}^6+2\,\mathtt{k}^7+24\,\mathtt{m}+101\,\mathtt{k}\,\mathtt{m}-131\,\mathtt{k}^2\,\mathtt{m}-136\,\mathtt{k}^3\,\mathtt{m}+66\,\mathtt{k}^4\,\mathtt{m}+$$
$$32\,\mathtt{k}^5\,\mathtt{m}-16\,\mathtt{k}^6\,\mathtt{m}+6\,\mathtt{m}^2+370\,\mathtt{k}\,\mathtt{m}^2-218\,\mathtt{k}^2\,\mathtt{m}^2-300\,\mathtt{k}^3\,\mathtt{m}^2+32\,\mathtt{k}^4\,\mathtt{m}^2+56\,\mathtt{k}^5\,\mathtt{m}^2-84\,\mathtt{m}^3+688\,\mathtt{k}\,\mathtt{m}^3-120\,\mathtt{k}^2\,\mathtt{m}^3-$$
$$304\,\mathtt{k}^3\,\mathtt{m}^3-24\,\mathtt{k}^4\,\mathtt{m}^3-120\,\mathtt{m}^4+648\,\mathtt{k}\,\mathtt{m}^4+120\,\mathtt{k}^2\,\mathtt{m}^4-144\,\mathtt{k}^3\,\mathtt{m}^4-48\,\mathtt{m}^5+240\,\mathtt{k}\,\mathtt{m}^5+144\,\mathtt{k}^2\,\mathtt{m}^5)\,\mathtt{h_2(k,m)}+$$
$$2^{\mathtt{k}}\,(3+4\,\mathtt{m})\,(30+31\,\mathtt{k}-95\,\mathtt{k}^2-59\,\mathtt{k}^3+43\,\mathtt{k}^4-5\,\mathtt{k}^5-5\,\mathtt{k}^6+6\,\mathtt{k}^7+120\,\mathtt{m}+289\,\mathtt{k}\,\mathtt{m}-303\,\mathtt{k}^2\,\mathtt{m}-410\,\mathtt{k}^3\,\mathtt{m}+208\,\mathtt{k}^4\,\mathtt{m}+$$
$$52\,\mathtt{k}^5\,\mathtt{m}-40\,\mathtt{k}^6\,\mathtt{m}+30\,\mathtt{m}^2+794\,\mathtt{k}\,\mathtt{m}^2+26\,\mathtt{k}^2\,\mathtt{m}^2-996\,\mathtt{k}^3\,\mathtt{m}^2+160\,\mathtt{k}^4\,\mathtt{m}^2+112\,\mathtt{k}^5\,\mathtt{m}^2-420\,\mathtt{m}^3+848\,\mathtt{k}\,\mathtt{m}^3+$$
$$864\,\mathtt{k}^2\,\mathtt{m}^3-1016\,\mathtt{k}^3\,\mathtt{m}^3-600\,\mathtt{m}^4+360\,\mathtt{k}\,\mathtt{m}^4+1032\,\mathtt{k}^2\,\mathtt{m}^4-432\,\mathtt{k}^3\,\mathtt{m}^4-240\,\mathtt{m}^5+48\,\mathtt{k}\,\mathtt{m}^5+432\,\mathtt{k}^2\,\mathtt{m}^5)\,\mathtt{h_2(1+k,m)}-$$
$$2^{\mathtt{k}}\,(1+\mathtt{k}-2\,\mathtt{m})\,(2+\mathtt{k}+2\,\mathtt{m})\,(3+4\,\mathtt{m})\,(6-4\,\mathtt{k}-14\,\mathtt{k}^2+14\,\mathtt{k}^3-7\,\mathtt{k}^4+2\,\mathtt{k}^5+$$
$$30\,\mathtt{m}-2\,\mathtt{k}\,\mathtt{m}-69\,\mathtt{k}^2\,\mathtt{m}+50\,\mathtt{k}^3\,\mathtt{m}-12\,\mathtt{k}^4\,\mathtt{m}+48\,\mathtt{m}^2+24\,\mathtt{k}\,\mathtt{m}^2-96\,\mathtt{k}^2\,\mathtt{m}^2+36\,\mathtt{k}^3\,\mathtt{m}^2+24\,\mathtt{m}^3+24\,\mathtt{k}\,\mathtt{m}^3-36\,\mathtt{k}^2\,\mathtt{m}^3)$$
$$\mathtt{h_2(2+k,m)}\big).$$

Finally, define $f(k,m) = \binom{2m+k-2}{2m-1}\, h_2(k,m)$, then Sigma produces the certificate recurrence (127) given by

$$\begin{aligned}
c_0(m) &= -864(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(2+3m)(4+3m)(1+4m)(3+4m)\\
&\quad (1+6m)(5+6m)(7+6m)(11+6m)(389987325+1563210054m+2677225618m^2+2567677266m^3\\
&\quad +1512020037m^4+560774016m^5+128075112m^6+16483104m^7+915728m^8),\\
c_1(m) &= 96(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(1+4m)(3+4m)(7+6m)(11+6m)\\
&\quad (188256814200+3218702456970m+17894686721409m^2+51759068834382m^3+91965480308606m^4\\
&\quad +108563491759548m^5+88575755915749m^6+50780656348884m^7+20427064666140m^8\\
&\quad +5647619363232m^9+1022091737392m^{10}+108997272384m^{11}+5190346304m^{12}),\\
c_2(m) &= -128(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(1+4m)(3+4m)(5+4m)(7+4m)\\
&\quad (83466403020+2609224361478m+16370962889247m^2+49594957358604m^3+90028764304859m^4\\
&\quad +107424612589224m^5+88133765355661m^6+50671172949012m^7+20411335550892m^8\\
&\quad +5646620689248m^9+1022091737392m^{10}+108997272384m^{11}+5190346304m^{12}),\\
c_3(m) &= 512(1+m)^2(2+m)^2(3+m)^2(1+2m)^2(3+2m)^2(5+2m)^2(1+4m)(3+4m)(5+4m)(7+4m)\\
&\quad (9+4m)(11+4m)^2(13+4m)(79380+7185780m+39195762m^2+91462910m^3+116468957m^4\\
&\quad +87187760m^5+38333768m^6+9157280m^7+915728m^8),
\end{aligned}$$

and

$$g(k,m) = \frac{-(k-1)\big[p_0(k,m)h(k,m)+p_1(k,m)h(k+1,m)+p_2(k,m)h(k+2,m)\big]}{(k+1)^2\prod_{i=1}^{5}(2m-k+i)\prod_{i=3}^{6}(2m+k+i)}\binom{2m+k-1}{2m}$$

where the polynomials $p_i(k,m)$ are given in Appendix D. ◊

Hence identity (47) follows together with Section 5. Moreover, identity (11) is immediate with the relation (70) and the fact that (11) holds for $m=1$.

Finally, a proof of identity (16) is given in

Remark 16. Define

$$p(s,r,m) := \frac{(-1)^s}{4^s}\frac{(m)_r(-3m-1)_r}{r!(1/2-2m)_r}.$$

Then observe that

$$h_2(0,m) = \sum_{r=0}^{m}\frac{m-r}{m}p(m,r,m)$$

and

$$h_2(2,m) = \sum_{r=0}^{m-1}\frac{m-r}{m}p(m-1,r,m)+\frac{1}{2}\sum_{r=0}^{m-2}\frac{m-r}{m}p(m-2,r,m)$$

which gives

$$4h_2(0,m)-h_2(2,m) = \frac{2(2m+1)}{(m+1)(2m-1)}p(m,m,m).$$

Therefore with Lemma 2 we get

$$3(6m+1)(6m+5)T_2(m)-4(4m+1)(4m+3)T_2(m+1) = -\frac{12(2m+1)(4m+3)}{m+1}p(m,m,m). \quad (133)$$

Now note that [PS95] or Sigma gives the same recurrence (133) for $h_2(0,m)$. Since $t_1(2m)$ is a solution of the homogeneous version of (133) and (16) holds for $m=1$, identity (16) holds for all $m \geq 1$. ◊

9. Conclusion

There are two essential observations arising from this work. First, as noted in the introduction, the challenge of proving equation (6) has led to significant new discoveries in methods of summation [Sch04a] and [PS04].

Second, and most tantalizing, Okada's theorem [Oka89, Sec. 4] as stated in the introduction was, in fact, originally given for the general q case not just $q=1$ as presented here. So one would very much like to produce the q-analog of (6) which would then complete the proof of all the classical finite plane partition product formulas [Sta86b]. Unfortunately, preliminary study suggests that the q-analog of the matrix $W(n)$ in (6) is substantially more intricate than the already very elaborate $W(n)$ constructed for Theorem 1 in this paper. In addition, it is also reasonable to suppose that the solution of the full Okada conjecture for general q along the lines suggested would again lead to further development and refinement of current summations methods.

Appendix A. The certificates needed in Remark 5

$$g_2(k,m) := \frac{1}{2\,(1+k)\,m\,(-1+2\,m)}\big(-2\,(3\,k^2+k^3+3\,(1-2\,m)\,m+k\,(2+m-2\,m^2))\,h_1(k,m)+(9\,k^2+3\,k^3+2\,(1-2\,m)\,m+k\,(6+4\,m-8\,m^2))\,h_1(k+1,m)-k\,(2+3\,k+k^2+2\,m-4\,m^2)\,h_1(k+2,m)\big)$$

$$\begin{aligned}g_3(k,m) := {}& \big(-2\,(2\,k^5\,m\,(-1+2\,m)+k^4\,(3-8\,m+16\,m^2)+12\,m\,(1-m-4\,m^2+4\,m^3)+\\ & k^2\,(15+16\,m-40\,m^2+32\,m^3-32\,m^4)+k^3\,(12-4\,m+6\,m^2+8\,m^3-8\,m^4)+k\,(6+26\,m-50\,m^2-8\,m^3+8\,m^4))\,h_1(k,m)+\\ & (6\,k^5\,m\,(-1+2\,m)+k^4\,(9-20\,m+40\,m^2)+16\,m\,(1-m-4\,m^2+4\,m^3)+k^2\,(45+68\,m-144\,m^2+32\,m^3-32\,m^4)+\\ & 2\,k^3\,(18+m-6\,m^2+16\,m^3-16\,m^4)+2\,k\,(9+34\,m-58\,m^2-40\,m^3+40\,m^4))\,h_1(k+1,m)+\\ & (2\,k^5\,(1-2\,m)\,m-3\,k^4\,(1-2\,m+4\,m^2)+k^2\,(-15-26\,m+52\,m^2)-\\ & 4\,m\,(1-m-4\,m^2+4\,m^3)-6\,k\,(1+4\,m-6\,m^2-8\,m^3+8\,m^4)+2\,k^3\,(-6-m+4\,m^2-8\,m^3+8\,m^4))\,h_1(k+2,m))/\\ & (4\,(1+k)^2\,(-1+m)\,m\,(-1+2\,m)\,(1+2\,m))\end{aligned}$$

$$\begin{aligned}g_4(k,m) := {}& \big(-2\,(k^6\,m\,(1-m-4\,m^2+4\,m^3)+k^5\,m\,(1+3\,m-20\,m^2+20\,m^3)+4\,m\,(9-10\,m-33\,m^2+38\,m^3-12\,m^4+8\,m^5)+\\ & k^3\,(36-14\,m+11\,m^2+73\,m^3-98\,m^4+60\,m^5-40\,m^6)+k^4\,(9-11\,m+25\,m^2-11\,m^3+6\,m^4+12\,m^5-8\,m^6)+\\ & k^2\,(45+28\,m-92\,m^2+143\,m^3-138\,m^4-12\,m^5+8\,m^6)+k\,(18+67\,m-134\,m^2-13\,m^3+78\,m^4-156\,m^5+104\,m^6))\,h_1(k,m)+\\ & (48\,m\,(1-m-4\,m^2+4\,m^3)+3\,k^6\,m\,(1-m-4\,m^2+4\,m^3)+\\ & k^5\,m\,(-1+13\,m-44\,m^2+44\,m^3)+k^3\,(108-9\,m-41\,m^2+242\,m^3-272\,m^4+72\,m^5-48\,m^6)+\\ & k^4\,(27-38\,m+73\,m^2+16\,m^3-36\,m^4+48\,m^5-32\,m^6)+2\,k\,(27+92\,m-158\,m^2-113\,m^3+158\,m^4-108\,m^5+72\,m^6)+\\ & k^2\,(135+161\,m-378\,m^2+204\,m^3-104\,m^4-240\,m^5+160\,m^6))\,h_1(k+1,m)+\\ & (k^6\,m\,(-1+m+4\,m^2-4\,m^3)+k^5\,(m-5\,m^2+12\,m^3-12\,m^4)-\\ & 6\,k\,(3+11\,m-16\,m^2-24\,m^3+24\,m^4)+k^3\,(-36-m+25\,m^2-92\,m^3+92\,m^4)+4\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)+\\ & k^4\,(-9+12\,m-23\,m^2-6\,m^3+16\,m^4-24\,m^5+16\,m^6)-k^2\,(45+65\,m-142\,m^2+38\,m^3+12\,m^4-120\,m^5+80\,m^6))\,h_1(k+2,m))/\\ & (4\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m))\end{aligned}$$

$$\begin{aligned}g_5(k,m) := {}& \big(-2\,(2\,k^7\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)-144\,m\,(-9+7\,m+42\,m^2-32\,m^3-24\,m^4+16\,m^5)+\\ & k^6\,m\,(-33+19\,m+174\,m^2-104\,m^3-168\,m^4+112\,m^5)+k^5\,m\,(-231+379\,m+310\,m^2-202\,m^3-248\,m^4+128\,m^5+64\,m^6-32\,m^7)+\\ & k^4\,(324-609\,m+1261\,m^2-134\,m^3-70\,m^4+568\,m^5-640\,m^6+448\,m^7-224\,m^8)+\\ & k^3\,(1296+21\,m-47\,m^2+104\,m^3-534\,m^4+1088\,m^5-912\,m^6+320\,m^7-160\,m^8)+\\ & 24\,k\,(27+117\,m-203\,m^2-116\,m^3+80\,m^4+64\,m^5+32\,m^6-128\,m^7+64\,m^8)+\\ & 2\,k^2\,(810+969\,m-2188\,m^2+1006\,m^3-999\,m^4-224\,m^5+840\,m^6-1184\,m^7+592\,m^8))\,h_1(k,m)+\\ & (6\,k^7\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)+3\,k^6\,m\,(-21+11\,m+114\,m^2-64\,m^3-120\,m^4+80\,m^5)+\\ & k^5\,m\,(-585+1071\,m+370\,m^2-248\,m^3-248\,m^4+16\,m^5+256\,m^6-128\,m^7)-\\ & 72\,m\,(-24+21\,m+106\,m^2-95\,m^3-32\,m^4+40\,m^5-32\,m^6+16\,m^7)-\\ & 8\,k\,(-243-954\,m+1491\,m^2+1648\,m^3-1547\,m^4-248\,m^5+184\,m^6-32\,m^7+16\,m^8)-\\ & 3\,k^4\,(-324+549\,m-1091\,m^2-74\,m^3+312\,m^4-616\,m^5+560\,m^6-256\,m^7+128\,m^8)+\\ & 6\,k^2\,(810+1365\,m-2627\,m^2-404\,m^3+390\,m^4-112\,m^5+560\,m^6-832\,m^7+416\,m^8)+\\ & k^3\,(3888+1179\,m-2691\,m^2+1576\,m^3-2744\,m^4+2512\,m^5-704\,m^6-1664\,m^7+832\,m^8))\,h_1(k+1,m)+\\ & (k^6\,m\,(15-7\,m-84\,m^2+44\,m^3+96\,m^4-64\,m^5)+2\,k^7\,m\,(3-2\,m-15\,m^2+10\,m^3+12\,m^4-8\,m^5)+\\ & 24\,m\,(-18+15\,m+82\,m^2-71\,m^3-32\,m^4+40\,m^5-32\,m^6+16\,m^7)+k^5\,m\,(195-355\,m-128\,m^2+72\,m^3+112\,m^4-128\,m^6+64\,m^7)+\\ & 36\,k\,(-18-75\,m+102\,m^2+188\,m^3-167\,m^4-56\,m^5+56\,m^6-32\,m^7+16\,m^8)+\\ & k^4\,(-324+501\,m-997\,m^2-80\,m^3+384\,m^4-752\,m^5+576\,m^6-128\,m^7+64\,m^8)-\\ & 4\,k^2\,(405+777\,m-1421\,m^2-530\,m^3+530\,m^4-56\,m^5+224\,m^6-320\,m^7+160\,m^8)-\\ & k^3\,(1296+525\,m-1295\,m^2+1030\,m^3-1252\,m^4+376\,m^5+272\,m^6-896\,m^7+448\,m^8))\,h_1(k+2,m))/\\ & (24\,(1+k)^2\,(-2+m)\,(-1+m)\,m\,(1+m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m))\end{aligned}$$

$$\begin{aligned}g_6(k,m) := {}& \big(-2\,(-2\,k^5\,(1-2\,m)^2\,m^2\,(192-19\,m+43\,m^2-20\,m^3+20\,m^4)+10\,k^7\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)+\\ & k^8\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)+k^6\,m\,(-192+167\,m+842\,m^2-713\,m^3-304\,m^4+184\,m^5+32\,m^6-16\,m^7)-\\ & 288\,m\,(15-14\,m-62\,m^2+53\,m^3+16\,m^4+8\,m^5-32\,m^6+16\,m^7)+\\ & k^4\,(-1080+1521\,m-2790\,m^2-1003\,m^3+956\,m^4+236\,m^5-568\,m^6+704\,m^7-352\,m^8)-\\ & 6\,k\,(360+1434\,m-2575\,m^2-1282\,m^3+1855\,m^4-1504\,m^5+1432\,m^6-736\,m^7+368\,m^8)+\\ & 2\,k^3\,(-2160+501\,m-103\,m^2-3808\,m^3+4906\,m^4-2848\,m^5+2608\,m^6-1216\,m^7+608\,m^8)+\\ & k^2\,(-5400-4638\,m+12029\,m^2-10774\,m^3+9775\,m^4+1328\,m^5+2680\,m^6-6112\,m^7+3056\,m^8))\,h_1(k,m)+\\ & (22\,k^7\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)+\\ & 3\,k^8\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)-2\,k^6\,m\,(219-203\,m-922\,m^2+834\,m^3+200\,m^4-96\,m^5-64\,m^6+32\,m^7)-\\ & 48\,m\,(120-101\,m-530\,m^2+407\,m^3+256\,m^4-40\,m^5-224\,m^6+112\,m^7)-\\ & 2\,k^5\,m\,(-192+776\,m-1621\,m^2+1897\,m^3-724\,m^4+688\,m^5-352\,m^6+176\,m^7)+\\ & 2\,k^2\,(-8100-11346\,m+23215\,m^2-1664\,m^3-469\,m^4+5080\,m^5-3256\,m^6-224\,m^7+112\,m^8)+\\ & k^4\,(-3240+4563\,m-7674\,m^2-5881\,m^3+6260\,m^4-988\,m^5+920\,m^6-448\,m^7+224\,m^8)+\\ & 2\,k^3\,(-6480-579\,m+3932\,m^2-11298\,m^3+12425\,m^4-3360\,m^5+4424\,m^6-3744\,m^7+1872\,m^8)-\\ & 4\,k\,(1620+5874\,m-9055\,m^2-10570\,m^3+9703\,m^4+1280\,m^5+1816\,m^6-4576\,m^7+2288\,m^8))\,h_1(k+1,m)+\\ & (k^8\,m\,(3-2\,m-15\,m^2+10\,m^3+12\,m^4-8\,m^5)-\\ & 6\,k^7\,m\,(-3+2\,m+15\,m^2-10\,m^3-12\,m^4+8\,m^5)+48\,m\,(30-17\,m-158\,m^2+89\,m^3+160\,m^4-88\,m^5-32\,m^6+16\,m^7)+\\ & 2\,k^6\,m\,(69-64\,m-290\,m^2+261\,m^3+64\,m^4-24\,m^5-32\,m^6+16\,m^7)+\end{aligned}$$

$$
\begin{aligned}
&2\,k^5\,m\,(-96+292\,m-417\,m^2+505\,m^3-228\,m^4+208\,m^5-96\,m^6+48\,m^7)+\\
&k^4\,(1080-1401\,m+2374\,m^2+1807\,m^3-2296\,m^4+1252\,m^5-1096\,m^6+448\,m^7-224\,m^8)+\\
&12\,k^2\,(450+741\,m-1427\,m^2-251\,m^3+413\,m^4-428\,m^5+416\,m^6-224\,m^7+112\,m^8)+\\
&8\,k\,(270+1047\,m-1364\,m^2-2774\,m^3+2075\,m^4+1504\,m^5-424\,m^6-992\,m^7+496\,m^8)-\\
&2\,k^3\,(-2160-435\,m+1958\,m^2-4298\,m^3+4071\,m^4+304\,m^5+600\,m^6-1376\,m^7+688\,m^8))\;h_1(k+2,m))/\\
&(48\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(2+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m))
\end{aligned}
$$

$$
\begin{aligned}
g_7(k,m) := {}&(-2\,(2\,k^9\,m\,(-180+72\,m+1025\,m^2-410\,m^3-1365\,m^4+546\,m^5+600\,m^6-240\,m^7-80\,m^8+32\,m^9)+\\
&k^8\,m\,(-5490+2241\,m+31150\,m^2-12705\,m^3-41020\,m^4+16688\,m^5+17600\,m^6-7120\,m^7-2240\,m^8+896\,m^9)-\\
&2\,k^7\,m\,(22050-10515\,m-121197\,m^2+\\
&\quad 57164\,m^3+144335\,m^4-65525\,m^5-50836\,m^6+20972\,m^7+5840\,m^8-2160\,m^9-192\,m^{10}+64\,m^{11})-\\
&960\,m\,(-4050+2610\,m+20505\,m^2-13223\,m^3-17493\,m^4+11579\,m^5+684\,m^6-1668\,m^7+1680\,m^8-496\,m^9-192\,m^{10}+64\,m^{11})-\\
&4\,k^6\,m\,(28980-15477\,m-155139\,m^2+\\
&\quad 82558\,m^3+166545\,m^4-86131\,m^5-40332\,m^6+14484\,m^7+6960\,m^8-1552\,m^9-1344\,m^{10}+448\,m^{11})-\\
&2\,k^5\,m\,(288270-501843\,m-298671\,m^2+301047\,m^3-\\
&\quad 21275\,m^4+60096\,m^5-55088\,m^6-29864\,m^7+71440\,m^8-18368\,m^9-11136\,m^{10}+3712\,m^{11})-\\
&4\,k^2\,(-1215000-1571130\,m+3426987\,m^2-1003736\,m^3+295207\,m^4+\\
&\quad 1880310\,m^5-1834214\,m^6+919272\,m^7-264744\,m^8-273760\,m^9+137312\,m^{10}-30336\,m^{11}+10112\,m^{12})+\\
&k^4\,(972000-1497510\,m+3038199\,m^2-6810\,m^3-843115\,m^4+\\
&\quad 2152120\,m^5-1797428\,m^6+574000\,m^7-162560\,m^8-181760\,m^9+104384\,m^{10}-34560\,m^{11}+11520\,m^{12})-\\
&8\,k\,(-243000-1074600\,m+1740870\,m^2+1571949\,m^3-1269413\,m^4-\\
&\quad 697425\,m^5+342995\,m^6+312012\,m^7-401124\,m^8+288720\,m^9-39280\,m^{10}-83136\,m^{11}+27712\,m^{12})+\\
&2\,k^3\,(1944000+288900\,m-752550\,m^2+708823\,m^3-739111\,m^4-\\
&\quad 24555\,m^5+292005\,m^6-282156\,m^7-318108\,m^8+560240\,m^9-120080\,m^{10}-113472\,m^{11}+37824\,m^{12}))\\
&h_1(k,m)+\\
&(6\,k^9\,m\,(-180+72\,m+1025\,m^2-410\,m^3-1365\,m^4+546\,m^5+600\,m^6-240\,m^7-80\,m^8+32\,m^9)+k^8\,m\,(-12870+5283\,m+72950\,m^2-\\
&\quad 29915\,m^3-95760\,m^4+39144\,m^5+40800\,m^6-16560\,m^7-5120\,m^8+2048\,m^9)+2\,k^7\,m\,(-49050+24345\,m+\\
&\quad 267098\,m^2-130861\,m^3-308240\,m^4+144020\,m^5+101584\,m^6-41408\,m^7-12160\,m^8+4160\,m^9+768\,m^{10}-256\,m^{11})+480\,m\\
&\quad (10800-7740\,m-52599\,m^2+38137\,m^3+36111\,m^4-28525\,m^5+6732\,m^6-1092\,m^7-3120\,m^8+1424\,m^9-192\,m^{10}+64\,m^{11})-\\
&4\,k^6\,m\,(50040-26631\,m-268276\,m^2+\\
&\quad 143007\,m^3+287830\,m^4-150578\,m^5-65288\,m^6+19336\,m^7+16160\,m^8-3296\,m^9-3456\,m^{10}+1152\,m^{11})-\\
&2\,k^5\,m\,(774810-1452609\,m-428789\,m^2+635873\,m^3-\\
&\quad 514425\,m^4+392748\,m^5-52512\,m^6-54696\,m^7+104240\,m^8-34304\,m^9-8064\,m^{10}+2688\,m^{11})-\\
&48\,k\,(-121500-490050\,m+707805\,m^2+1058171\,m^3-896532\,m^4-\\
&\quad 423225\,m^5+390055\,m^6-146532\,m^7-20676\,m^8+118480\,m^9-33840\,m^{10}-14784\,m^{11}+4928\,m^{12})+\\
&4\,k^3\,(2916000+1270800\,m-2899125\,m^2+1372531\,m^3-1047282\,m^4-\\
&\quad 983105\,m^5+1288925\,m^6-921452\,m^7+55684\,m^8+520080\,m^9-168080\,m^{10}-43584\,m^{11}+14528\,m^{12})+\\
&k^4\,(2916000-4226130\,m+7816437\,m^2+2925370\,m^3-4814165\,m^4+\\
&\quad 4420240\,m^5-2618024\,m^6-380000\,m^7-242000\,m^8+491520\,m^9-27648\,m^{10}-184320\,m^{11}+61440\,m^{12})-\\
&4\,k^2\,(-3645000-6552990\,m+12031461\,m^2+4504816\,m^3-5512867\,m^4+\\
&\quad 2288850\,m^5-1168662\,m^6-309912\,m^7-564456\,m^8+844640\,m^9-108704\,m^{10}-249984\,m^{11}+83328\,m^{12}))\\
&h_1(k+1,m)+\\
&(k^8\,m\,(3690-1521\,m-20900\,m^2+8605\,m^3+27370\,m^4-11228\,m^5-11600\,m^6+4720\,m^7+1440\,m^8-576\,m^9)-\\
&\quad 2\,k^9\,m\,(-180+72\,m+1025\,m^2-410\,m^3-1365\,m^4+546\,m^5+600\,m^6-240\,m^7-80\,m^8+32\,m^9)-2\,k^5\,m\,(-264870+482583\,m+\\
&\quad 191515\,m^2-209565\,m^3+63095\,m^4-105166\,m^5+96920\,m^6-20320\,m^7-37520\,m^8+15008\,m^9)-960\,m\,(1350-855\,m-\\
&\quad 6849\,m^2+4240\,m^3+6015\,m^4-3253\,m^5-1044\,m^6-84\,m^7+720\,m^8-112\,m^9-192\,m^{10}+64\,m^{11})+2\,k^7\,m\,(14850-7365\,m-\\
&\quad 80854\,m^2+39483\,m^3+93420\,m^4-43070\,m^5-31512\,m^6+12264\,m^7+4480\,m^8-1440\,m^9-384\,m^{10}+128\,m^{11})+\\
&\quad 4\,k^6\,m\,(11880-6027\,m-64472\,m^2+\\
&\quad\quad 32859\,m^3+72110\,m^4-36046\,m^5-18616\,m^6+4472\,m^7+5920\,m^8-1312\,m^9-1152\,m^{10}+384\,m^{11})+\\
&\quad k^4\,(-972000+1283310\,m-2362119\,m^2-991880\,m^3+1827455\,m^4-\\
&\quad\quad 1822110\,m^5+639648\,m^6+845280\,m^7-83040\,m^8-415840\,m^9+92416\,m^{10}+80640\,m^{11}-26880\,m^{12})-\\
&\quad 40\,k\,(48600+207900\,m-257706\,m^2-602583\,m^3+\\
&\quad\quad 452603\,m^4+365379\,m^5-260129\,m^6+22140\,m^7+4620\,m^8-21360\,m^9+9424\,m^{10}-960\,m^{11}+320\,m^{12})+\\
&\quad 4\,k^3\,(-972000-523350\,m+1248150\,m^2-742213\,m^3+416581\,m^4+\\
&\quad\quad 823705\,m^5-684225\,m^6+209116\,m^7-47012\,m^8-81040\,m^9+41040\,m^{10}-9408\,m^{11}+3136\,m^{12})+\\
&\quad 12\,k^2\,(-405000-824010\,m+1431979\,m^2+881209\,m^3-955718\,m^4+\\
&\quad\quad 121605\,m^5+93087\,m^6-233508\,m^7-40404\,m^8+192560\,m^9-43056\,m^{10}-37056\,m^{11}+12352\,m^{12}))\\
&\quad h_1(k+2,m))/\\
&(480\,(1+k)^2\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m))
\end{aligned}
$$

$$
\begin{aligned}
g_8(k,m) := (&-2\,(k^{10}\,m\,(2700-900\,m-15807\,m^2+5269\,m^3+22935\,m^4-7645\,m^5-12276\,m^6+4092\,m^7+2640\,m^8-880\,m^9-192\,m^{10}+64\,m^{11})+\\
&k^9\,m\,(57780-19692\,m-337233\,m^2+114871\,m^3+\\
&\quad 484905\,m^4-164911\,m^5-254844\,m^6+86388\,m^7+53040\,m^8-17872\,m^9-3648\,m^{10}+1216\,m^{11})+\\
&k^6\,m\,(10534860-7160184\,m-51898347\,m^2+33606621\,m^3+41665687\,m^4-20601929\,m^5-\\
&\quad 11234536\,m^6+3069668\,m^7+3037056\,m^8-509520\,m^9-733312\,m^{10}+184256\,m^{11}+55552\,m^{12}-15872\,m^{13})+\\
&k^5\,m\,(-6032880+23045202\,m-45259965\,m^2+51887997\,m^3-15305633\,m^4+7385209\,m^5+8023910\,m^6-\\
&\quad 11721784\,m^7+9439416\,m^8-2098944\,m^9-1819360\,m^{10}+589952\,m^{11}+15232\,m^{12}-4352\,m^{13})+\\
&k^7\,m\,(2835000-1211490\,m-15935949\,m^2+6738150\,m^3+20454622\,m^4-8348042\,m^5-\\
&\quad 8622517\,m^6+3223310\,m^7+1388316\,m^8-432936\,m^9-127984\,m^{10}+33440\,m^{11}+8512\,m^{12}-2432\,m^{13})+\\
&k^8\,m\,(605070-225423\,m-3484494\,m^2+1292931\,m^3+4815763\,m^4-1764782\,m^5-\\
&\quad 2342407\,m^6+835538\,m^7+434244\,m^8-147192\,m^9-28624\,m^{10}+9056\,m^{11}+448\,m^{12}-128\,m^{13})+\\
&2880\,m\,(85050-54000\,m-422775\,m^2+226854\,m^3+403550\,m^4-\\
&\quad 56872\,m^5-323873\,m^6+63838\,m^7+126156\,m^8-41064\,m^9-10544\,m^{10}+4000\,m^{11}-448\,m^{12}+128\,m^{13})-\\
&432\,k\,(-283500-1221750\,m+1862535\,m^2+2281381\,m^3-2101781\,m^4-259369\,m^5-328406\,m^6+\\
&\quad 581798\,m^7+375932\,m^8-805112\,m^9+156112\,m^{10}+175776\,m^{11}-49856\,m^{12}-8064\,m^{13}+2304\,m^{14})+\\
&k^3\,(244944000+17720100\,m-130295700\,m^2+386039019\,m^3-418990570\,m^4+77121898\,m^5-37682848\,m^6-51734773\,m^7+\\
&\quad 93771830\,m^8-82525956\,m^9+17055736\,m^{10}+17221904\,m^{11}-5435360\,m^{12}-281792\,m^{13}+80512\,m^{14})+\\
&k^4\,(61236000-75573270\,m+107776503\,m^2+168572874\,m^3-142484821\,m^4-71219273\,m^5+75657692\,m^6-46082863\,m^7+\\
&\quad 16761962\,m^8+10577796\,m^9-8482328\,m^{10}+4232624\,m^{11}-735776\,m^{12}-623168\,m^{13}+178048\,m^{14})-\\
&6\,k^2\,(-51030000-60310440\,m+129386466\,m^2-28728909\,m^3-901280\,m^4+59825492\,m^5-3004244\,m^6-60104627\,m^7+\\
&\quad 21629050\,m^8+13146756\,m^9-8928760\,m^{10}+3592816\,m^{11}-512800\,m^{12}-632128\,m^{13}+180608\,m^{14}))\\
&h_1(k,m)+\\
&(3\,k^{10}\,m\,(2700-900\,m-15807\,m^2+5269\,m^3+22935\,m^4-7645\,m^5-12276\,m^6+4092\,m^7+2640\,m^8-880\,m^9-192\,m^{10}+64\,m^{11})+\\
&9\,k^9\,m\,(15660-5364\,m-91335\,m^2+\\
&\quad 31265\,m^3+131055\,m^4-44777\,m^5-68580\,m^6+23340\,m^7+14160\,m^8-4784\,m^9-960\,m^{10}+320\,m^{11})+\\
&k^8\,m\,(1353510-515349\,m-7767153\,m^2+2942643\,m^3+10621327\,m^4-3961574\,m^5-\\
&\quad 5060164\,m^6+1821704\,m^7+914496\,m^8-306240\,m^9-63808\,m^{10}+19328\,m^{11}+1792\,m^{12}-512\,m^{13})+\\
&1440\,m\,(226800-127980\,m-1178055\,m^2+583878\,m^3+1271810\,m^4-\\
&\quad 347644\,m^5-801911\,m^6+260386\,m^7+191892\,m^8-84888\,m^9+6832\,m^{10}+1120\,m^{11}-3136\,m^{12}+896\,m^{13})-\\
&6\,k^7\,m\,(-930150+410265\,m+5195127\,m^2-2261046\,m^3-6538432\,m^4+2717667\,m^5+\\
&\quad 2669171\,m^6-981838\,m^7-444436\,m^8+127016\,m^9+52752\,m^{10}-13216\,m^{11}-4032\,m^{12}+1152\,m^{13})-\\
&3\,k^6\,m\,(-7964460+5913864\,m+37832307\,m^2-26897589\,m^3-25126175\,m^4+13204977\,m^5+\\
&\quad 5382856\,m^6-1025732\,m^7-1994240\,m^8+351184\,m^9+469632\,m^{10}-127424\,m^{11}-26880\,m^{12}+7680\,m^{13})+\\
&9\,k^5\,m\,(-3585960+9305334\,m-8950481\,m^2+10956671\,m^3-4368411\,m^4+1363743\,m^5+\\
&\quad 2898132\,m^6-2066292\,m^7+846992\,m^8-378608\,m^9+22464\,m^{10}+27456\,m^{11}-32256\,m^{12}+9216\,m^{13})+\\
&48\,k\,(7654500+30176550\,m-39256245\,m^2-79013457\,m^3+53454327\,m^4+54189803\,m^5-13406458\,m^6-34505876\,m^7+\\
&\quad 6319096\,m^8+14035584\,m^9-4514304\,m^{10}-1236032\,m^{11}+472192\,m^{12}-55552\,m^{13}+15872\,m^{14})+\\
&3\,k^4\,(61236000-72530370\,m+92772063\,m^2+203199981\,m^3-173332591\,m^4-72403839\,m^5+51830688\,m^6-10547932\,m^7+\\
&\quad 15596672\,m^8-11349152\,m^9-203680\,m^{10}+4956736\,m^{11}-1289216\,m^{12}-335104\,m^{13}+95744\,m^{14})-6\,k^3\\
&\quad(-122472000-47808900\,m+134524530\,m^2-151585827\,m^3+132081006\,m^4+43098422\,m^5-17623682\,m^6-13933611\,m^7-\\
&\quad 5959902\,m^8+17675236\,m^9-7824536\,m^{10}+505328\,m^{11}+361056\,m^{12}-488768\,m^{13}+139648\,m^{14})-4\,k^2\\
&\quad(-229635000-393630030\,m+691905537\,m^2+379143432\,m^3-369473559\,m^4-14899475\,m^5-10513724\,m^6+6928661\,m^7+\\
&\quad 62967458\,m^8-76638540\,m^9+7210632\,m^{10}+24482672\,m^{11}-6338464\,m^{12}-1682240\,m^{13}+480640\,m^{14}))\\
&h_1(k+1,m)+\\
&(k^9\,m\,(-41580+14292\,m+242391\,m^2-\\
&\quad 83257\,m^3-347295\,m^4+119041\,m^5+181188\,m^6-61836\,m^7-37200\,m^8+12592\,m^9+2496\,m^{10}-832\,m^{11})+\\
&k^{10}\,m\,(-2700+900\,m+15807\,m^2-5269\,m^3-22935\,m^4+7645\,m^5+12276\,m^6-4092\,m^7-2640\,m^8+880\,m^9+192\,m^{10}-64\,m^{11})+\\
&k^5\,m\,(13744080-30529602\,m+13926465\,m^2-22297959\,m^3+17708731\,m^4-4073867\,m^5-12454060\,m^6+\\
&\quad 4415828\,m^7+1833168\,m^8+390000\,m^9-1122880\,m^{10}+203456\,m^{11}+157696\,m^{12}-45056\,m^{13})-\\
&2880\,m\,(28350-9045\,m-167688\,m^2+56703\,m^3+245807\,m^4-\\
&\quad 95512\,m^5-116369\,m^6+55414\,m^7+5916\,m^8-6696\,m^9+4432\,m^{10}-992\,m^{11}-448\,m^{12}+128\,m^{13})+\\
&k^8\,m\,(-402570+153063\,m+2310453\,m^2-873177\,m^3-3162041\,m^4+\\
&\quad 1172470\,m^5+1514534\,m^6-537556\,m^7-282168\,m^8+91536\,m^9+22688\,m^{10}-6592\,m^{11}-896\,m^{12}+256\,m^{13})+\\
&2\,k^7\,m\,(-757350+332775\,m+4231512\,m^2-1827834\,m^3-5342140\,m^4+2174675\,m^5+\\
&\quad 2237266\,m^6-778112\,m^7-424680\,m^8+112320\,m^9+59872\,m^{10}-15104\,m^{11}-4480\,m^{12}+1280\,m^{13})+\\
&k^6\,m\,(-7586460+5549904\,m+36211239\,m^2-25089345\,m^3-25152707\,m^4+11876869\,m^5+\\
&\quad 7437992\,m^6-1426900\,m^7-2818176\,m^8+656400\,m^9+500864\,m^{10}-145600\,m^{11}-19712\,m^{12}+5632\,m^{13})+\\
&k^4\,(-61236000+65083770\,m-83576943\,m^2-185166723\,m^3+178657387\,m^4+18810991\,m^5-24435720\,m^6+25312466\,m^7-\\
&\quad 24308884\,m^8+13525848\,m^9-1425136\,m^{10}-4253728\,m^{11}+1261632\,m^{12}+144256\,m^{13}-41216\,m^{14})-\\
&24\,k\,(5103000+21392100\,m-23125230\,m^2-71672661\,m^3+38307336\,m^4+72077684\,m^5-22139884\,m^6-41410823\,m^7+\\
&\quad 13171378\,m^8+10068852\,m^9-4046232\,m^{10}-51536\,m^{11}+137056\,m^{12}-110656\,m^{13}+31616\,m^{14})+\\
&12\,k^2\,(-25515000-49955670\,m+82807623\,m^2+66962142\,m^3-60379133\,m^4-12211229\,m^5-2495052\,m^6+13041881\,m^7+\\
&\quad 5984186\,m^8-15211932\,m^9+3191720\,m^{10}+3079472\,m^{11}-898848\,m^{12}-117824\,m^{13}+33664\,m^{14})+\\
&4\,k^3\,(-61236000-30700350\,m+83766060\,m^2-82981056\,m^3+51700237\,m^4+68862175\,m^5-25613295\,m^6-33066313\,m^7+\\
&\quad 11865626\,m^8+7302540\,m^9-4982680\,m^{10}+2003024\,m^{11}-257568\,m^{12}-378560\,m^{13}+108160\,m^{14}))\\
&h_1(k+2,m))/\\
&(1440\,(1+k)^2\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\\
&\quad(3+2\,m)\,(5+2\,m))
\end{aligned}
$$

$$
\begin{aligned}
g_9(k,m) :=\ & \big(-2\,\big(2\,k^{11}\,m\,\big(18900-11700\,m-96249\,m^2+\\
&\quad 56497\,m^3+89841\,m^4-41665\,m^5-29988\,m^6+11868\,m^7+4080\,m^8-1456\,m^9-192\,m^{10}+64\,m^{11}\big)+\\
&\quad k^{10}\,m\,\big(1077300-673200\,m-5469393\,m^2+3243212\,m^3+\\
&\quad\quad 5038449\,m^4-2361080\,m^5-1644048\,m^6+656904\,m^7+215760\,m^8-77504\,m^9-9600\,m^{10}+3200\,m^{11}\big)+\\
&\quad k^9\,m\,\big(14556780-9351792\,m-73215663\,m^2+44715468\,m^3+64753085\,m^4-31196092\,m^5-\\
&\quad\quad 19786718\,m^6+8064412\,m^7+2390328\,m^8-859824\,m^9-104480\,m^{10}+33856\,m^{11}+896\,m^{12}-256\,m^{13}\big)+\\
&\quad 181440\,m\,\big(-176400+159600\,m+756490\,m^2-674843\,m^3-196933\,m^4+\\
&\quad\quad 137133\,m^5-20045\,m^6+33826\,m^7-27844\,m^8+3464\,m^9+8080\,m^{10}-2208\,m^{11}-448\,m^{12}+128\,m^{13}\big)-\\
&\quad 2\,k^8\,m\,\big(-49809060+33611724\,m+246104421\,m^2-158300597\,m^3-200718040\,m^4+100919767\,m^5+\\
&\quad\quad 54485683\,m^6-22542926\,m^7-6103380\,m^8+2117864\,m^9+345616\,m^{10}-103072\,m^{11}-11200\,m^{12}+3200\,m^{13}\big)-\\
&\quad 7\,k^6\,m\,\big(-119338380+110040912\,m+503865255\,m^2-452143120\,m^3-115091327\,m^4+48386372\,m^5+41482376\,m^6-\\
&\quad\quad 3389272\,m^7-21882768\,m^8+6264256\,m^9+2670848\,m^{10}-816512\,m^{11}-68096\,m^{12}+19456\,m^{13}\big)-\\
&\quad 2\,k^7\,m\,\big(-226497600+173548620\,m+1060263528\,m^2-777501095\,m^3-651751710\,m^4+346636967\,m^5+143787042\,m^6-\\
&\quad\quad 55163856\,m^7-21182856\,m^8+6253184\,m^9+2353632\,m^{10}-670976\,m^{11}-104832\,m^{12}+29952\,m^{13}\big)+\\
&\quad k^5\,m\,\big(4125790620-9084595248\,m+3226307565\,m^2-2640652672\,m^3-1719540003\,m^4+2164447336\,m^5-1631269962\,m^6+\\
&\quad\quad 530028948\,m^7+382377480\,m^8-155585360\,m^9+1589664\,m^{10}+1610432\,m^{11}-1975680\,m^{12}+564480\,m^{13}\big)+\\
&\quad 864\,k\,\big(-18522000-76734000\,m+141953070\,m^2+43835777\,m^3-31475444\,m^4-36458663\,m^5+46031644\,m^6-33570880\,m^7+\\
&\quad\quad 7638200\,m^8+12078928\,m^9-4582144\,m^{10}-309376\,m^{11}+165248\,m^{12}-57344\,m^{13}+16384\,m^{14}\big)-96\,k^2\\
&\qquad \big(416745000+423084060\,m-1057960674\,m^2+854405667\,m^3-891391718\,m^4+92082328\,m^5-65241249\,m^6-21685090\,m^7+\\
&\quad\quad 79288328\,m^8-84204984\,m^9+19703264\,m^{10}+14909984\,m^{11}-4404096\,m^{12}-522368\,m^{13}+149248\,m^{14}\big)+\\
&\quad 2\,k^4\,\big(-4000752000+6885186840\,m-15915276036\,m^2+8782225686\,m^3-\\
&\quad\quad 9771692147\,m^4+2234585635\,m^5-975802383\,m^6-1064879977\,m^7+977030714\,m^8-546038580\,m^9+\\
&\quad\quad 125618024\,m^{10}+99449936\,m^{11}-30177312\,m^{12}-2744000\,m^{13}+784000\,m^{14}\big)-4\,k^3\,\big(8001504000-\\
&\quad\quad 947286900\,m+1171099800\,m^2+2814339039\,m^3-2311045009\,m^4-1790099602\,m^5+3081940119\,m^6-3028083485\,m^7+\\
&\quad\quad 986138242\,m^8+717030348\,m^9-311584280\,m^{10}+23089168\,m^{11}-1138080\,m^{12}-6053824\,m^{13}+1729664\,m^{14}\big)\big)\\
&\ h_1(k,m)+\\
&\big(6\,k^{11}\,m\,\big(18900-11700\,m-96249\,m^2+\\
&\quad 56497\,m^3+89841\,m^4-41665\,m^5-29988\,m^6+11868\,m^7+4080\,m^8-1456\,m^9-192\,m^{10}+64\,m^{11}\big)+\\
&\quad k^{10}\,m\,\big(2702700-1692000\,m-13713207\,m^2+8147720\,m^3+\\
&\quad\quad 12599799\,m^4-5916620\,m^5-4092480\,m^6+1638408\,m^7+533040\,m^8-191744\,m^9-23424\,m^{10}+7808\,m^{11}\big)+\\
&\quad k^9\,m\,\big(33086340-21389976\,m-166048149\,m^2+102087790\,m^3+145436141\,m^4-70468082\,m^5-\\
&\quad\quad 43792784\,m^6+17888704\,m^7+5231472\,m^8-1869280\,m^9-241280\,m^{10}+76544\,m^{11}+3584\,m^{12}-1024\,m^{13}\big)+\\
&\quad 201600\,m\,\big(-211680+203238\,m+870321\,m^2-822115\,m^3-101938\,m^4+\\
&\quad\quad 27837\,m^5+46243\,m^6+31450\,m^7-67284\,m^8+16072\,m^9+11536\,m^{10}-3360\,m^{11}-448\,m^{12}+128\,m^{13}\big)+\\
&\quad 7\,k^6\,m\,\big(205076340-197359056\,m-841426953\,m^2+795151448\,m^3+97718365\,m^4-23329084\,m^5-61131748\,m^6+\\
&\quad\quad 4564592\,m^7+33475008\,m^8-10995488\,m^9-2632000\,m^{10}+910336\,m^{11}-30464\,m^{12}+8704\,m^{13}\big)-\\
&\quad 2\,k^8\,m\,\big(-102592980+70001712\,m+504744525\,m^2-328345206\,m^3-403571965\,m^4+204087494\,m^5+107413852\,m^6-\\
&\quad\quad 43972952\,m^7-12565824\,m^8+4209408\,m^9+869056\,m^{10}-252800\,m^{11}-34048\,m^{12}+9728\,m^{13}\big)-\\
&\quad 2\,k^7\,m\,\big(-478396800+378286020\,m+2205519672\,m^2-1671960829\,m^3-1230795030\,m^4+658415605\,m^5+263263266\,m^6-\\
&\quad\quad 95810712\,m^7-45296040\,m^8+13239520\,m^9+5191392\,m^{10}-1512064\,m^{11}-201600\,m^{12}+57600\,m^{13}\big)+k^5\,m\\
&\qquad \big(11651309460-26455840344\,m+12268772895\,m^2-10491426358\,m^3-4585047555\,m^4+4129502362\,m^5-1868188644\,m^6+\\
&\quad\quad 992327304\,m^7-32902752\,m^8-74908736\,m^9+92480448\,m^{10}-24833920\,m^{11}-5531904\,m^{12}+1580544\,m^{13}\big)-\\
&\quad 4\,k^4\,\big(6001128000-9946359360\,m+21247603884\,m^2-\\
&\quad\quad 5491676520\,m^3+5890208793\,m^4-1159111915\,m^5+1392385163\,m^6-840592991\,m^7-126776954\,m^8+\\
&\quad\quad 707344212\,m^9-238330344\,m^{10}-49765328\,m^{11}+18426400\,m^{12}-1696576\,m^{13}+484736\,m^{14}\big)+\\
&\quad 8\,k^2\,\big(-15002820000-22896223560\,m+47280581244\,m^2-5670984414\,m^3+\\
&\quad\quad 4498759153\,m^4+1529148847\,m^5+3053404701\,m^6-6157266913\,m^7+1045919066\,m^8+2619504108\,m^9-\\
&\quad\quad 865737880\,m^{10}-199888432\,m^{11}+68751840\,m^{12}-1959104\,m^{13}+559744\,m^{14}\big)+48\,k\,\big(-1000188000-\\
&\quad\quad 3760495200\,m+6213977280\,m^2+5201196198\,m^3-5048261729\,m^4-473279927\,m^5+680014935\,m^6-664395319\,m^7+\\
&\quad\quad 422399174\,m^8-93562380\,m^9-15077608\,m^{10}+54959024\,m^{11}-14458848\,m^{12}-3563840\,m^{13}+1018240\,m^{14}\big)-\\
&\quad 4\,k^3\,\big(24004512000+4138287300\,m-12749524380\,m^2+\\
&\quad\quad 17770245393\,m^3-17086592950\,m^4-2079619261\,m^5+2930960370\,m^6-2963823800\,m^7+2223702448\,m^8-\\
&\quad\quad 840053520\,m^9+42349216\,m^{10}+309350656\,m^{11}-84751872\,m^{12}-16952320\,m^{13}+4843520\,m^{14}\big)\big)\\
&\ h_1(k+1,m)+\big(-2\,k^{11}\,m\,\big(18900-\\
&\quad 11700\,m-96249\,m^2+56497\,m^3+89841\,m^4-41665\,m^5-29988\,m^6+11868\,m^7+4080\,m^8-1456\,m^9-192\,m^{10}+64\,m^{11}\big)-\\
&\quad 3\,k^{10}\,m\,\big(270900-169800\,m-1373969\,m^2+817418\,m^3+\\
&\quad\quad 1260225\,m^4-592590\,m^5-408072\,m^6+163584\,m^7+52880\,m^8-19040\,m^9-2304\,m^{10}+768\,m^{11}\big)-\\
&\quad 20160\,m\,\big(-529200+452340\,m+2319636\,m^2-1828315\,m^3-1005763\,m^4+\\
&\quad\quad 116973\,m^5+851923\,m^6-187790\,m^7-304164\,m^8+94984\,m^9+28816\,m^{10}-9120\,m^{11}-448\,m^{12}+128\,m^{13}\big)+\\
&\quad k^9\,m\,\big(-9793980+6327792\,m+49160715\,m^2-30195228\,m^3-43106071\,m^4+20825768\,m^5+\\
&\quad\quad 13064044\,m^6-5297792\,m^7-1606656\,m^8+565152\,m^9+83392\,m^{10}-25856\,m^{11}-1792\,m^{12}+512\,m^{13}\big)+\\
&\quad 4\,k^8\,m\,\big(-14386680+9792072\,m+70835088\,m^2-45906553\,m^3-56933703\,m^4+28468921\,m^5+\\
&\quad\quad 15619317\,m^6-6225210\,m^7-2034300\,m^8+658840\,m^9+164592\,m^{10}-47584\,m^{11}-6720\,m^{12}+1920\,m^{13}\big)+\\
&\quad 14\,k^7\,m\,\big(-20412000+16035300\,m+94350168\,m^2-70788573\,m^3-53998148\,m^4+27708865\,m^5+\\
&\quad\quad 13384352\,m^6-4666300\,m^7-2573040\,m^8+775248\,m^9+271232\,m^{10}-80704\,m^{11}-8960\,m^{12}+2560\,m^{13}\big)-\\
&\quad 7\,k^6\,m\,\big(48160980-46746072\,m-196225569\,m^2+186594834\,m^3+18958949\,m^4+630914\,m^5-20434796\,m^6+4444120\,m^7+\\
&\quad\quad 7656960\,m^8-2969856\,m^9-130496\,m^{10}+92032\,m^{11}-44800\,m^{12}+12800\,m^{13}\big)-k^5\,m\,\big(4055407020-\\
&\quad\quad 8856333648\,m+3003539649\,m^2-3090872792\,m^3+96038503\,m^4+324583044\,m^5-705198424\,m^6+444411416\,m^7-\\
&\quad\quad 104735664\,m^8+6923840\,m^9+37154816\,m^{10}-10727040\,m^{11}-1530368\,m^{12}+437248\,m^{13}\big)-336\,k\,\big(-47628000-\\
&\quad\quad 190776600\,m+273493980\,m^2+423168306\,m^3-379514261\,m^4-95734799\,m^5+36215019\,m^6+37464113\,m^7+5172902\,m^8-\\
&\quad\quad 29536812\,m^9+7192472\,m^{10}+4913456\,m^{11}-1422816\,m^{12}-198464\,m^{13}+56704\,m^{14}\big)+12\,k^3\,\big(2667168000+
\end{aligned}
$$

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\begin{aligned}
&736199100\,m - 2255122680\,m^2 + 3070362415\,m^3 - 2780249927\,m^4 - 630189578\,m^5 + 225283925\,m^6 + 234052763\,m^7 + \\
&125082218\,m^8 - 298684660\,m^9 + 71708168\,m^{10} + 50866064\,m^{11} - 14885408\,m^{12} - 1910720\,m^{13} + 545920\,m^{14}) + \\
&2\,k^4\,(4000752000 - 6138882540\,m + 13077117216\,m^2 - 2992877121\,m^3 + \\
&2044864416\,m^4 + 1864333301\,m^5 + 84135488\,m^6 - 2104600064\,m^7 + 627641320\,m^8 + 562629840\,m^9 - \\
&217183680\,m^{10} - 10377344\,m^{11} + 6448768\,m^{12} - 2759680\,m^{13} + 788480\,m^{14}) - 8\,k^2\,(-5000940000 - \\
&8823612420\,m + 17242362828\,m^2 + 1844965095\,m^3 - 2853747344\,m^4 + 1766306629\,m^5 + 941916744\,m^6 - 3381158284\,m^7 + \\
&1061356976\,m^8 + 834760800\,m^9 - 324468256\,m^{10} - 12920512\,m^{11} + 8684544\,m^{12} - 4040960\,m^{13} + 1154560\,m^{14}))\\
&h_1(k+2,m))/\\
&(20160\,(1+k)^2\,(-4+m)\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(2+m)\,(3+m)\,(-7+2\,m)\,(-5+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\\
&(5+2\,m)\,(7+2\,m))
\end{aligned}
$$

$$
\begin{aligned}
g_{10}(k,m) := (&-2\,(32\,k^{11}\,m\,(1814400 - 461700\,m - 10953504\,m^2 + 2786697\,m^3 + 17365712\,m^4 - 4415502\,m^5 - 11147136\,m^6 + 2831257\,m^7 + \\
&3402256\,m^8 - 862576\,m^9 - 518336\,m^{10} + 131040\,m^{11} + 37632\,m^{12} - 9472\,m^{13} - 1024\,m^{14} + 256\,m^{15}) + \\
&k^{12}\,m\,(1587600 - 396900\,m - 9600516\,m^2 + 2400129\,m^3 + 15291640\,m^4 - 3822910\,m^5 - 9901892\,m^6 + 2475473\,m^7 + \\
&3065920\,m^8 - 766480\,m^9 - 477568\,m^{10} + 119392\,m^{11} + 35840\,m^{12} - 8960\,m^{13} - 1024\,m^{14} + 256\,m^{15}) + \\
&k^8\,m\,(54285986160 - 17849254284\,m - 318185975088\,m^2 + 103772190339\,m^3 + 464461703229\,m^4 - 147820162076\,m^5 - \\
&256570600762\,m^6 + 77522360799\,m^7 + 63601516717\,m^8 - 17462805946\,m^9 - 8232326352\,m^{10} + 1975344320\,m^{11} + \\
&675853024\,m^{12} - 145416384\,m^{13} - 37090048\,m^{14} + 7950592\,m^{15} + 933120\,m^{16} - 207360\,m^{17}) + \\
&k^{10}\,m\,(992703600 - 263349900\,m - 5968170576\,m^2 + 1581703059\,m^3 + 9354610805\,m^4 - \\
&2472344232\,m^5 - 5879073326\,m^6 + 1545570835\,m^7 + 1735311369\,m^8 - 452125938\,m^9 - 252301520\,m^{10} + \\
&64801088\,m^{11} + 17394272\,m^{12} - 4370368\,m^{13} - 476928\,m^{14} + 115968\,m^{15} + 2304\,m^{16} - 512\,m^{17}) + \\
&241920\,m\,(57153600 - 32281200\,m - 294688800\,m^2 + 137297502\,m^3 + \\
&324649721\,m^4 - 46340124\,m^5 - 264094168\,m^6 + 63312892\,m^7 + 91480559\,m^8 - 35679470\,m^9 - \\
&2455184\,m^{10} + 3179680\,m^{11} - 2026976\,m^{12} + 362432\,m^{13} + 170752\,m^{14} - 39424\,m^{15} - 2304\,m^{16} + 512\,m^{17}) - \\
&2\,k^9\,m\,(-4584399120 + 1306148868\,m + 27351546732\,m^2 - 7770795873\,m^3 - 41971575298\,m^4 + 11828793524\,m^5 + \\
&25369616622\,m^6 - 7037035319\,m^7 - 7061880488\,m^8 + 1905164480\,m^9 + 959258080\,m^{10} - 247464928\,m^{11} - \\
&64740480\,m^{12} + 15681536\,m^{13} + 2210816\,m^{14} - 500480\,m^{15} - 36864\,m^{16} + 8192\,m^{17}) - 4\,k^7\,m\,(-45298990800 + \\
&18950806980\,m + 255431488176\,m^2 - 104891331465\,m^3 - 332389359087\,m^4 + 128021127306\,m^5 + 149487158408\,m^6 - \\
&48408757047\,m^7 - 32159038409\,m^8 + 7174027442\,m^9 + 5657756976\,m^{10} - 992293376\,m^{11} - 778858976\,m^{12} + \\
&157418688\,m^{13} + 50896640\,m^{14} - 11232512\,m^{15} - 1052928\,m^{16} + 233984\,m^{17}) + k^6\,m\,(533201492880 - \\
&384728957172\,m - 2565670008816\,m^2 + 1776166820109\,m^3 + 1810845092651\,m^4 - 948157686224\,m^5 - 364034204954\,m^6 - \\
&4316501483\,m^7 + 205985694687\,m^8 - 192070878\,m^9 - 82806119984\,m^{10} + 16287251904\,m^{11} + 10156246688\,m^{12} - \\
&2488865344\,m^{13} - 332656896\,m^{14} + 86065920\,m^{15} - 2048256\,m^{16} + 455168\,m^{17}) + 2\,k^5\,m\,(-275677985520 + \\
&671127368508\,m - 572407062300\,m^2 + 1034645953473\,m^3 - 1080332948282\,m^4 + 596209401228\,m^5 + 319848426674\,m^6 - \\
&420811240273\,m^7 + 324813734580\,m^8 - 82176994968\,m^9 - 56227379488\,m^{10} + 25247347552\,m^{11} - \\
&5165882624\,m^{12} + 292299520\,m^{13} + 920100864\,m^{14} - 196484352\,m^{15} - 23675904\,m^{16} + 5261312\,m^{17}) + \\
&k^4\,(3456649728000 - 4210994511360\,m + 5654349571824\,m^2 + 10808435865924\,m^3 - 9376988512548\,m^4 - \\
&4113282068729\,m^5 + 5004974554266\,m^6 - 3872338578626\,m^7 + 1828695439448\,m^8 + 419279808583\,m^9 - \\
&697815557854\,m^{10} + 522827691920\,m^{11} - 82360945312\,m^{12} - 83674371424\,m^{13} + 22163495104\,m^{14} + \\
&1528716032\,m^{15} - 549605888\,m^{16} + 118183680\,m^{17} - 26263040\,m^{18}) + 144\,k\,(48009024000 + \\
&210928536000\,m - 309097736640\,m^2 - 435552994404\,m^3 + 366102616512\,m^4 + 124553980857\,m^5 + 42483316598\,m^6 - \\
&156646319046\,m^7 - 63099062628\,m^8 + 173231852961\,m^9 - 38353158066\,m^{10} - 34403043216\,m^{11} + 12387575840\,m^{12} - \\
&497769888\,m^{13} - 350922432\,m^{14} + 378076416\,m^{15} - 73972224\,m^{16} - 14503680\,m^{17} + 3223040\,m^{18}) + \\
&4\,k^2\,(4320812160000 + 5466485465280\,m - 11224225600992\,m^2 + \\
&636839647308\,m^3 + 1834214361408\,m^4 - 4972908681879\,m^5 + 89370376814\,m^6 + 5632907943810\,m^7 - \\
&2929189446348\,m^8 - 287772852039\,m^9 + 953052255774\,m^{10} - 817376931984\,m^{11} + 112943120672\,m^{12} + \\
&143507222880\,m^{13} - 35587630272\,m^{14} - 4470000384\,m^{15} + 1290567168\,m^{16} - 122164992\,m^{17} + 27147776\,m^{18}) - \\
&4\,k^3\,(-3456649728000 - 514312999200\,m + 2204369839800\,m^2 - \\
&4807556573508\,m^3 + 5327882415426\,m^4 - 1217926648777\,m^5 + 611239487018\,m^6 + 731779284194\,m^7 - \\
&1317602907422\,m^8 + 1210916107547\,m^9 - 356052576230\,m^{10} - 170317436144\,m^{11} + 104864253216\,m^{12} - \\
&39104723168\,m^{13} + 3716881856\,m^{14} + 5781899008\,m^{15} - 1216952320\,m^{16} - 161309952\,m^{17} + 35846656\,m^{18}))\\
&h_1(k,m) + \\
&(3\,k^{12}\,m\,(1587600 - 396900\,m - 9600516\,m^2 + 2400129\,m^3 + 15291640\,m^4 - 3822910\,m^5 - 9901892\,m^6 + 2475473\,m^7 + 3065920\,m^8 - \\
&766480\,m^9 - 477568\,m^{10} + 119392\,m^{11} + 35840\,m^{12} - 8960\,m^{13} - 1024\,m^{14} + 256\,m^{15}) + 16\,k^{11}\,m\,(9298800 - \\
&2373300\,m - 56120508\,m^2 + 14320053\,m^3 + 88902632\,m^4 - 22670102\,m^5 - 56980924\,m^6 + 14512069\,m^7 + 17347616\,m^8 - \\
&4408976\,m^9 - 2632448\,m^{10} + 666848\,m^{11} + 189952\,m^{12} - 47872\,m^{13} - 5120\,m^{14} + 1280\,m^{15}) + k^8\,m\,(112658046480 - \\
&38248923492\,m - 657375259788\,m^2 + 221041616337\,m^3 + 947395904466\,m^4 - 309272300810\,m^5 - 511731431960\,m^6 + \\
&156502928049\,m^7 + 124196892194\,m^8 - 33524588852\,m^9 - 16567450656\,m^{10} + 3801023136\,m^{11} + \\
&1509120704\,m^{12} - 318284928\,m^{13} - 87927296\,m^{14} + 18998528\,m^{15} + 2105856\,m^{16} - 467968\,m^{17}) + \\
&k^{10}\,m\,(2298618000 - 614960100\,m - 13807357380\,m^2 + 3689473041\,m^3 + 21590092360\,m^4 - \\
&5749520858\,m^5 - 13509468168\,m^6 + 3573993029\,m^7 + 3961539956\,m^8 - 1036213112\,m^9 - 571704064\,m^{10} + \\
&146781664\,m^{11} + 39408768\,m^{12} - 9823232\,m^{13} - 1138688\,m^{14} + 271616\,m^{15} + 9216\,m^{16} - 2048\,m^{17}) - \\
&161280\,m\,(-114307200 + 58038120\,m + 611712810\,m^2 - 273874293\,m^3 - \\
&726792144\,m^4 + 208615451\,m^5 + 455486907\,m^6 - 180067848\,m^7 - 75354261\,m^8 + 52361370\,m^9 - \\
&20820864\,m^{10} + 1325120\,m^{11} + 5004384\,m^{12} - 1172928\,m^{13} - 208128\,m^{14} + 55296\,m^{15} - 2304\,m^{16} + 512\,m^{17}) - \\
&2\,k^9\,m\,(-9720693360 + 2809394604\,m + 57900776256\,m^2 - 16674948099\,m^3 - 88450486441\,m^4 +
\end{aligned}
$$

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\begin{aligned}
&25215442730\,m^5+53050223884\,m^6-14818109177\,m^7-14634006627\,m^8+3941167590\,m^9+1989789904\,m^{10}-\\
&504813824\,m^{11}-141143968\,m^{12}+33088064\,m^{13}+5657856\,m^{14}-1248000\,m^{15}-117504\,m^{16}+26112\,m^{17})-\\
&4\,k^7\,m\,(-88059409200+38930639580\,m+491233767048\,m^2-212787310095\,m^3-617861726991\,m^4+248733027842\,m^5+\\
&260708170492\,m^6-84630969465\,m^7-55391069845\,m^8+11406686122\,m^9+10728787632\,m^{10}-1945169600\,m^{11}-\\
&1435699552\,m^{12}+310941120\,m^{13}+78224128\,m^{14}-18077440\,m^{15}-1043712\,m^{16}+231936\,m^{17})+2\,k^5\,m\\
&(-1104540086160+2368385341524\,m-839078167680\,m^2+1824805873395\,m^3-2232408668287\,m^4+1034858416830\,m^5+\\
&816395906508\,m^6-520378284095\,m^7+192215811187\,m^8-154534652806\,m^9+65513136304\,m^{10}+9426359040\,m^{11}-\\
&27297795168\,m^{12}+5575718336\,m^{13}+1766144768\,m^{14}-425565440\,m^{15}-11273472\,m^{16}+2505216\,m^{17})+k^6\,m\\
&(1243670909040-969617575836\,m-5785269760908\,m^2+4373091666351\,m^3+3304856724876\,m^4-1951339535894\,m^5-\\
&318391671752\,m^6-106868555325\,m^7+300734632088\,m^8-14035597664\,m^9-111344100672\,m^{10}+29344869792\,m^{11}+\\
&7482062336\,m^{12}-2565970176\,m^{13}+315538432\,m^{14}-45069568\,m^{15}-23869440\,m^{16}+5304320\,m^{17})+\\
&96\,k\,(216040608000+869039539200\,m-1087590732480\,m^2-\\
&2418078313188\,m^3+1539487432692\,m^4+1884537711087\,m^5-535448999410\,m^6-1130841411798\,m^7+\\
&299084482116\,m^8+365887511547\,m^9-163852976838\,m^{10}+8955256080\,m^{11}+13400495072\,m^{12}-\\
&14045710560\,m^{13}+2483845056\,m^{14}+1199398656\,m^{15}-269882880\,m^{16}-21153024\,m^{17}+4700672\,m^{18})-\\
&8\,k^3\,(-5184974592000-2403550875600\,m+6152222887380\,m^2-\\
&5245110485820\,m^3+4580902412271\,m^4+1522533681862\,m^5-774665962480\,m^6-406017445694\,m^7+\\
&129727687013\,m^8+272289151012\,m^9-434627587288\,m^{10}+296358559328\,m^{11}-2694564704\,m^{12}-\\
&84013646080\,m^{13}+17337753856\,m^{14}+5279553536\,m^{15}-1253185280\,m^{16}-47084544\,m^{17}+10463232\,m^{18})+\\
&24\,k^2\,(2160406080000+3873942858240\,m-6565700748816\,m^2-\\
&4627997415108\,m^3+4148970362172\,m^4+1025549084561\,m^5-353289753762\,m^6-242092487350\,m^7-\\
&649660492616\,m^8+902938360409\,m^9-112275583586\,m^{10}-270012198416\,m^{11}+86315364256\,m^{12}+\\
&5881232224\,m^{13}-5138615488\,m^{14}+2618631424\,m^{15}-469683712\,m^{16}-130569984\,m^{17}+29015552\,m^{18})-\\
&2\,k^4\,(-5184974592000+5974887875040\,m-7142860128456\,m^2-\\
&18682595644068\,m^3+15905159497242\,m^4+6970974994643\,m^5-5742408764850\,m^6+2175775869022\,m^7-\\
&1773097725386\,m^8+723200459427\,m^9+93850174794\,m^{10}-422927021360\,m^{11}+146569511456\,m^{12}+\\
&551681312\,m^{13}-8153955904\,m^{14}+6082957056\,m^{15}-1128102912\,m^{16}-277155072\,m^{17}+61590016\,m^{18}))\\
&h_1(k+1,m)+\\
&(241920\,m\,(3+m-2\,m^2)^2\,(-2116800+2002560\,m+8828684\,m^2-8331868\,m^3-1300561\,m^4+1223848\,m^5-\\
&595985\,m^6+257830\,m^7+56476\,m^8-28328\,m^9+5968\,m^{10}-1504\,m^{11}-448\,m^{12}+128\,m^{13})+k^{12}\,m\\
&(-1587600+396900\,m+9600516\,m^2-2400129\,m^3-15291640\,m^4+3822910\,m^5+9901892\,m^6-2475473\,m^7-3065920\,m^8+\\
&766480\,m^9+477568\,m^{10}-119392\,m^{11}-35840\,m^{12}+8960\,m^{13}+1024\,m^{14}-256\,m^{15})-8\,k^{11}\,m\,(5670000-\\
&1449900\,m-34213500\,m^2+8746659\,m^3+54171208\,m^4-13839098\,m^5-34686652\,m^6+8849555\,m^7+10543104\,m^8-\\
&2683824\,m^9-1595776\,m^{10}+404768\,m^{11}+114688\,m^{12}-28928\,m^{13}-3072\,m^{14}+768\,m^{15})+k^6\,m\,(-401043318480+\\
&302624000532\,m+1889676202860\,m^2-1360657162413\,m^3-1205544127214\,m^4+603488327302\,m^5+290464486452\,m^6-\\
&18343029313\,m^7-153695348866\,m^8+30548625652\,m^9+33577740896\,m^{10}-10819642400\,m^{11}-\\
&558269376\,m^{12}+459743872\,m^{13}-229918208\,m^{14}+44638976\,m^{15}+9063936\,m^{16}-2014208\,m^{17})+\\
&k^{10}\,m\,(-681534000+182155500\,m+4094188740\,m^2-1092683955\,m^3-6403790878\,m^4+\\
&1702155002\,m^5+4010634572\,m^6-1057641575\,m^7-1179440482\,m^8+306782740\,m^9+171746656\,m^{10}-\\
&43667040\,m^{11}-12189632\,m^{12}+2989184\,m^{13}+389632\,m^{14}-90880\,m^{15}-4608\,m^{16}+1024\,m^{17})+2\,k^9\,m\\
&(-2768184720+796075668\,m+16496782128\,m^2-4723668861\,m^3-25241806275\,m^4+7138291426\,m^5+15208363688\,m^6-\\
&4194550275\,m^7-4250346389\,m^8+1121175626\,m^9+599583600\,m^{10}-147411712\,m^{11}-46478432\,m^{12}+\\
&10542528\,m^{13}+2134784\,m^{14}-465152\,m^{15}-48384\,m^{16}+10752\,m^{17})+4\,k^7\,m\,(-24066428400+\\
&10586853900\,m+134313101208\,m^2-57595091931\,m^3-169624893399\,m^4+66331219034\,m^5+73981598236\,m^6-\\
&22193571141\,m^7-17850677437\,m^8+3501809722\,m^9+3667781424\,m^{10}-728052800\,m^{11}-439564384\,m^{12}+\\
&101459904\,m^{13}+19186432\,m^{14}-4649728\,m^{15}-103680\,m^{16}+23040\,m^{17})+k^8\,m\,(-32700976560+\\
&10980933804\,m+191069868924\,m^2-63406985571\,m^3-276651262980\,m^4+88544772114\,m^5+151619185172\,m^6-\\
&44832232419\,m^7-38455808924\,m^8+9843807272\,m^9+5649144384\,m^{10}-1242079904\,m^{11}-562067072\,m^{12}+\\
&118752768\,m^{13}+32645120\,m^{14}-7129856\,m^{15}-728064\,m^{16}+161792\,m^{17})+2\,k^5\,m\,(442236276720-\\
&854807925708\,m-36545711184\,m^2-357328567989\,m^3+854364607789\,m^4-281174896198\,m^5-438434087840\,m^6+\\
&92041377005\,m^7+130834343107\,m^8+21837633050\,m^9-72590416528\,m^{10}+10514973952\,m^{11}+12216415904\,m^{12}-\\
&2897367104\,m^{13}-480187648\,m^{14}+127244032\,m^{15}-5080320\,m^{16}+1128960\,m^{17})+576\,k\,(-12002256000-\\
&51266779200\,m+53153762760\,m^2+178911501372\,m^3-89605370010\,m^4-193655867483\,m^5+61526788044\,m^6+\\
&110306988064\,m^7-40733532700\,m^8-20935646561\,m^9+11548937138\,m^{10}-2849372944\,m^{11}-56545120\,m^{12}+\\
&892568096\,m^{13}-202922048\,m^{14}-41972992\,m^{15}+10862080\,m^{16}-260352\,m^{17}+57856\,m^{18})-24\,k^3\,(576108288000+\\
&330493942800\,m-835620004980\,m^2+639973822848\,m^3-356475413279\,m^4-627902921835\,m^5+240637266730\,m^6+\\
&314935359060\,m^7-172818007401\,m^8-7691761545\,m^9+56815752530\,m^{10}-52104954384\,m^{11}+6636391104\,m^{12}+\\
&9616538400\,m^{13}-2329439040\,m^{14}-342100224\,m^{15}+95006976\,m^{16}-6693120\,m^{17}+1487360\,m^{18})-\\
&32\,k^2\,(540101520000+1099567496160\,m-1758066850944\,m^2-1705005060660\,m^3+\\
&1425604055346\,m^4+580216777614\,m^5-112475768053\,m^6-364153546323\,m^7-78034540539\,m^8+\\
&329883106761\,m^9-95734758354\,m^{10}-45318743424\,m^{11}+24231589280\,m^{12}-7239796704\,m^{13}+\\
&403036608\,m^{14}+1285016832\,m^{15}-262048512\,m^{16}-41792256\,m^{17}+9287168\,m^{18})+4\,k^4\,(-864162432000+\\
&887639815440\,m-1061208482436\,m^2-2832604381260\,m^3+2695946299905\,m^4+430718247250\,m^5-632676984636\,m^6+\\
&605024718354\,m^7-335825962177\,m^8+48209769656\,m^9-22653218528\,m^{10}-6780706528\,m^{11}+18881129184\,m^{12}-\\
&13619340672\,m^{13}+1464828928\,m^{14}+1875702784\,m^{15}-387142912\,m^{16}-57729024\,m^{17}+12828672\,m^{18}))\\
&h_1(k+2,m))\Big/\\
&(80640\,(1+k)^2\,(-4+m)\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(4+m)\,(-9+2\,m)\,(-7+2\,m)\,(-5+2\,m)\\
&(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m)\,(7+2\,m))
\end{aligned}
$$

Appendix B. The certificates needed in Remark 10

$$g_2(k,m) := \frac{1}{2\,(1+k)\,m\,(1+2\,m)}\big(-2\,\big(3\,k^2+k^3-3\,m\,(1+2\,m)-k\,\big(-2+m+2\,m^2\big)\big)\,h2(k,m)+\big(9\,k^2+3\,k^3-2\,m\,(1+2\,m)+k\,\big(6-4\,m-8\,m^2\big)\big)\,h_2(k+1,m)-k\,\big(2+3\,k+k^2-2\,m-4\,m^2\big)\,h_2(k+2,m)\big)$$

$$g_3(k,m) := \big(-2\,\big(2\,k^5\,m\,(1+2\,m)+k^4\,\big(3+8\,m+16\,m^2\big)+12\,m\,\big(-1-m+4\,m^2+4\,m^3\big)+k^3\,\big(12+4\,m+6\,m^2-8\,m^3-8\,m^4\big)+k\,\big(6-26\,m-50\,m^2+8\,m^3+8\,m^4\big)-k^2\,\big(-15+16\,m+40\,m^2+32\,m^3+32\,m^4\big)\big)\,h2(k,m)+\big(6\,k^5\,m\,(1+2\,m)+k^4\,\big(9+20\,m+40\,m^2\big)+16\,m\,\big(-1-m+4\,m^2+4\,m^3\big)-2\,k^3\,\big(-18+m+6\,m^2+16\,m^3+16\,m^4\big)-k^2\,\big(-45+68\,m+144\,m^2+32\,m^3+32\,m^4\big)+2\,k\,\big(9-34\,m-58\,m^2+40\,m^3+40\,m^4\big)\big)\,h_2(k+1,m)-\big(2\,k^5\,m\,(1+2\,m)+k^2\,\big(15-26\,m-52\,m^2\big)+3\,k^4\,\big(1+2\,m+4\,m^2\big)+4\,m\,\big(-1-m+4\,m^2+4\,m^3\big)+6\,k\,\big(1-4\,m-6\,m^2+8\,m^3+8\,m^4\big)-2\,k^3\,\big(-6+m+4\,m^2+8\,m^3+8\,m^4\big)\big)\,h_2(k+2,m)\big)/\big(4\,(1+k)^2\,m\,(1+m)\,(-1+2\,m)\,(1+2\,m)\big)$$

$$g_4(k,m) := \big(-2\,\big(k^6\,m\,\big(-1-m+4\,m^2+4\,m^3\big)+k^5\,m\,\big(-1+3\,m+20\,m^2+20\,m^3\big)+4\,m\,\big(-9-10\,m+33\,m^2+38\,m^3+12\,m^4+8\,m^5\big)+k^3\,\big(36+14\,m+11\,m^2-73\,m^3-98\,m^4-60\,m^5-40\,m^6\big)+k^4\,\big(9+11\,m+25\,m^2+11\,m^3+6\,m^4-12\,m^5-8\,m^6\big)+k^2\,\big(45-28\,m-92\,m^2-143\,m^3-138\,m^4+12\,m^5+8\,m^6\big)+k\,\big(18-67\,m-134\,m^2+13\,m^3+78\,m^4+156\,m^5+104\,m^6\big)\big)\,h2(k,m)+\big(48\,m\,\big(-1-m+4\,m^2+4\,m^3\big)+3\,k^6\,m\,\big(-1-m+4\,m^2+4\,m^3\big)+k^5\,m\,\big(1+13\,m+44\,m^2+44\,m^3\big)+k^4\,\big(27+38\,m+73\,m^2-16\,m^3-36\,m^4-48\,m^5-32\,m^6\big)-k^3\,\big(-108-9\,m+41\,m^2+242\,m^3+272\,m^4+72\,m^5+48\,m^6\big)+2\,k\,\big(27-92\,m-158\,m^2+113\,m^3+158\,m^4+108\,m^5+72\,m^6\big)+k^2\,\big(135-161\,m-378\,m^2-204\,m^3-104\,m^4+240\,m^5+160\,m^6\big)\big)\,h_2(k+1,m)+\big(-k^5\,m\,\big(1+5\,m+12\,m^2+12\,m^3\big)+k^6\,\big(m+m^2-4\,m^3-4\,m^4\big)-6\,k\,\big(3-11\,m-16\,m^2+24\,m^3+24\,m^4\big)+k^3\,\big(-36+m+25\,m^2+92\,m^3+92\,m^4\big)+4\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)+k^2\,\big(-45+65\,m+142\,m^2+38\,m^3-12\,m^4-120\,m^5-80\,m^6\big)+k^4\,\big(-9-12\,m-23\,m^2+6\,m^3+16\,m^4+24\,m^5+16\,m^6\big)\big)\,h_2(k+2,m)\big)/\big(4\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\big)$$

$$g_5(k,m) := \big(-2\,\big(2\,k^7\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)-144\,m\,\big(9+7\,m-42\,m^2-32\,m^3+24\,m^4+16\,m^5\big)+k^6\,m\,\big(33+19\,m-174\,m^2-104\,m^3+168\,m^4+112\,m^5\big)+k^5\,m\,\big(231+379\,m-310\,m^2-202\,m^3+248\,m^4+128\,m^5-64\,m^6-32\,m^7\big)+k^4\,\big(324+609\,m+1261\,m^2+134\,m^3-70\,m^4-568\,m^5-640\,m^6-448\,m^7-224\,m^8\big)+24\,k\,\big(27-117\,m-203\,m^2+116\,m^3+80\,m^4-64\,m^5+32\,m^6+128\,m^7+64\,m^8\big)-k^3\,\big(-1296+21\,m+47\,m^2+104\,m^3+534\,m^4+1088\,m^5+912\,m^6+320\,m^7+160\,m^8\big)+2\,k^2\,\big(810-969\,m-2188\,m^2-1006\,m^3-999\,m^4+224\,m^5+840\,m^6+1184\,m^7+592\,m^8\big)\big)\,h2(k,m)+\big(6\,k^7\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)+3\,k^6\,m\,\big(21+11\,m-114\,m^2-64\,m^3+120\,m^4+80\,m^5\big)+k^5\,m\,\big(585+1071\,m-370\,m^2-248\,m^3+248\,m^4+16\,m^5-256\,m^6-128\,m^7\big)-72\,m\,\big(24+21\,m-106\,m^2-95\,m^3+32\,m^4+40\,m^5+32\,m^6+16\,m^7\big)-8\,k\,\big(-243+954\,m+1491\,m^2-1648\,m^3-1547\,m^4+248\,m^5+184\,m^6+32\,m^7+16\,m^8\big)-3\,k^4\,\big(-324-549\,m-1091\,m^2+74\,m^3+312\,m^4+616\,m^5+560\,m^6+256\,m^7+128\,m^8\big)+6\,k^2\,\big(810-1365\,m-2627\,m^2+404\,m^3+390\,m^4+112\,m^5+560\,m^6+832\,m^7+416\,m^8\big)+k^3\,\big(3888-1179\,m-2691\,m^2-1576\,m^3-2744\,m^4-2512\,m^5-704\,m^6+1664\,m^7+832\,m^8\big)\big)\,h_2(k+1,m)+\big(-2\,k^7\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)-k^6\,m\,\big(15+7\,m-84\,m^2-44\,m^3+96\,m^4+64\,m^5\big)+24\,m\,\big(18+15\,m-82\,m^2-71\,m^3+32\,m^4+40\,m^5+32\,m^6+16\,m^7\big)+k^5\,m\,\big(-195-355\,m+128\,m^2+72\,m^3-112\,m^4+128\,m^6+64\,m^7\big)+k^3\,\big(-1296+525\,m+1295\,m^2+1030\,m^3+1252\,m^4+376\,m^5-272\,m^6-896\,m^7-448\,m^8\big)+36\,k\,\big(-18+75\,m+102\,m^2-188\,m^3-167\,m^4+56\,m^5+56\,m^6+32\,m^7+16\,m^8\big)+k^4\,\big(-324-501\,m-997\,m^2+80\,m^3+384\,m^4+752\,m^5+576\,m^6+128\,m^7+64\,m^8\big)-4\,k^2\,\big(405-777\,m-1421\,m^2+530\,m^3+530\,m^4+56\,m^5+224\,m^6+320\,m^7+160\,m^8\big)\big)\,h_2(k+2,m)\big)/\big(24\,(1+k)^2\,(-1+m)\,m\,(1+m)\,(2+m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\big)$$

$$g_6(k,m) := \big(-2\,\big(-2\,k^5\,m^2\,(1+2\,m)^2\,\big(192+19\,m+43\,m^2+20\,m^3+20\,m^4\big)+10\,k^7\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)+k^8\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)+k^6\,m\,\big(192+167\,m-842\,m^2-713\,m^3+304\,m^4+184\,m^5-32\,m^6-16\,m^7\big)-288\,m\,\big(-15-14\,m+62\,m^2+53\,m^3-16\,m^4+8\,m^5+32\,m^6+16\,m^7\big)-k^4\,\big(1080+1521\,m+2790\,m^2-1003\,m^3-956\,m^4+236\,m^5+568\,m^6+704\,m^7+352\,m^8\big)-6\,k\,\big(360-1434\,m-2575\,m^2+1282\,m^3+1855\,m^4+1504\,m^5+1432\,m^6+736\,m^7+368\,m^8\big)+2\,k^3\,\big(-2160-501\,m-103\,m^2+3808\,m^3+4906\,m^4+2848\,m^5+2608\,m^6+1216\,m^7+608\,m^8\big)+k^2\,\big(-5400+4638\,m+12029\,m^2+10774\,m^3+9775\,m^4-1328\,m^5+2680\,m^6+6112\,m^7+3056\,m^8\big)\big)\,h2(k,m)+\big(22\,k^7\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)+3\,k^8\,m\,\big(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\big)-2\,k^6\,m\,\big(-219-203\,m+922\,m^2+834\,m^3-200\,m^4-96\,m^5+64\,m^6+32\,m^7\big)-48\,m\,\big(-120-101\,m+530\,m^2+407\,m^3-256\,m^4-40\,m^5+224\,m^6+112\,m^7\big)-2\,k^5\,m\,\big(192+776\,m+1621\,m^2+1897\,m^3+724\,m^4+688\,m^5+352\,m^6+176\,m^7\big)+2\,k^2\,\big(-8100+11346\,m+23215\,m^2+1664\,m^3-469\,m^4-5080\,m^5-3256\,m^6+224\,m^7+112\,m^8\big)+k^4\,\big(-3240-4563\,m-7674\,m^2+5881\,m^3+6260\,m^4+988\,m^5+920\,m^6+448\,m^7+224\,m^8\big)+2\,k^3\,\big(-6480+579\,m+3932\,m^2+11298\,m^3+12425\,m^4+3360\,m^5+4424\,m^6+3744\,m^7+1872\,m^8\big)-$$

$$
\begin{aligned}
&4\,k\,\left(1620-5874\,m-9055\,m^2+10570\,m^3+9703\,m^4-1280\,m^5+1816\,m^6+4576\,m^7+2288\,m^8\right)\right)\,h_2(k+1,m)+\\
&\left(-6\,k^7\,m\,\left(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\right)-\right.\\
&\quad k^8\,m\,\left(3+2\,m-15\,m^2-10\,m^3+12\,m^4+8\,m^5\right)+48\,m\,\left(-30-17\,m+158\,m^2+89\,m^3-160\,m^4-88\,m^5+32\,m^6+16\,m^7\right)+\\
&\quad 2\,k^6\,m\,\left(-69-64\,m+290\,m^2+261\,m^3-64\,m^4-24\,m^5+32\,m^6+16\,m^7\right)+\\
&\quad 2\,k^5\,m\,\left(96+292\,m+417\,m^2+505\,m^3+228\,m^4+208\,m^5+96\,m^6+48\,m^7\right)+\\
&\quad 12\,k^2\,\left(450-741\,m-1427\,m^2+251\,m^3+413\,m^4+428\,m^5+416\,m^6+224\,m^7+112\,m^8\right)-\\
&\quad k^4\,\left(-1080-1401\,m-2374\,m^2+1807\,m^3+2296\,m^4+1252\,m^5+1096\,m^6+448\,m^7+224\,m^8\right)+\\
&\quad 8\,k\,\left(270-1047\,m-1364\,m^2+2774\,m^3+2075\,m^4-1504\,m^5-424\,m^6+992\,m^7+496\,m^8\right)-\\
&\quad \left.2\,k^3\,\left(-2160+435\,m+1958\,m^2+4298\,m^3+4071\,m^4-304\,m^5+600\,m^6+1376\,m^7+688\,m^8\right)\right)\,h_2(k+2,m)\Big)\Big/\\
&\left(48\,{\left(1+k\right)}^2\,\left(-2+m\right)\,\left(-1+m\right)\,m\,\left(1+m\right)\,\left(-1+2\,m\right)\,\left(1+2\,m\right)\,\left(3+2\,m\right)\,\left(5+2\,m\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
g_7(k,m):=\;&\left(-2\,\left(2\,k^9\,m\,\left(180+72\,m-1025\,m^2-410\,m^3+1365\,m^4+546\,m^5-600\,m^6-240\,m^7+80\,m^8+32\,m^9\right)+\right.\right.\\
&k^8\,m\,\left(5490+2241\,m-31150\,m^2-12705\,m^3+41020\,m^4+16688\,m^5-17600\,m^6-7120\,m^7+2240\,m^8+896\,m^9\right)-\\
&2\,k^7\,m\,\left(-22050-10515\,m+121197\,m^2+\right.\\
&\qquad\left.57164\,m^3-144335\,m^4-65525\,m^5+50836\,m^6+20972\,m^7-5840\,m^8-2160\,m^9+192\,m^{10}+64\,m^{11}\right)-\\
&960\,m\,\left(4050+2610\,m-20505\,m^2-13223\,m^3+17493\,m^4+11579\,m^5-684\,m^6-1668\,m^7-1680\,m^8-496\,m^9+192\,m^{10}+64\,m^{11}\right)-\\
&4\,k^6\,m\,\left(-28980-15477\,m+155139\,m^2+\right.\\
&\qquad\left.82558\,m^3-166545\,m^4-86131\,m^5+40332\,m^6+14484\,m^7-6960\,m^8-1552\,m^9+1344\,m^{10}+448\,m^{11}\right)-\\
&2\,k^5\,m\,\left(-288270-501843\,m+298671\,m^2+301047\,m^3+\right.\\
&\qquad\left.21275\,m^4+60096\,m^5+55088\,m^6-29864\,m^7-71440\,m^8-18368\,m^9+11136\,m^{10}+3712\,m^{11}\right)-\\
&4\,k^2\,\left(-1215000+1571130\,m+3426987\,m^2+1003736\,m^3+295207\,m^4-\right.\\
&\qquad\left.1880310\,m^5-1834214\,m^6-919272\,m^7-264744\,m^8+273760\,m^9+137312\,m^{10}+30336\,m^{11}+10112\,m^{12}\right)+\\
&k^4\,\left(972000+1497510\,m+3038199\,m^2+6810\,m^3-843115\,m^4-\right.\\
&\qquad\left.2152120\,m^5-1797428\,m^6-574000\,m^7-162560\,m^8+181760\,m^9+104384\,m^{10}+34560\,m^{11}+11520\,m^{12}\right)-\\
&8\,k\,\left(-243000+1074600\,m+1740870\,m^2-1571949\,m^3-1269413\,m^4+\right.\\
&\qquad\left.697425\,m^5+342995\,m^6-312012\,m^7-401124\,m^8-288720\,m^9-39280\,m^{10}+83136\,m^{11}+27712\,m^{12}\right)+\\
&\left.2\,k^3\,\left(1944000-288900\,m-752550\,m^2-708823\,m^3-739111\,m^4+\right.\right.\\
&\qquad\left.\left.24555\,m^5+292005\,m^6+282156\,m^7-318108\,m^8-560240\,m^9-120080\,m^{10}+113472\,m^{11}+37824\,m^{12}\right)\right)\\
&h2(k,m)+\\
&\left(6\,k^9\,m\,\left(180+72\,m-1025\,m^2-410\,m^3+1365\,m^4+546\,m^5-600\,m^6-240\,m^7+80\,m^8+32\,m^9\right)+k^8\,m\,\left(12870+5283\,m-72950\,m^2-\right.\right.\\
&\qquad\left.29915\,m^3+95760\,m^4+39144\,m^5-40800\,m^6-16560\,m^7+5120\,m^8+2048\,m^9\right)+2\,k^7\,m\,\left(49050+24345\,m-\right.\\
&\qquad\left.267098\,m^2-130861\,m^3+308240\,m^4+144020\,m^5-101584\,m^6-41408\,m^7+12160\,m^8+4160\,m^9-768\,m^{10}-256\,m^{11}\right)+480\,m\,\\
&\qquad\left(-10800-7740\,m+52599\,m^2+38137\,m^3-36111\,m^4-28525\,m^5-6732\,m^6-1092\,m^7+3120\,m^8+1424\,m^9+192\,m^{10}+64\,m^{11}\right)-\\
&4\,k^6\,m\,\left(-50040-26631\,m+268276\,m^2+\right.\\
&\qquad\left.143007\,m^3-287830\,m^4-150578\,m^5+65288\,m^6+19336\,m^7-16160\,m^8-3296\,m^9+3456\,m^{10}+1152\,m^{11}\right)-\\
&2\,k^5\,m\,\left(-774810-1452609\,m+428789\,m^2+635873\,m^3+\right.\\
&\qquad\left.514425\,m^4+392748\,m^5+52512\,m^6-54696\,m^7-104240\,m^8-34304\,m^9+8064\,m^{10}+2688\,m^{11}\right)-\\
&48\,k\,\left(-121500+490050\,m+707805\,m^2-1058171\,m^3-896532\,m^4+\right.\\
&\qquad\left.423225\,m^5+390055\,m^6+146532\,m^7-20676\,m^8-118480\,m^9-33840\,m^{10}+14784\,m^{11}+4928\,m^{12}\right)+\\
&4\,k^3\,\left(2916000-1270800\,m-2899125\,m^2-1372531\,m^3-1047282\,m^4+\right.\\
&\qquad\left.983105\,m^5+1288925\,m^6+921452\,m^7+55684\,m^8-520080\,m^9-168080\,m^{10}+43584\,m^{11}+14528\,m^{12}\right)+\\
&k^4\,\left(2916000+4226130\,m+7816437\,m^2-2925370\,m^3-4814165\,m^4-\right.\\
&\qquad\left.4420240\,m^5-2618024\,m^6+380000\,m^7-242000\,m^8-491520\,m^9-27648\,m^{10}+184320\,m^{11}+61440\,m^{12}\right)-\\
&4\,k^2\,\left(-3645000+6552990\,m+12031461\,m^2-4504816\,m^3-5512867\,m^4-\right.\\
&\qquad\left.\left.2288850\,m^5-1168662\,m^6+309912\,m^7-564456\,m^8-844640\,m^9-108704\,m^{10}+249984\,m^{11}+83328\,m^{12}\right)\right)\\
&h_2(k+1,m)-\\
&\left(2\,k^9\,m\,\left(180+72\,m-1025\,m^2-410\,m^3+1365\,m^4+546\,m^5-600\,m^6-240\,m^7+80\,m^8+32\,m^9\right)+k^8\,m\,\left(3690+1521\,m-20900\,m^2-\right.\right.\\
&\qquad\left.8605\,m^3+27370\,m^4+11228\,m^5-11600\,m^6-4720\,m^7+1440\,m^8+576\,m^9\right)+2\,k^5\,m\,\left(264870+482583\,m-\right.\\
&\qquad\left.191515\,m^2-209565\,m^3-63095\,m^4-105166\,m^5-96920\,m^6-20320\,m^7+37520\,m^8+15008\,m^9\right)+960\,m\,\left(-1350-855\,m+\right.\\
&\qquad\left.6849\,m^2+4240\,m^3-6015\,m^4-3253\,m^5+1044\,m^6-84\,m^7-720\,m^8-112\,m^9+192\,m^{10}+64\,m^{11}\right)-2\,k^7\,m\,\left(-14850-7365\,m+\right.\\
&\qquad\left.80854\,m^2+39483\,m^3-93420\,m^4-43070\,m^5+31512\,m^6+12264\,m^7-4480\,m^8-1440\,m^9+384\,m^{10}+128\,m^{11}\right)-\\
&4\,k^6\,m\,\left(-11880-6027\,m+64472\,m^2+\right.\\
&\qquad\left.32859\,m^3-72110\,m^4-36046\,m^5+18616\,m^6+4472\,m^7-5920\,m^8-1312\,m^9+1152\,m^{10}+384\,m^{11}\right)+\\
&40\,k\,\left(48600-207900\,m-257706\,m^2+602583\,m^3+\right.\\
&\qquad\left.452603\,m^4-365379\,m^5-260129\,m^6-22140\,m^7+4620\,m^8+21360\,m^9+9424\,m^{10}+960\,m^{11}+320\,m^{12}\right)-\\
&4\,k^3\,\left(-972000+523350\,m+1248150\,m^2+742213\,m^3+416581\,m^4-\right.\\
&\qquad\left.823705\,m^5-684225\,m^6-209116\,m^7-47012\,m^8+81040\,m^9+41040\,m^{10}+9408\,m^{11}+3136\,m^{12}\right)-\\
&12\,k^2\,\left(-405000+824010\,m+1431979\,m^2-881209\,m^3-955718\,m^4-\right.\\
&\qquad\left.121605\,m^5+93087\,m^6+233508\,m^7-40404\,m^8-192560\,m^9-43056\,m^{10}+37056\,m^{11}+12352\,m^{12}\right)+\\
&k^4\,\left(972000+1283310\,m+2362119\,m^2-991880\,m^3-1827455\,m^4-\right.\\
&\qquad\left.\left.1822110\,m^5-639648\,m^6+845280\,m^7+83040\,m^8-415840\,m^9-92416\,m^{10}+80640\,m^{11}+26880\,m^{12}\right)\right)\\
&h_2(k+2,m)\Big)\Big/\\
&\left(480\,{\left(1+k\right)}^2\,\left(-2+m\right)\,\left(-1+m\right)\,m\,\left(1+m\right)\,\left(2+m\right)\,\left(3+m\right)\,\left(-5+2\,m\right)\,\left(-3+2\,m\right)\,\left(-1+2\,m\right)\,\left(1+2\,m\right)\,\left(3+2\,m\right)\,\left(5+2\,m\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
g_8(k,m) := (&-2\,(k^{10}\,m\,(-2700-900\,m+15807\,m^2+5269\,m^3-22935\,m^4-7645\,m^5+12276\,m^6+4092\,m^7-2640\,m^8-880\,m^9+192\,m^{10}+64\,m^{11})+\\
&k^9\,m\,(-57780-19692\,m+337233\,m^2+114871\,m^3-\\
&\quad 484905\,m^4-164911\,m^5+254844\,m^6+86388\,m^7-53040\,m^8-17872\,m^9+3648\,m^{10}+1216\,m^{11})+\\
&k^5\,m\,(6032880+23045202\,m+45259965\,m^2+51887997\,m^3+15305633\,m^4+7385209\,m^5-8023910\,m^6-\\
&\quad 11721784\,m^7-9439416\,m^8-2098944\,m^9+1819360\,m^{10}+589952\,m^{11}-15232\,m^{12}-4352\,m^{13})-\\
&k^8\,m\,(605070+225423\,m-3484494\,m^2-1292931\,m^3+4815763\,m^4+1764782\,m^5-\\
&\quad 2342407\,m^6-835538\,m^7+434244\,m^8+147192\,m^9-28624\,m^{10}-9056\,m^{11}+448\,m^{12}+128\,m^{13})+\\
&2880\,m\,(-85050-54000\,m+422775\,m^2+226854\,m^3-403550\,m^4-\\
&\quad 56872\,m^5+323873\,m^6+63838\,m^7-126156\,m^8-41064\,m^9+10544\,m^{10}+4000\,m^{11}+448\,m^{12}+128\,m^{13})-\\
&k^7\,m\,(2835000+1211490\,m-15935949\,m^2-6738150\,m^3+20454622\,m^4+8348042\,m^5-\\
&\quad 8622517\,m^6-3223310\,m^7+1388316\,m^8+432936\,m^9-127984\,m^{10}-33440\,m^{11}+8512\,m^{12}+2432\,m^{13})-\\
&k^6\,m\,(10534860+7160184\,m-51898347\,m^2-33606621\,m^3+41665687\,m^4+20601929\,m^5-\\
&\quad 11234536\,m^6-3069668\,m^7+3037056\,m^8+509520\,m^9-733312\,m^{10}-184256\,m^{11}+55552\,m^{12}+15872\,m^{13})-\\
&432\,k\,(-283500+1221750\,m+1862535\,m^2-2281381\,m^3-2101781\,m^4+259369\,m^5-328406\,m^6-\\
&\quad 581798\,m^7+375932\,m^8+805112\,m^9+156112\,m^{10}-175776\,m^{11}-49856\,m^{12}+8064\,m^{13}+2304\,m^{14})+\\
&k^3\,(244944000-17720100\,m-130295700\,m^2-386039019\,m^3-418990570\,m^4-77121898\,m^5-37682848\,m^6+51734773\,m^7+\\
&\quad 93771830\,m^8+82525956\,m^9+17055736\,m^{10}-17221904\,m^{11}-5435360\,m^{12}+281792\,m^{13}+80512\,m^{14})+\\
&k^4\,(61236000+75573270\,m+107776503\,m^2-168572874\,m^3-142484821\,m^4+71219273\,m^5+75657692\,m^6+46082863\,m^7+\\
&\quad 16761962\,m^8-10577796\,m^9-8482328\,m^{10}-4232624\,m^{11}-735776\,m^{12}+623168\,m^{13}+178048\,m^{14})-\\
&6\,k^2\,(-51030000+60310440\,m+129386466\,m^2+28728909\,m^3-901280\,m^4-59825492\,m^5-3004244\,m^6+60104627\,m^7+\\
&\quad 21629050\,m^8-13146756\,m^9-8928760\,m^{10}-3592816\,m^{11}-512800\,m^{12}+632128\,m^{13}+180608\,m^{14}))\\
&h2(k,m)+\\
&(3\,k^{10}\,m\,(-2700-900\,m+15807\,m^2+5269\,m^3-22935\,m^4-7645\,m^5+12276\,m^6+4092\,m^7-2640\,m^8-880\,m^9+192\,m^{10}+64\,m^{11})+\\
&9\,k^9\,m\,(-15660-5364\,m+91335\,m^2+\\
&\quad 31265\,m^3-131055\,m^4-44777\,m^5+68580\,m^6+23340\,m^7-14160\,m^8-4784\,m^9+960\,m^{10}+320\,m^{11})-\\
&k^8\,m\,(1353510+515349\,m-7767153\,m^2-2942643\,m^3+10621327\,m^4+3961574\,m^5-\\
&\quad 5060164\,m^6-1821704\,m^7+914496\,m^8+306240\,m^9-63808\,m^{10}-19328\,m^{11}+1792\,m^{12}+512\,m^{13})+\\
&1440\,m\,(-226800-127980\,m+1178055\,m^2+583878\,m^3-1271810\,m^4-\\
&\quad 347644\,m^5+801911\,m^6+260386\,m^7-191892\,m^8-84888\,m^9-6832\,m^{10}+1120\,m^{11}+3136\,m^{12}+896\,m^{13})-\\
&6\,k^7\,m\,(930150+410265\,m-5195127\,m^2-2261046\,m^3+6538432\,m^4+2717667\,m^5-\\
&\quad 2669171\,m^6-981838\,m^7+444436\,m^8+127016\,m^9-52752\,m^{10}-13216\,m^{11}+4032\,m^{12}+1152\,m^{13})-\\
&3\,k^6\,m\,(7964460+5913864\,m-37832307\,m^2-26897589\,m^3+25126175\,m^4+13204977\,m^5-\\
&\quad 5382856\,m^6-1025732\,m^7+1994240\,m^8+351184\,m^9-469632\,m^{10}-127424\,m^{11}+26880\,m^{12}+7680\,m^{13})+\\
&9\,k^5\,m\,(3585960+9305334\,m+8950481\,m^2+10956671\,m^3+4368411\,m^4+1363743\,m^5-\\
&\quad 2898132\,m^6-2066292\,m^7-846992\,m^8-378608\,m^9-22464\,m^{10}+27456\,m^{11}+32256\,m^{12}+9216\,m^{13})+\\
&48\,k\,(7654500-30176550\,m-39256245\,m^2+79013457\,m^3+53454327\,m^4-54189803\,m^5-13406458\,m^6+34505876\,m^7+\\
&\quad 6319096\,m^8-14035584\,m^9-4514304\,m^{10}+1236032\,m^{11}+472192\,m^{12}+55552\,m^{13}+15872\,m^{14})+\\
&3\,k^4\,(61236000+72530370\,m+92772063\,m^2-203199981\,m^3-173332591\,m^4+72403839\,m^5+51830688\,m^6+10547932\,m^7+\\
&\quad 15596672\,m^8+11349152\,m^9-203680\,m^{10}-4956736\,m^{11}-1289216\,m^{12}+335104\,m^{13}+95744\,m^{14})-6\,k^3\\
&\quad(-122472000+47808900\,m+134524530\,m^2+151585827\,m^3+132081006\,m^4-43098422\,m^5-17623682\,m^6+13933611\,m^7-\\
&\quad 5959902\,m^8-17675236\,m^9-7824536\,m^{10}-505328\,m^{11}+361056\,m^{12}+488768\,m^{13}+139648\,m^{14})-4\,k^2\\
&\quad(-229635000+393630030\,m+691905537\,m^2-379143432\,m^3-369473559\,m^4+14899475\,m^5-10513724\,m^6-6928661\,m^7+\\
&\quad 62967458\,m^8+76638540\,m^9+7210632\,m^{10}-24482672\,m^{11}-6338464\,m^{12}+1682240\,m^{13}+480640\,m^{14}))\\
&h_2(k+1,m)-\\
&(k^{10}\,m\,(-2700-900\,m+15807\,m^2+5269\,m^3-22935\,m^4-7645\,m^5+12276\,m^6+4092\,m^7-2640\,m^8-880\,m^9+192\,m^{10}+64\,m^{11})+\\
&k^9\,m\,(-41580-14292\,m+242391\,m^2+\\
&\quad 83257\,m^3-347295\,m^4-119041\,m^5+181188\,m^6+61836\,m^7-37200\,m^8-12592\,m^9+2496\,m^{10}+832\,m^{11})+\\
&2880\,m\,(-28350-9045\,m+167688\,m^2+56703\,m^3-245807\,m^4-\\
&\quad 95512\,m^5+116369\,m^6+55414\,m^7-5916\,m^8-6696\,m^9-4432\,m^{10}-992\,m^{11}+448\,m^{12}+128\,m^{13})-\\
&k^8\,m\,(402570+153063\,m-2310453\,m^2-873177\,m^3+3162041\,m^4+\\
&\quad 1172470\,m^5-1514534\,m^6-537556\,m^7+282168\,m^8+91536\,m^9-22688\,m^{10}-6592\,m^{11}+896\,m^{12}+256\,m^{13})-\\
&2\,k^7\,m\,(757350+332775\,m-4231512\,m^2-1827834\,m^3+5342140\,m^4+2174675\,m^5-\\
&\quad 2237266\,m^6-778112\,m^7+424680\,m^8+112320\,m^9-59872\,m^{10}-15104\,m^{11}+4480\,m^{12}+1280\,m^{13})-\\
&k^6\,m\,(7586460+5549904\,m-36211239\,m^2-25089345\,m^3+25152707\,m^4+11876869\,m^5-\\
&\quad 7437992\,m^6-1426900\,m^7+2818176\,m^8+656400\,m^9-500864\,m^{10}-145600\,m^{11}+19712\,m^{12}+5632\,m^{13})+\\
&k^5\,m\,(13744080+30529602\,m+13926465\,m^2+22297959\,m^3+17708731\,m^4+4073867\,m^5-12454060\,m^6-\\
&\quad 4415828\,m^7+1833168\,m^8-390000\,m^9-1122880\,m^{10}-203456\,m^{11}+157696\,m^{12}+45056\,m^{13})+\\
&24\,k\,(5103000-21392100\,m-23125230\,m^2+71672661\,m^3+38307336\,m^4-72077684\,m^5-22139884\,m^6+41410823\,m^7+\\
&\quad 13171378\,m^8-10068852\,m^9-4046232\,m^{10}+51536\,m^{11}+137056\,m^{12}+110656\,m^{13}+31616\,m^{14})-\\
&12\,k^2\,(-25515000+49955670\,m+82807623\,m^2-66962142\,m^3-60379133\,m^4+12211229\,m^5-2495052\,m^6-13041881\,m^7+\\
&\quad 5984186\,m^8+15211932\,m^9+3191720\,m^{10}-3079472\,m^{11}-898848\,m^{12}+117824\,m^{13}+33664\,m^{14})+\\
&k^4\,(61236000+65083770\,m+83576943\,m^2-185166723\,m^3-178657387\,m^4+18810991\,m^5+24435720\,m^6+25312466\,m^7+\\
&\quad 24308884\,m^8+13525848\,m^9+1425136\,m^{10}-4253728\,m^{11}-1261632\,m^{12}+144256\,m^{13}+41216\,m^{14})-\\
&4\,k^3\,(-61236000+30700350\,m+83766060\,m^2+82981056\,m^3+51700237\,m^4-68862175\,m^5-25613295\,m^6+33066313\,m^7+\\
&\quad 11865626\,m^8-7302540\,m^9-4982680\,m^{10}-2003024\,m^{11}-257568\,m^{12}+378560\,m^{13}+108160\,m^{14}))\\
&h_2(k+2,m))/\\
&(1440\,(1+k)^2\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\\
&\quad(5+2\,m)\,(7+2\,m))
\end{aligned}
$$

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g9(k,m) := ( - 2 (2 k^11 m ( - 18900 - 11700 m + 96249 m^2 +
                56497 m^3 - 89841 m^4 - 41665 m^5 + 29988 m^6 + 11868 m^7 - 4080 m^8 - 1456 m^9 + 192 m^10 + 64 m^11) +
          k^10 m ( - 1077300 - 673200 m + 5469393 m^2 + 3243212 m^3 -
                5038449 m^4 - 2361080 m^5 + 1644048 m^6 + 656904 m^7 - 215760 m^8 - 77504 m^9 + 9600 m^10 + 3200 m^11) +
          181440 m (176400 + 159600 m - 756490 m^2 - 674843 m^3 + 196933 m^4 +
                137133 m^5 + 20045 m^6 + 33826 m^7 + 27844 m^8 + 3464 m^9 - 8080 m^10 - 2208 m^11 + 448 m^12 + 128 m^13) -
          k^9 m (14556780 + 9351792 m - 73215663 m^2 - 44715468 m^3 + 64753085 m^4 + 31196092 m^5 -
                19786718 m^6 - 8064412 m^7 + 2390328 m^8 + 859824 m^9 - 104480 m^10 - 33856 m^11 + 896 m^12 + 256 m^13) -
          2 k^8 m (49809060 + 33611724 m - 246104421 m^2 - 158300597 m^3 + 200718040 m^4 + 100919767 m^5 -
                54485683 m^6 - 22542926 m^7 + 6103380 m^8 + 2117864 m^9 - 345616 m^10 - 103072 m^11 + 11200 m^12 + 3200 m^13) -
          7 k^6 m (119338380 + 110040912 m - 503865255 m^2 - 452143120 m^3 + 115091327 m^4 + 48386372 m^5 - 41482376 m^6 -
                3389272 m^7 + 21882768 m^8 + 6264256 m^9 - 2670848 m^10 - 816512 m^11 + 68096 m^12 + 19456 m^13) -
          2 k^7 m (226497600 + 173548620 m - 1060263528 m^2 - 777501095 m^3 + 651751710 m^4 + 346636967 m^5 - 143787042 m^6 -
                55163856 m^7 + 21182856 m^8 + 6253184 m^9 - 2353632 m^10 - 670976 m^11 + 104832 m^12 + 29952 m^13) +
          k^5 m ( - 4125790620 - 9084595248 m - 3226307565 m^2 - 2640652672 m^3 + 1719540003 m^4 + 2164447336 m^5 + 1631269962 m^6 +
                530028948 m^7 - 382377480 m^8 - 155585360 m^9 - 1589664 m^10 + 1610432 m^11 + 1975680 m^12 + 564480 m^13) +
          864 k ( - 18522000 + 76734000 m + 141953070 m^2 - 43835777 m^3 - 31475444 m^4 + 36458663 m^5 + 46031644 m^6 + 33570880 m^7 +
                7638200 m^8 - 12078928 m^9 - 4582144 m^10 + 309376 m^11 + 165248 m^12 + 57344 m^13 + 16384 m^14) - 96 k^2
             (416745000 - 423084060 m - 1057960674 m^2 - 854405667 m^3 - 891391718 m^4 - 92082328 m^5 - 65241249 m^6 + 21685090 m^7 +
                79288328 m^8 + 84204984 m^9 + 19703264 m^10 - 14909984 m^11 - 4404096 m^12 + 522368 m^13 + 149248 m^14) +
          2 k^4 ( - 4000752000 - 6885186840 m - 15915276036 m^2 - 8782225686 m^3 -
                9771692147 m^4 - 2234585635 m^5 - 975802383 m^6 + 1064879977 m^7 + 977030714 m^8 + 546038580 m^9 +
                125618024 m^10 - 99449936 m^11 - 30177312 m^12 + 2744000 m^13 + 784000 m^14) - 4 k^3 (8001504000 +
                947286900 m + 1171099800 m^2 - 2814339039 m^3 - 2311045009 m^4 + 1790099602 m^5 + 3081940119 m^6 + 3028083485 m^7 +
                986138242 m^8 - 717030348 m^9 - 311584280 m^10 - 23089168 m^11 - 1138080 m^12 + 6053824 m^13 + 1729664 m^14)) )
     h2(k,m) +
     (6 k^11 m ( - 18900 - 11700 m + 96249 m^2 +
                56497 m^3 - 89841 m^4 - 41665 m^5 + 29988 m^6 + 11868 m^7 - 4080 m^8 - 1456 m^9 + 192 m^10 + 64 m^11) +
          k^10 m ( - 2702700 - 1692000 m + 13713207 m^2 + 8147720 m^3 -
                12599799 m^4 - 5916620 m^5 + 4092480 m^6 + 1638408 m^7 - 533040 m^8 - 191744 m^9 + 23424 m^10 + 7808 m^11) +
          201600 m (211680 + 203238 m - 870321 m^2 - 822115 m^3 + 101938 m^4 +
                27837 m^5 - 46243 m^6 + 31450 m^7 + 67284 m^8 + 16072 m^9 - 11536 m^10 - 3360 m^11 + 448 m^12 + 128 m^13) -
          k^9 m (33086340 + 21389976 m - 166048149 m^2 - 102087790 m^3 + 145436141 m^4 + 70468082 m^5 -
                43792784 m^6 - 17888704 m^7 + 5231472 m^8 + 1869280 m^9 - 241280 m^10 - 76544 m^11 + 3584 m^12 + 1024 m^13) +
          7 k^6 m ( - 205076340 - 197359056 m + 841426953 m^2 + 795151448 m^3 - 97718365 m^4 - 23329084 m^5 + 61131748 m^6 +
                4564592 m^7 - 33475008 m^8 - 10995488 m^9 + 2632000 m^10 + 910336 m^11 + 30464 m^12 + 8704 m^13) - 2 k^8 m
             (102592980 + 70001712 m - 504744525 m^2 - 328345206 m^3 + 403571965 m^4 + 204087494 m^5 - 107413852 m^6 - 43972952 m^7 +
                12565824 m^8 + 4209408 m^9 - 869056 m^10 - 252800 m^11 + 34048 m^12 + 9728 m^13) - 2 k^7 m (478396800 +
                378286020 m - 2205519672 m^2 - 1671960829 m^3 + 1230795030 m^4 + 658415605 m^5 - 263263266 m^6 - 95810712 m^7 +
                45296040 m^8 + 13239520 m^9 - 5191392 m^10 - 1512064 m^11 + 201600 m^12 + 57600 m^13) + k^5 m ( - 11651309460 -
                26455840344 m - 12268772895 m^2 - 10491426358 m^3 + 4585047555 m^4 + 4129502362 m^5 + 1868188644 m^6 +
                992327304 m^7 + 32902752 m^8 - 74908736 m^9 - 92480448 m^10 - 24833920 m^11 + 5531904 m^12 + 1580544 m^13) -
          4 k^4 (6001128000 + 9946359360 m + 21247603884 m^2 +
                5491676520 m^3 + 5890208793 m^4 + 1159111915 m^5 + 1392385163 m^6 + 840592991 m^7 - 126776954 m^8 -
                707344212 m^9 - 238330344 m^10 + 49765328 m^11 + 18426400 m^12 + 1696576 m^13 + 484736 m^14) +
          8 k^2 ( - 15002820000 + 22896223560 m + 47280581244 m^2 + 5670984414 m^3 +
                4498759153 m^4 - 1529148847 m^5 + 3053404701 m^6 + 6157266913 m^7 + 1045919066 m^8 - 2619504108 m^9 -
                865737880 m^10 + 199888432 m^11 + 68751840 m^12 + 1959104 m^13 + 559744 m^14) + 48 k ( - 1000188000 +
                3760495200 m + 6213977280 m^2 - 5201196198 m^3 - 5048261729 m^4 + 473279927 m^5 + 680014935 m^6 + 664395319 m^7 +
                422399174 m^8 + 93562380 m^9 - 15077608 m^10 - 54959024 m^11 - 14458848 m^12 + 3563840 m^13 + 1018240 m^14) -
          4 k^3 (24004512000 - 4138287300 m - 12749524380 m^2 -
                17770245393 m^3 - 17086592950 m^4 + 2079619261 m^5 + 2930960370 m^6 + 2963823800 m^7 + 2223702448 m^8 +
                840053520 m^9 + 42349216 m^10 - 309350656 m^11 - 84751872 m^12 + 16952320 m^13 + 4843520 m^14)) )
     h2(k+1,m) + ( - 2 k^11 m ( - 18900 -
                11700 m + 96249 m^2 + 56497 m^3 - 89841 m^4 - 41665 m^5 + 29988 m^6 + 11868 m^7 - 4080 m^8 - 1456 m^9 + 192 m^10 + 64 m^11) -
          3 k^10 m ( - 270900 - 169800 m + 1373969 m^2 + 817418 m^3 -
                1260225 m^4 - 592590 m^5 + 408072 m^6 + 163584 m^7 - 52880 m^8 - 19040 m^9 + 2304 m^10 + 768 m^11) +
          k^5 m (4055407020 + 8856333648 m + 3003539649 m^2 + 3090872792 m^3 + 96038503 m^4 - 324583044 m^5 - 705198424 m^6 -
                444411416 m^7 - 104735664 m^8 - 6923840 m^9 + 37154816 m^10 + 10727040 m^11 - 1530368 m^12 - 437248 m^13) -
          20160 m (529200 + 452340 m - 2319636 m^2 - 1828315 m^3 + 1005763 m^4 +
                116973 m^5 - 851923 m^6 - 187790 m^7 + 304164 m^8 + 94984 m^9 - 28816 m^10 - 9120 m^11 + 448 m^12 + 128 m^13) +
          k^9 m (9793980 + 6327792 m - 49160715 m^2 - 30195228 m^3 + 43106071 m^4 + 20825768 m^5 -
                13064044 m^6 - 5297792 m^7 + 1606656 m^8 + 565152 m^9 - 83392 m^10 - 25856 m^11 + 1792 m^12 + 512 m^13) +
          4 k^8 m (14386680 + 9792072 m - 70835088 m^2 - 45906553 m^3 + 56933703 m^4 + 28468921 m^5 -
                15619317 m^6 - 6225210 m^7 + 2034300 m^8 + 658840 m^9 - 164592 m^10 - 47584 m^11 + 6720 m^12 + 1920 m^13) +
          14 k^7 m (20412000 + 16035300 m - 94350168 m^2 - 70788573 m^3 + 53998148 m^4 + 27708865 m^5 -
                13384352 m^6 - 4666300 m^7 + 2573040 m^8 + 775248 m^9 - 271232 m^10 - 80704 m^11 + 8960 m^12 + 2560 m^13) -
          7 k^6 m ( - 48160980 - 46746072 m + 196225569 m^2 + 186594834 m^3 - 18958949 m^4 + 630914 m^5 + 20434796 m^6 + 4444120 m^7 -
                7656960 m^8 - 2969856 m^9 + 130496 m^10 + 92032 m^11 + 44800 m^12 + 12800 m^13) - 336 k ( - 47628000 +
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\begin{aligned}
&190776600\,m+273493980\,m^2-423168306\,m^3-379514261\,m^4+95734799\,m^5+36215019\,m^6-37464113\,m^7+5172902\,m^8+\\
&29536812\,m^9+7192472\,m^{10}-4913456\,m^{11}-1422816\,m^{12}+198464\,m^{13}+56704\,m^{14})+12\,k^3\,(2667168000-\\
&736199100\,m-2255122680\,m^2-3070362415\,m^3-2780249927\,m^4+630189578\,m^5+225283925\,m^6-234052763\,m^7+\\
&125082218\,m^8+298684660\,m^9+71708168\,m^{10}-50866064\,m^{11}-14885408\,m^{12}+1910720\,m^{13}+545920\,m^{14})+\\
&2\,k^4\,(4000752000+6138882540\,m+13077117216\,m^2+2992877121\,m^3+\\
&2044864416\,m^4-1864333301\,m^5+84135488\,m^6+2104600064\,m^7+627641320\,m^8-562629840\,m^9-\\
&217183680\,m^{10}+10377344\,m^{11}+6448768\,m^{12}+2759680\,m^{13}+788480\,m^{14})-8\,k^2\,(-5000940000+\\
&8823612420\,m+17242362828\,m^2-1844965095\,m^3-2853747344\,m^4-1766306629\,m^5+941916744\,m^6+3381158284\,m^7+\\
&1061356976\,m^8-834760800\,m^9-324468256\,m^{10}+12920512\,m^{11}+8684544\,m^{12}+4040960\,m^{13}+1154560\,m^{14}))\\
&h_2(k+2,m))/\\
&(20160\,(1+k)^2\,(-3+m)\,(-2+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(4+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\,(-1+2\,m)\,(1+2\,m)\\
&(5+2\,m)\,(7+2\,m))
\end{aligned}
$$

$$
\begin{aligned}
g_{10}(k,m):=\;&(-2\,(32\,k^{11}\,m\,(-1814400-461700\,m+10953504\,m^2+2786697\,m^3-17365712\,m^4-4415502\,m^5+11147136\,m^6+2831257\,m^7-\\
&3402256\,m^8-862576\,m^9+518336\,m^{10}+131040\,m^{11}-37632\,m^{12}-9472\,m^{13}+1024\,m^{14}+256\,m^{15})+\\
&k^{12}\,m\,(-1587600-396900\,m+9600516\,m^2+2400129\,m^3-15291640\,m^4-3822910\,m^5+9901892\,m^6+2475473\,m^7-\\
&3065920\,m^8-766480\,m^9+477568\,m^{10}+119392\,m^{11}-35840\,m^{12}-8960\,m^{13}+1024\,m^{14}+256\,m^{15})-\\
&k^{10}\,m\,(992703600+263349900\,m-5968170576\,m^2-1581703059\,m^3+9354610805\,m^4+\\
&2472344232\,m^5-5879073326\,m^6-1545570835\,m^7+1735311369\,m^8+452125938\,m^9-252301520\,m^{10}-\\
&64801088\,m^{11}+17394272\,m^{12}+4370368\,m^{13}-476928\,m^{14}-115968\,m^{15}+2304\,m^{16}+512\,m^{17})+\\
&241920\,m\,(-57153600-32281200\,m+294688800\,m^2+137297502\,m^3-\\
&324649721\,m^4-46340124\,m^5+264094168\,m^6+63312892\,m^7-91480559\,m^8-35679470\,m^9+\\
&2455184\,m^{10}+3179680\,m^{11}+2026976\,m^{12}+362432\,m^{13}-170752\,m^{14}-39424\,m^{15}+2304\,m^{16}+512\,m^{17})-\\
&2\,k^9\,m\,(4584399120+1306148868\,m-27351546732\,m^2-7770795873\,m^3+41971575298\,m^4+\\
&11828793524\,m^5-25369616622\,m^6-7037035319\,m^7+7061880488\,m^8+1905164480\,m^9-959258080\,m^{10}-\\
&247464928\,m^{11}+64740480\,m^{12}+15681536\,m^{13}-2210816\,m^{14}-500480\,m^{15}+36864\,m^{16}+8192\,m^{17})-\\
&k^8\,m\,(54285986160+17849254284\,m-318185975088\,m^2-103772190339\,m^3+464461703229\,m^4+147820162076\,m^5-\\
&256570600762\,m^6-77522360799\,m^7+63601516717\,m^8+17462805946\,m^9-8232326352\,m^{10}-1975344320\,m^{11}+\\
&675853024\,m^{12}+145416384\,m^{13}-37090048\,m^{14}-7950592\,m^{15}+933120\,m^{16}+207360\,m^{17})-\\
&4\,k^7\,m\,(45298990800+18950806980\,m-255431488176\,m^2-104891331465\,m^3+332389359087\,m^4+128021127306\,m^5-\\
&149487158408\,m^6-48408757047\,m^7+32159038409\,m^8+7174027442\,m^9-5657756976\,m^{10}-992293376\,m^{11}+\\
&778858976\,m^{12}+157418688\,m^{13}-50896640\,m^{14}-11232512\,m^{15}+1052928\,m^{16}+233984\,m^{17})+\\
&k^6\,m\,(-533201492880-384728957172\,m+2565670008816\,m^2+1776166820109\,m^3-1810845092651\,m^4-948157686224\,m^5+\\
&364034204954\,m^6-4316501483\,m^7-205985694687\,m^8-192070878\,m^9+82806119984\,m^{10}+16287251904\,m^{11}-\\
&10156246688\,m^{12}-2488865344\,m^{13}+332656896\,m^{14}+86065920\,m^{15}+2048256\,m^{16}+455168\,m^{17})+\\
&2\,k^5\,m\,(275677985520+671127368508\,m+572407062300\,m^2+1034645953473\,m^3+1080332948282\,m^4+596209401228\,m^5-\\
&319848426674\,m^6-420811240273\,m^7-324813734580\,m^8-82176994968\,m^9+56227379488\,m^{10}+25247347552\,m^{11}+\\
&5165882624\,m^{12}+292299520\,m^{13}-920100864\,m^{14}-196484352\,m^{15}+23675904\,m^{16}+5261312\,m^{17})+\\
&k^4\,(3456649728000+4210994511360\,m+5654349571824\,m^2-10808435865924\,m^3-9376988512548\,m^4+\\
&4113282068729\,m^5+5004974554266\,m^6+3872338578626\,m^7+1828695439448\,m^8-419279808583\,m^9-\\
&697815557854\,m^{10}-522827691920\,m^{11}-82360945312\,m^{12}+83674371424\,m^{13}+22163495104\,m^{14}-\\
&1528716032\,m^{15}-549605888\,m^{16}-118183680\,m^{17}-26263040\,m^{18})+144\,k\,(48009024000-\\
&210928536000\,m-309097736640\,m^2+435552994404\,m^3+366102616512\,m^4-124553980857\,m^5+42483316598\,m^6+\\
&156646319046\,m^7-63099062628\,m^8-173231852961\,m^9-38353158066\,m^{10}+34403043216\,m^{11}+12387575840\,m^{12}+\\
&497769888\,m^{13}-350922432\,m^{14}-378076416\,m^{15}-73972224\,m^{16}+14503680\,m^{17}+3223040\,m^{18})+\\
&4\,k^2\,(4320812160000-5466485465280\,m-11224225600992\,m^2-636839647308\,m^3+\\
&1834214361408\,m^4+4972908681879\,m^5+89370376814\,m^6-5632907943810\,m^7-2929189446348\,m^8+\\
&287772852039\,m^9+953052255774\,m^{10}+817376931984\,m^{11}+112943120672\,m^{12}-143507222880\,m^{13}-\\
&35587630272\,m^{14}+4470000384\,m^{15}+1290567168\,m^{16}+122164992\,m^{17}+27147776\,m^{18})-\\
&4\,k^3\,(-3456649728000+514312999200\,m+2204369839800\,m^2+\\
&4807556573508\,m^3+5327882415426\,m^4+1217926648777\,m^5+611239487018\,m^6-731779284194\,m^7-\\
&1317602907422\,m^8-1210916107547\,m^9-356052576230\,m^{10}+170317436144\,m^{11}+104864253216\,m^{12}+\\
&39104723168\,m^{13}+3716881856\,m^{14}-5781899008\,m^{15}-1216952320\,m^{16}+161309952\,m^{17}+35846656\,m^{18}))\\
&h2(k,m)+\\
&(3\,k^{12}\,m\,(-1587600-396900\,m+9600516\,m^2+2400129\,m^3-15291640\,m^4-3822910\,m^5+9901892\,m^6+2475473\,m^7-\\
&3065920\,m^8-766480\,m^9+477568\,m^{10}+119392\,m^{11}-35840\,m^{12}-8960\,m^{13}+1024\,m^{14}+256\,m^{15})+\\
&16\,k^{11}\,m\,(-9298800-2373300\,m+56120508\,m^2+14320053\,m^3-88902632\,m^4-22670102\,m^5+56980924\,m^6+14512069\,m^7-\\
&17347616\,m^8-4408976\,m^9+2632448\,m^{10}+666848\,m^{11}-189952\,m^{12}-47872\,m^{13}+5120\,m^{14}+1280\,m^{15})-\\
&161280\,m\,(114307200+58038120\,m-611712810\,m^2-273874293\,m^3+\\
&726792144\,m^4+208615451\,m^5-455486907\,m^6-180067848\,m^7+75354261\,m^8+52361370\,m^9+\\
&20820864\,m^{10}+1325120\,m^{11}-5004384\,m^{12}-1172928\,m^{13}+208128\,m^{14}+55296\,m^{15}+2304\,m^{16}+512\,m^{17})-\\
&k^{10}\,m\,(2298618000+614960100\,m-13807357380\,m^2-3689473041\,m^3+21590092360\,m^4+5749520858\,m^5-\\
&13509468168\,m^6-3573993029\,m^7+3961539956\,m^8+1036213112\,m^9-571704064\,m^{10}-146781664\,m^{11}+\\
&39408768\,m^{12}+9823232\,m^{13}-1138688\,m^{14}-271616\,m^{15}+9216\,m^{16}+2048\,m^{17})-2\,k^9\,m\,(9720693360+\\
&2809394604\,m-57900776256\,m^2-16674948099\,m^3+88450486441\,m^4+25215442730\,m^5-53050223884\,m^6-\\
&14818109177\,m^7+14634006627\,m^8+3941167590\,m^9-1989789904\,m^{10}-504813824\,m^{11}+141143968\,m^{12}+
\end{aligned}
$$

$$
\begin{aligned}
&33088064\,m^{13} - 5657856\,m^{14} - 1248000\,m^{15} + 117504\,m^{16} + 26112\,m^{17}) - 4\,k^7\,m\,(88059409200+\\
&38930639580\,m - 491233767048\,m^2 - 212787310095\,m^3 + 617861726991\,m^4 + 248733027842\,m^5 - 260708170492\,m^6 -\\
&84630969465\,m^7 + 55391069845\,m^8 + 11406686122\,m^9 - 10728787632\,m^{10} - 1945169600\,m^{11} + 1435699552\,m^{12} +\\
&310941120\,m^{13} - 78224128\,m^{14} - 18077440\,m^{15} + 1043712\,m^{16} + 231936\,m^{17}) - k^8\,m\,(112658046480+\\
&38248923492\,m - 657375259788\,m^2 - 221041616337\,m^3 + 947395904466\,m^4 + 309272300810\,m^5 - 511731431960\,m^6 -\\
&156502928049\,m^7 + 124196892194\,m^8 + 33524588852\,m^9 - 16567450656\,m^{10} - 3801023136\,m^{11} +\\
&1509120704\,m^{12} + 318284928\,m^{13} - 87927296\,m^{14} - 18998528\,m^{15} + 2105856\,m^{16} + 467968\,m^{17}) + 2\,k^5\,m\\
&(1104540086160 + 2368385341524\,m + 839078167680\,m^2 + 1824805873395\,m^3 + 2232408668287\,m^4 + 1034858416830\,m^5 -\\
&816395906508\,m^6 - 520378284095\,m^7 - 192215811187\,m^8 - 154534652806\,m^9 - 65513136304\,m^{10} + 9426359040\,m^{11} +\\
&27297795168\,m^{12} + 5575718336\,m^{13} - 1766144768\,m^{14} - 425565440\,m^{15} + 11273472\,m^{16} + 2505216\,m^{17}) + k^6\,m\\
&(-1243670909040 - 969617575836\,m + 5785269760908\,m^2 + 4373091666351\,m^3 - 3304856724876\,m^4 - 1951339535894\,m^5 +\\
&318391671752\,m^6 - 106868555325\,m^7 - 300734632088\,m^8 - 14035597664\,m^9 + 111344100672\,m^{10} + 29344869792\,m^{11} -\\
&7482062336\,m^{12} - 2565970176\,m^{13} - 315538432\,m^{14} - 45069568\,m^{15} + 23869440\,m^{16} + 5304320\,m^{17}) +\\
&96\,k\,(216040608000 - 869039539200\,m - 1087590732480\,m^2 +\\
&2418078313188\,m^3 + 1539487432692\,m^4 - 1884537711087\,m^5 - 535448999410\,m^6 + 1130841411798\,m^7 +\\
&299084482116\,m^8 - 365887511547\,m^9 - 163852976838\,m^{10} - 8955256080\,m^{11} + 13400495072\,m^{12} +\\
&14045710560\,m^{13} + 2483845056\,m^{14} - 1199398656\,m^{15} - 269882880\,m^{16} + 21153024\,m^{17} + 4700672\,m^{18}) -\\
&8\,k^3\,(-5184974592000 + 2403550875600\,m + 6152222887380\,m^2 +\\
&5245110485820\,m^3 + 4580902412271\,m^4 - 1522533681862\,m^5 - 774665962480\,m^6 + 406017445694\,m^7 +\\
&129727687013\,m^8 - 272289151012\,m^9 - 434627587288\,m^{10} - 296358559328\,m^{11} - 2694564704\,m^{12} +\\
&84013646080\,m^{13} + 17337753856\,m^{14} - 5279553536\,m^{15} - 1253185280\,m^{16} + 47084544\,m^{17} + 10463232\,m^{18}) +\\
&24\,k^2\,(2160406080000 - 3873942858240\,m - 6565700748816\,m^2 +\\
&4627997415108\,m^3 + 4148970362172\,m^4 - 1025549084561\,m^5 - 353289753762\,m^6 + 242092487350\,m^7 -\\
&649660492616\,m^8 - 902938360409\,m^9 - 112275583586\,m^{10} + 270012198416\,m^{11} + 86315364256\,m^{12} -\\
&5881232224\,m^{13} - 5138615488\,m^{14} - 2618631424\,m^{15} - 469683712\,m^{16} + 130569984\,m^{17} + 29015552\,m^{18}) -\\
&2\,k^4\,(-5184974592000 - 5974887875040\,m - 7142860128456\,m^2 +\\
&18682595644068\,m^3 + 15905159497242\,m^4 - 6970974994643\,m^5 - 5742408764850\,m^6 - 2175775869022\,m^7 -\\
&1773097725386\,m^8 - 723200459427\,m^9 + 93850174794\,m^{10} + 422927021360\,m^{11} + 146569511456\,m^{12} -\\
&551681312\,m^{13} - 8153955904\,m^{14} - 6082957056\,m^{15} - 1128102912\,m^{16} + 277155072\,m^{17} + 61590016\,m^{18}))\\
&h_2(k+1,m) +\\
&(241920\,m\,(-3+m+2\,m^2)^2\,(2116800 + 2002560\,m - 8828684\,m^2 - 8331868\,m^3 + 1300561\,m^4 + 1223848\,m^5 +\\
&595985\,m^6 + 257830\,m^7 - 56476\,m^8 - 28328\,m^9 - 5968\,m^{10} - 1504\,m^{11} + 448\,m^{12} + 128\,m^{13}) + k^{12}\,m\\
&(1587600 + 396900\,m - 9600516\,m^2 - 2400129\,m^3 + 15291640\,m^4 + 3822910\,m^5 - 9901892\,m^6 - 2475473\,m^7 + 3065920\,m^8 +\\
&766480\,m^9 - 477568\,m^{10} - 119392\,m^{11} + 35840\,m^{12} + 8960\,m^{13} - 1024\,m^{14} - 256\,m^{15}) - 8\,k^{11}\,m\,(-5670000 -\\
&1449900\,m + 34213500\,m^2 + 8746659\,m^3 - 54171208\,m^4 - 13839098\,m^5 + 34686652\,m^6 + 8849555\,m^7 - 10543104\,m^8 -\\
&2683824\,m^9 + 1595776\,m^{10} + 404768\,m^{11} - 114688\,m^{12} - 28928\,m^{13} + 3072\,m^{14} + 768\,m^{15}) + k^6\,m\,(401043318480+\\
&302624000532\,m - 1889676202860\,m^2 - 1360657162413\,m^3 + 1205544127214\,m^4 + 603488327302\,m^5 - 290464486452\,m^6 -\\
&18343029313\,m^7 + 153695348866\,m^8 + 30548625652\,m^9 - 33577740896\,m^{10} - 10819642400\,m^{11} +\\
&558269376\,m^{12} + 459743872\,m^{13} + 229918208\,m^{14} + 44638976\,m^{15} - 9063936\,m^{16} - 2014208\,m^{17}) +\\
&k^{10}\,m\,(681534000 + 182155500\,m - 4094188740\,m^2 - 1092683955\,m^3 + 6403790878\,m^4 +\\
&1702155002\,m^5 - 4010634572\,m^6 - 1057641575\,m^7 + 1179440482\,m^8 + 306782740\,m^9 - 171746656\,m^{10} -\\
&43667040\,m^{11} + 12189632\,m^{12} + 2989184\,m^{13} - 389632\,m^{14} - 90880\,m^{15} + 4608\,m^{16} + 1024\,m^{17}) + 2\,k^9\,m\\
&(2768184720 + 796075668\,m - 16496782128\,m^2 - 4723668861\,m^3 + 25241806275\,m^4 + 7138291426\,m^5 - 15208363688\,m^6 -\\
&4194550275\,m^7 + 4250346389\,m^8 + 1121175626\,m^9 - 599583600\,m^{10} - 147411712\,m^{11} + 46478432\,m^{12} +\\
&10542528\,m^{13} - 2134784\,m^{14} - 465152\,m^{15} + 48384\,m^{16} + 10752\,m^{17}) + 4\,k^7\,m\,(24066428400+\\
&10586853900\,m - 134313101208\,m^2 - 57595091931\,m^3 + 169624893399\,m^4 + 66331219034\,m^5 - 73981598236\,m^6 -\\
&22193571141\,m^7 + 17850677437\,m^8 + 3501809722\,m^9 - 3667781424\,m^{10} - 728052800\,m^{11} + 439564384\,m^{12} +\\
&101459904\,m^{13} - 19186432\,m^{14} - 4649728\,m^{15} + 103680\,m^{16} + 23040\,m^{17}) + k^8\,m\,(32700976560+\\
&10980933804\,m - 191069868924\,m^2 - 63406985571\,m^3 + 276651262980\,m^4 + 88544772114\,m^5 - 151619185172\,m^6 -\\
&44832232419\,m^7 + 38455808924\,m^8 + 9843807272\,m^9 - 5649144384\,m^{10} - 1242079904\,m^{11} + 562067072\,m^{12} +\\
&118752768\,m^{13} - 32645120\,m^{14} - 7129856\,m^{15} + 728064\,m^{16} + 161792\,m^{17}) + 2\,k^5\,m\,(-442236276720 -\\
&854807925708\,m + 36545711184\,m^2 - 357328567989\,m^3 - 854364607789\,m^4 - 281174896198\,m^5 + 438434087840\,m^6 +\\
&92041377005\,m^7 - 130834343107\,m^8 + 21837633050\,m^9 + 72590416528\,m^{10} + 10514973952\,m^{11} - 12216415904\,m^{12} -\\
&2897367104\,m^{13} + 480187648\,m^{14} + 127244032\,m^{15} + 5080320\,m^{16} + 1128960\,m^{17}) + 576\,k\,(-12002256000 +\\
&51266779200\,m + 53153762760\,m^2 - 178911501372\,m^3 - 89605370010\,m^4 + 193655867483\,m^5 + 61526788044\,m^6 -\\
&110306988064\,m^7 - 40733532700\,m^8 + 20935646561\,m^9 + 11548937138\,m^{10} + 2849372944\,m^{11} - 56545120\,m^{12} -\\
&892568096\,m^{13} - 202922048\,m^{14} + 41972992\,m^{15} + 10862080\,m^{16} + 260352\,m^{17} + 57856\,m^{18}) - 24\,k^3\,(576108288000 -\\
&330493942800\,m - 835620004980\,m^2 - 639973822848\,m^3 - 356475413279\,m^4 + 627902921835\,m^5 + 240637266730\,m^6 -\\
&314935359060\,m^7 - 172818007401\,m^8 + 7691761545\,m^9 + 56815752530\,m^{10} + 52104954384\,m^{11} + 6636391104\,m^{12} -\\
&9616538400\,m^{13} - 2329439040\,m^{14} + 342100224\,m^{15} + 95006976\,m^{16} + 6693120\,m^{17} + 1487360\,m^{18}) -\\
&32\,k^2\,(540101520000 - 1099567496160\,m - 1758066850944\,m^2 + 1705005060660\,m^3 +\\
&1425604055346\,m^4 - 580216777614\,m^5 - 112475768053\,m^6 + 364153546323\,m^7 - 78034540539\,m^8 -\\
&329883106761\,m^9 - 95734758354\,m^{10} + 45318743424\,m^{11} + 24231589280\,m^{12} + 7239796704\,m^{13} +\\
&403036608\,m^{14} - 1285016832\,m^{15} - 262048512\,m^{16} + 41792256\,m^{17} + 9287168\,m^{18}) + 4\,k^4\,(-864162432000 -\\
&887639815440\,m - 1061208482436\,m^2 + 2832604381260\,m^3 + 2695946299905\,m^4 - 430718247250\,m^5 - 632676984636\,m^6 -\\
&605024718354\,m^7 - 335825962177\,m^8 - 48209769656\,m^9 - 22653218528\,m^{10} + 6780706528\,m^{11} + 18881129184\,m^{12} +\\
&13619340672\,m^{13} + 1464828928\,m^{14} - 1875702784\,m^{15} - 387142912\,m^{16} + 57729024\,m^{17} + 12828672\,m^{18}))\\
&h_2(k+2,m))\Big/\\
&(80640\,(1+k)^2\,(-4+m)\,(-3+m)\,(-2+m)\,(-1+m)\,m\,(1+m)\,(2+m)\,(3+m)\,(4+m)\,(-7+2\,m)\,(-5+2\,m)\,(-3+2\,m)\\
&(-1+2\,m)\,(1+2\,m)\,(3+2\,m)\,(5+2\,m)\,(7+2\,m)\,(9+2\,m))
\end{aligned}
$$

```
g11(k, m) := ( − 2 (2 k^13 m (57153600 + 12700800 m − 349190676 m^2 − 77597928 m^3 +
        572100201 m^4 + 127133378 m^5 − 390874302 m^6 − 86860956 m^7 + 132652377 m^8 + 29478306 m^9 −
        24090768 m^10 − 5353504 m^11 + 2364768 m^12 + 525504 m^13 − 117504 m^14 − 26112 m^15 + 2304 m^16 + 512 m^17) +
    5 k^12 m (1043053200 + 235361700 m − 6364692612 m^2 − 1435977297 m^3 + 10392226056 m^4 +
        2343789758 m^5 − 7056042084 m^6 − 1590288609 m^7 + 2370777552 m^8 + 533737776 m^9 − 424135296 m^10 −
        95326816 m^11 + 40739328 m^12 + 9133824 m^13 − 1963008 m^14 − 438528 m^15 + 36864 m^16 + 8192 m^17) +
    k^10 m (1317237390000 + 323239077900 m − 7978547133900 m^2 − 1954750190139 m^3 +
        12769556507640 m^4 + 3114841217978 m^5 − 8362137476044 m^6 − 2022895242347 m^7 + 2657754473136 m^8 +
        634137813312 m^9 − 441091534272 m^10 − 103014235744 m^11 + 38991323648 m^12 + 8824613632 m^13 −
        1804637184 m^14 − 391545600 m^15 + 41496576 m^16 + 8572928 m^17 − 409600 m^18 − 81920 m^19) −
    2 k^11 m ( − 55300190400 − 12875922000 m + 336540626064 m^2 + 78313889520 m^3 − 545556174696 m^4 − 126753273295 m^5 +
        365596027604 m^6 + 84694143082 m^7 − 120345051216 m^8 − 27746582799 m^9 + 20911867860 m^10 + 4786233140 m^11 −
        1935269440 m^12 − 438141152 m^13 + 89876352 m^14 + 20025984 m^15 − 1717248 m^16 − 373504 m^17 + 5120 m^18 + 1024 m^19) −
    2903040 m (3857868000 + 2028952800 m − 20715424380 m^2 − 10801168317 m^3 + 22478775192 m^4 + 10904421720 m^5 −
        5857755932 m^6 − 410451709 m^7 + 2364550032 m^8 − 932848302 m^9 − 1778142936 m^10 − 151556088 m^11 + 383543680 m^12 +
        75961664 m^13 − 26510592 m^14 − 6134784 m^15 + 362496 m^16 + 96768 m^17 + 10240 m^18 + 2048 m^19) −
    6 k^9 m ( − 1676174245200 − 452423978820 m + 10057309337124 m^2 + 2703158051373 m^3 −
        15688023347988 m^4 − 4166426692147 m^5 + 9813033686952 m^6 + 2545842269979 m^7 − 2921799405252 m^8 −
        728116737165 m^9 + 453141921564 m^10 + 106102726364 m^11 − 39533278272 m^12 − 8548277664 m^13 +
        2114878080 m^14 + 426342528 m^15 − 70668288 m^16 − 13928704 m^17 + 1121280 m^18 + 224256 m^19) −
    3 k^8 m ( − 16029908974800 − 5090017697460 m + 94367654130564 m^2 + 29664378532053 m^3 −
        139603734037692 m^4 − 42574926492986 m^5 + 79534435155548 m^6 + 22723566820477 m^7 − 21120819142884 m^8 −
        5334687073812 m^9 + 3159901020432 m^10 + 669473951536 m^11 − 331210992256 m^12 − 62286624128 m^13 +
        24685518336 m^14 + 4696815360 m^15 − 1018426368 m^16 − 201380864 m^17 + 15749120 m^18 + 3149824 m^19) −
    2 k^7 m ( − 83065584823200 − 36022119071400 m + 465011310397032 m^2 + 197098230179502 m^3 − 592313509001736 m^4 −
        231303517582909 m^5 + 259496355649092 m^6 + 79697398545540 m^7 − 59954103622224 m^8 −
        11003055107865 m^9 + 12700399567500 m^10 + 1872815971436 m^11 − 2030733349824 m^12 − 371515849248 m^13 +
        160990655616 m^14 + 32831387520 m^15 − 5170646016 m^16 − 1077507328 m^17 + 45173760 m^18 + 9034752 m^19) −
    2 k^5 m ( − 653934812554800 − 1289041997951340 m + 178879252371912 m^2 + 70075476376148 7 m^3 + 1270019392062849 m^4 +
        874938386486155 m^5 − 70668233783990 m^6 − 326560033803559 m^7 − 367305781072911 m^8 − 102792001658019 m^9 +
        53421002366604 m^10 + 26106536208316 m^11 + 6595371175840 m^12 + 382605628448 m^13 − 1162723898496 m^14 −
        242142319488 m^15 + 34521814272 m^16 + 7959552256 m^17 + 181918720 m^18 + 36383744 m^19) +
    k^6 m (241807922773200 + 168148007595540 m − 1185997165177716 m^2 − 810899464766445 m^3 + 878786534825280 m^4 +
        528812181371534 m^5 − 42778387557444 m^6 + 103795560063027 m^7 + 129197924200008 m^8 − 21213159074664 m^9 −
        73091920680864 m^10 − 13752052421248 m^11 + 9577972932864 m^12 + 2418311003136 m^13 − 268136441856 m^14 −
        83622293760 m^15 − 11166308352 m^16 − 1878071296 m^17 + 381050880 m^18 + 76210176 m^19) +
    5184 k (1080203040000 − 4872915936000 m − 7353486730800 m^2 + 9125433377460 m^3 + 6872987597292 m^4 −
        5559523820073 m^5 − 4041925268757 m^6 + 127510213874 m^7 + 1802168024002 m^8 + 2249562315663 m^9 +
        725704274355 m^10 − 282290059788 m^11 − 238894069404 m^12 − 114282477472 m^13 − 5561148704 m^14 +
        20575338624 m^15 + 3780476544 m^16 − 948442368 m^17 − 195289344 m^18 + 9774080 m^19 + 1954816 m^20) −
    24 k^3 ( − 466647713280000 + 118875076966800 m + 308277178524900 m^2 + 250983538312752 m^3 + 90647601156099 m^4 −
        408598297723809 m^5 − 336112708527003 m^6 − 46978953970898 m^7 + 131383807565729 m^8 + 203395223044335 m^9 +
        72304838893887 m^10 − 16936311253980 m^11 − 16844993877708 m^12 − 9127088562848 m^13 − 905142351520 m^14 +
        1311052799616 m^15 + 264050502528 m^16 − 44860777728 m^17 − 9843627264 m^18 + 79221760 m^19 + 15844352 m^20) +
    144 k^2 (97218273600000 − 134531234056800 m − 280746272383560 m^2 − 30757843230516 m^3 + 48233363281146 m^4 +
        185381893995705 m^5 + 114326690191097 m^6 + 1894881253382 m^7 + 40291091190940 m^8 + 36944508135849 m^9 −
        15152179950147 m^10 − 29860046834676 m^11 − 5310672958372 m^12 + 4263030925472 m^13 + 1176357898528 m^14 −
        50851029120 m^15 − 34231632000 m^16 − 14540894976 m^17 − 2510311168 m^18 + 455367680 m^19 + 91073536 m^20) −
    4 k^4 ( − 699971569920000 − 912698304008400 m − 1750193661877380 m^2 + 353055037231512 m^3 + 634920517530009 m^4 +
        793624145870889 m^5 + 927769647385007 m^6 + 710816610624914 m^7 + 399616947530731 m^8 − 14897476920207 m^9 −
        148703848821291 m^10 − 143092064493204 m^11 − 26488740151972 m^12 + 19895500333472 m^13 + 6075847764256 m^14 +
        1959149095 68 m^15 − 95347455360 m^16 − 98254965504 m^17 − 17809614592 m^18 + 2541992960 m^19 + 508398592 m^20)) )
  h2(k, m) +
 (6 k^13 m (57153600 + 12700800 m − 349190676 m^2 − 77597928 m^3 + 572100201 m^4 +
        127133378 m^5 − 390874302 m^6 − 86860956 m^7 + 132652377 m^8 + 29478306 m^9 − 24090768 m^10 −
        5353504 m^11 + 2364768 m^12 + 525504 m^13 − 117504 m^14 − 26112 m^15 + 2304 m^16 + 512 m^17) + 3 k^12 m
     (4529422800 + 1024398900 m − 27633174948 m^2 − 6248711349 m^3 + 45095927868 m^4 + 10193348254 m^5 − 30589718796 m^6 −
        6909111573 m^7 + 10262059236 m^8 + 2314949208 m^9 − 1831587264 m^10 − 412392032 m^11 + 175319424 m^12 +
        39363072 m^13 − 8404992 m^14 − 1879296 m^15 + 156672 m^16 + 34816 m^17) − 2 k^11 m ( − 130408185600 −
        30540531600 m + 793222965396 m^2 + 185636961756 m^3 − 1284114383037 m^4 − 299948863422 m^5 + 858423032150 m^6 +
        199802121544 m^7 − 281531770221 m^8 − 65144442246 m^9 + 48690868512 m^10 + 11164024560 m^11 − 4487529952 m^12 −
        1014692672 m^13 + 209171712 m^14 + 46316544 m^15 − 4189440 m^16 − 898560 m^17 + 20480 m^18 + 4096 m^19) −
    1451520 m (10287648000 + 5866408800 m − 53843585220 m^2 − 29636753202 m^3 + 54095930253 m^4 + 23990602377 m^5 −
        15875214914 m^6 + 3121545848 m^7 + 11711636157 m^8 − 1153784955 m^9 − 5789033652 m^10 − 973804836 m^11 +
        858152992 m^12 + 209805344 m^13 − 29817984 m^14 − 9297024 m^15 − 1220352 m^16 − 182016 m^17 + 56320 m^18 + 11264 m^19) −
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\begin{aligned}
&3\,k^{10}\,m\,\big(-953054650800-236126105100\,m+5767441395228\,m^2+1426100444091\,m^3-\\
&\quad 9208284710076\,m^4-2264478048314\,m^5+6004955700644\,m^6+1461389130763\,m^7-1898258080884\,m^8-\\
&\quad 453915264000\,m^9+313831298208\,m^{10}+72990279808\,m^{11}-27950464384\,m^{12}-6240505088\,m^{13}+\\
&\quad 1354767360\,m^{14}+287198976\,m^{15}-35705856\,m^{16}-7221248\,m^{17}+450560\,m^{18}+90112\,m^{19}\big)+\\
&6\,k^7\,m\,\big(58319161941600+26923065913800\,m-322374518496396\,m^2-145512881733690\,m^3+393796174379805\,m^4+\\
&\quad 163396666778710\,m^5-157820767043422\,m^6-49919399824574\,m^7+34643464241565\,m^8+\\
&\quad 6149363318970\,m^9-7618558072848\,m^{10}-1252266358688\,m^{11}+1124489888480\,m^{12}+231002860480\,m^{13}-\\
&\quad 70760642304\,m^{14}-15854548992\,m^{15}+1306348800\,m^{16}+302103040\,m^{17}+7454720\,m^{18}+1490944\,m^{19}\big)-\\
&2\,k^9\,m\,\big(-10468986166800-2880583736580\,m+62685667298016\,m^2+17159570589609\,m^3-\\
&\quad 97233277141215\,m^4-26228297381022\,m^5+60245617012318\,m^6+15782810041505\,m^7-17744604606519\,m^8-\\
&\quad 4416441371328\,m^9+2750667059976\,m^{10}+632389319832\,m^{11}-249113425952\,m^{12}-52219159168\,m^{13}+\\
&\quad 14548369920\,m^{14}+2873740800\,m^{15}-526827264\,m^{16}-103729152\,m^{17}+8427520\,m^{18}+1685504\,m^{19}\big)-\\
&9\,k^8\,m\,\big(-10648783227600-3500908116660\,m+62398603819332\,m^2+20275790466213\,m^3-\\
&\quad 91106469845388\,m^4-28563945022286\,m^5+50748533351980\,m^6+14696165348485\,m^7-13204425657620\,m^8-\\
&\quad 3280112954696\,m^9+2022108226944\,m^{10}+412236703008\,m^{11}-225885968256\,m^{12}-42406737920\,m^{13}+\\
&\quad 16955561984\,m^{14}+3308532992\,m^{15}-644699136\,m^{16}-129906688\,m^{17}+8437760\,m^{18}+1687552\,m^{19}\big)+\\
&3\,k^6\,m\,\big(138627614235600+108190833003540\,m-648693555742980\,m^2-508663771170381\,m^3+349248823950420\,m^4+\\
&\quad 281434560244742\,m^5+101136177718884\,m^6+91112697387747\,m^7+41667118388124\,m^8-7904025864432\,m^9-\\
&\quad 26222521150368\,m^{10}-7658784611776\,m^{11}+1235690343552\,m^{12}+680569951488\,m^{13}+241803531264\,m^{14}+\\
&\quad 35372056320\,m^{15}-17402397696\,m^{16}-3533529088\,m^{17}+210739200\,m^{18}+42147840\,m^{19}\big)-2\,k^5\,m\\
&\quad\big(-1909838955704400-3797094058896420\,m+396833152851036\,m^2+1917019692332793\,m^3+3575950905504582\,m^4+\\
&\quad 2120320590809412\,m^5-542154433057256\,m^6-506201326253347\,m^7-341756845776834\,m^8-186880178878734\,m^9-\\
&\quad 38761084563816\,m^{10}+13235740284120\,m^{11}+23111578927552\,m^{12}+5097722864192\,m^{13}-1260500830464\,m^{14}-\\
&\quad 382870588416\,m^{15}-44039846400\,m^{16}-5586163200\,m^{17}+2652928000\,m^{18}+530585600\,m^{19}\big)-\\
&1152\,k\,\big(-14582741040000+60235464996000\,m+80089522367400\,m^2-152368084977660\,m^3-107263756327356\,m^4+\\
&\quad 104589133396191\,m^5+58463242703379\,m^6-15872783410358\,m^7-8044711084 06\,m^8+5133362514159\,m^9-\\
&\quad 6155398946685\,m^{10}-7593103444044\,m^{11}-321830355804\,m^{12}+1909244436064\,m^{13}+361121317088\,m^{14}-\\
&\quad 144930154368\,m^{15}-32973889920\,m^{16}+2373102336\,m^{17}+608333568\,m^{18}+51143680\,m^{19}+10228736\,m^{20}\big)+\\
&288\,k^2\,\big(145827410400000-276752911604400\,m-482131235826540\,m^2+290058174937728\,m^3+309499870867317\,m^4+\\
&\quad 29004451025724\,m^5-32733462789906\,m^6-62725047844700\,m^7+32298731102921\,m^8+7644620979 4512\,m^9+\\
&\quad 13396707743712\,m^{10}-19473155264064\,m^{11}-7522991785056\,m^{12}-563031297536\,m^{13}+379740011264\,m^{14}+\\
&\quad 403016395776\,m^{15}+64241193216\,m^{16}-25242316800\,m^{17}-5018849280\,m^{18}+372981760\,m^{19}+74596352\,m^{20}\big)-\\
&12\,k^4\,\big(-699971569920000-874975041939600\,m-1470258132325380\,m^2+1119839242492764\,m^3+1130975612640333\,m^4+\\
&\quad 61346723333148\,m^5+169542208168898\,m^6+271684113867436\,m^7+255376361248613\,m^8+133001441025804\,m^9-\\
&\quad 1011095863536\,m^{10}-62597132962464\,m^{11}-25884302232064\,m^{12}-3055285404032\,m^{13}+1319601920768\,m^{14}+\\
&\quad 1641230429184\,m^{15}+270392017152\,m^{16}-96973624320\,m^{17}-19560448000\,m^{18}+1256366080\,m^{19}+251273216\,m^{20}\big)+\\
&8\,k^3\,\big(4199829419520000-2302850819883600\,m-5141794159059300\,m^2-1757385883095444\,m^3+157766635350009\,m^4+\\
&\quad 4489820647363062\,m^5+2660598583835832\,m^6-495001340061656\,m^7-125422343554411\,m^8-35687704647714\,m^9-\\
&\quad 387582373725834\,m^{10}-347754396751896\,m^{11}-18602059314264\,m^{12}+85858530196672\,m^{13}+18801525225152\,m^{14}-\\
&\quad 4684896701184\,m^{15}-1292124664320\,m^{16}-75892369920\,m^{17}-6365913600\,m^{18}+6630983680\,m^{19}+1326196736\,m^{20}\big)\big)\\
&h_2(k+1,m)-\\
&\big(2\,k^{13}\,m\,\big(57153600+12700800\,m-349190676\,m^2-77597928\,m^3+\\
&\quad 572100201\,m^4+127133378\,m^5-390874302\,m^6-86860956\,m^7+132652377\,m^8+29478306\,m^9-\\
&\quad 24090768\,m^{10}-5353504\,m^{11}+2364768\,m^{12}+525504\,m^{13}-117504\,m^{14}-26112\,m^{15}+2304\,m^{16}+512\,m^{17}\big)+\\
&\quad k^{12}\,m\,\big(4186501200+948194100\,m-25538030892\,m^2-5783123781\,m^3+41663326662\,m^4+\\
&\quad 9430547986\,m^5-28244472984\,m^6-6387945837\,m^7+9466144974\,m^8+2138079372\,m^9-1687042656\,m^{10}-\\
&\quad 380271008\,m^{11}+161130816\,m^{12}+36210048\,m^{13}-7699968\,m^{14}-1722624\,m^{15}+142848\,m^{16}+31744\,m^{17}\big)+\\
&\quad k^{10}\,m\,\big(822401521200+202857266700\,m-4978700294292\,m^2-1225137924747\,m^3+7958058577794\,m^4+\\
&\quad 1945378416782\,m^5-5203578360344\,m^6-1256078784451\,m^7+1655269215258\,m^8+391155855444\,m^9-\\
&\quad 277778532576\,m^{10}-63493772320\,m^{11}+25629026752\,m^{12}+5588249216\,m^{13}-1340686848\,m^{14}-\\
&\quad 277196544\,m^{15}+40106496\,m^{16}+8004608\,m^{17}-573440\,m^{18}-114688\,m^{19}\big)-4354560\,m\,\big(857304000+\\
&\quad 387147600\,m-4708887480\,m^2-1867089042\,m^3+5943144699\,m^4+1247642767\,m^5-3836730070\,m^6+28385632\,m^7+\\
&\quad 2303417103\,m^8+346584159\,m^9-601935900\,m^{10}-157744364\,m^{11}+41835872\,m^{12}+14978656\,m^{13}+\\
&\quad 2110080\,m^{14}+144000\,m^{15}-263424\,m^{16}-50432\,m^{17}+5120\,m^{18}+1024\,m^{19}\big)-2\,k^{11}\,m\,\big(-38839953600-\\
&\quad 9089496000\,m+236261463876\,m^2+55245594960\,m^3-382539348333\,m^4-89249843896\,m^5+25584384095 8\,m^6+\\
&\quad 59440056968\,m^7-84012342765\,m^8-19382726376\,m^9+14578934136\,m^{10}+3326782472\,m^{11}-1356105056\,m^{12}-\\
&\quad 304270720\,m^{13}+64919040\,m^{14}+14199552\,m^{15}-1418496\,m^{16}-299008\,m^{17}+10240\,m^{18}+2048\,m^{19}\big)-\\
&\quad 6\,k^9\,m\,\big(-1002333301200-273003179220\,m+6007698417384\,m^2+1626097779837\,m^3-\\
&\quad 9347212665621\,m^4-2485305180646\,m^5+5835407834162\,m^6+1497744907045\,m^7-1750678820973\,m^8-\\
&\quad 422844077808\,m^9+283121462424\,m^{10}+62596439432\,m^{11}-27697058656\,m^{12}-5618013056\,m^{13}+\\
&\quad 1759127040\,m^{14}+344153088\,m^{15}-66028800\,m^{16}-13035520\,m^{17}+1034240\,m^{18}+206848\,m^{19}\big)-\\
&\quad 3\,k^8\,m\,\big(-8940690738000-2902847403060\,m+52465183440156\,m^2+16791717388581\,m^3-\\
&\quad 76988664289386\,m^4-23584619594966\,m^5+43560361205536\,m^6+12123420363557\,m^7-11858332127538\,m^8-\\
&\quad 2775246537648\,m^9+1979939261040\,m^{10}+388963564912\,m^{11}-234324203456\,m^{12}-44686944256\,m^{13}+\\
&\quad 17117902848\,m^{14}+3419855616\,m^{15}-597189120\,m^{16}-122040320\,m^{17}+6737920\,m^{18}+1347584\,m^{19}\big)+\\
&\quad 2\,k^7\,m\,\big(52369300720800+23658209379000\,m-290617992125028\,m^2-127632581426574\,m^3+360772426660527\,m^4+
\end{aligned}
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\begin{aligned}
&142690225294948\ m^5 - 154087514472162\ m^6 - 44451308810406\ m^7 + 39065965947423\ m^8 + 7138081703760\ m^9 - \\
&8521359206472\ m^{10} - 1630735928504\ m^{11} + 1070842283808\ m^{12} + 240641191296\ m^{13} - 51976078848\ m^{14} - \\
&12632218368\ m^{15} + 285288192\ m^{16} + 96618496\ m^{17} + 20981760\ m^{18} + 4196352\ m^{19}) + k^6\ m\ (95247003070800 + \\
&79854154328340\ m - 430604909262252\ m^2 - 367658556019677\ m^3 + 169133367246390\ m^4 + 174676429428026\ m^5 + \\
&112890178192008\ m^6 + 72730609617099\ m^7 + 19728504629982\ m^8 + 3115898570244\ m^9 - 7189166828160\ m^{10} - \\
&5492419795648\ m^{11} - 2424891827904\ m^{12} - 86188749696\ m^{13} + 463121155584\ m^{14} + 88138758912\ m^{15} - \\
&19356000768\ m^{16} - 4128369664\ m^{17} + 109240320\ m^{18} + 21848064\ m^{19}) - 2\ k^5\ m\ (-672941242234800 - \\
&1263023864604540\ m + 386068443176412\ m^2 + 663550762873023\ m^3 + 689056114695804\ m^4 + 529909173756616\ m^5 + \\
&94177299258092\ m^6 - 23836566445949\ m^7 - 106838661790524\ m^8 - 65966714498070\ m^9 - 21971201622360\ m^{10} - \\
&1611429593528\ m^{11} + 5460185497856\ m^{12} + 1797974942272\ m^{13} + 135530063616\ m^{14} - 33480428544\ m^{15} - \\
&45662094336\ m^{16} - 8049110528\ m^{17} + 1325066240\ m^{18} + 265013248\ m^{19}) - 5184\ k\ (-1080203040000 + \\
&4728031560000\ m + 5316424470000\ m^2 - 15272058440700\ m^3 - 9062698930776\ m^4 + 13704815798475\ m^5 + \\
&5447665798895\ m^6 - 5158726160790\ m^7 + 119930599974\ m^8 + 2949161853795\ m^9 - 70279662465\ m^{10} - \\
&1237692191100\ m^{11} - 205779763276\ m^{12} + 184330011360\ m^{13} + 43210612320\ m^{14} - 7749098880\ m^{15} - \\
&2160210816\ m^{16} - 143642880\ m^{17} - 16672000\ m^{18} + 9630720\ m^{19} + 1926144\ m^{20}) + 216\ k^3\ (51849745920000 - \\
&33841705942800\ m - 77651998701300\ m^2 - 32147582999652\ m^3 + 5248196223213\ m^4 + 83125341213490\ m^5 + \\
&35133499861522\ m^6 - 27906354511080\ m^7 + 1750520200333\ m^8 + 16486705492154\ m^9 - 1990945011592\ m^{10} - \\
&8560015112464\ m^{11} - 1516158112208\ m^{12} + 1219701796032\ m^{13} + 325248604928\ m^{14} - 22937252864\ m^{15} - \\
&11061996800\ m^{16} - 3585701376\ m^{17} - 602359808\ m^{18} + 122818560\ m^{19} + 24563712\ m^{20}) + 144\ k^2\ (97218273600000 - \\
&207739680372000\ m - 341408576793960\ m^2 + 294673416721596\ m^3 + 283230453260466\ m^4 - 45665824155291\ m^5 - \\
&86292673522087\ m^6 - 76488318744306\ m^7 + 9590019452868\ m^8 + 62620353980325\ m^9 + 21902077369533\ m^{10} - \\
&5480072394516\ m^{11} - 5055043415428\ m^{12} - 2585861007072\ m^{13} - 186701787360\ m^{14} + 421278132096\ m^{15} + \\
&79924253568\ m^{16} - 17723987712\ m^{17} - 3714117376\ m^{18} + 141818880\ m^{19} + 28363776\ m^{20}) - 4\ k^4\ (-699971569920000 - \\
&790397201437200\ m - 1317147125655780\ m^2 + 1098480550109760\ m^3 + 1298053001192769\ m^4 + 372846269500515\ m^5 - \\
&74043859951909\ m^6 - 429800732330858\ m^7 + 86872348543643\ m^8 + 381671201662347\ m^9 + 107779340948049\ m^{10} - \\
&57581803189524\ m^{11} - 32628510470596\ m^{12} - 10443726068384\ m^{13} - 241130560096\ m^{14} + 2094505878912\ m^{15} + \\
&399385291392\ m^{16} - 87011766528\ m^{17} - 18571627264\ m^{18} + 482728960\ m^{19} + 96545792\ m^{20}))) \\
&h_2(k+2,m))/ \\
&(1451520\ (1+k)^2\ (-4+m)\ (-3+m)\ (-2+m)\ (-1+m)\ m\ (1+m)\ (2+m)\ (3+m)\ (4+m)\ (5+m)\ (-9+2\,m)\ (-7+2\,m) \\
&(-5+2\,m)\ (-3+2\,m)\ (-1+2\,m)\ (1+2\,m)\ (3+2\,m)\ (5+2\,m)\ (7+2\,m)\ (9+2\,m))
\end{aligned}
$$

Appendix C. The certificate polynomials of $B_0(m)$

$$
\begin{aligned}
p_0(k,m) = 2(\ &- 2124950729932800000\ k^2 - 12586293967887360000\ k^3 - 3405991268203776000\ k^4 + \\
&30328948861743168000\ k^5 + 15987851881683292800\ k^6 - 25732414446531508800\ k^7 - 15910597606431729600\ k^8 + \\
&9222734774942640000\ k^9 + 6598812735685333800\ k^{10} - 1205224605696453300\ k^{11} - 1230061332511695600\ k^{12} - \\
&50556484783570500\ k^{13} + 83244714131161800\ k^{14} + 24601910976745200\ k^{15} + 2168870393894400\ k^{16} - \\
&1824906284901000\ k^{17} - 494128178296200\ k^{18} + 27243552854700\ k^{19} + 16890164096400\ k^{20} + 1669327663500\ k^{21} - \\
&22598692200\ k^{22} - 49359277800\ k^{23} - 4200789600\ k^{24} - 8499802919731200000\ k\,m - 8082183597239808000\ k^2\,m - \\
&312865120550942208000\ k^3\,m - 5211054253023782400\ k^4\,m + 793761751751409849600\ k^5\,m + 335725729165331202240\ k^6\,m - \\
&66750978685124297664 0\ k^7\,m - 37327661432098406928 0\ k^8\,m + 230702133164152946400\ k^9\,m + \\
&156433267926110770740\ k^{10}\,m - 28520975904780249540\ k^{11}\,m - 28313287167851417880\ k^{12}\,m - \\
&1187882260535467800\ k^{13}\,m + 1795334869592781840\ k^{14}\,m + 531178903010928960\ k^{15}\,m + 53366367942220320\ k^{16}\,m - \\
&37171375237692000\ k^{17}\,m - 10385303649304860\ k^{18}\,m + 496106473177260\ k^{19}\,m + 331273092372120\ k^{20}\,m + \\
&33036072463800\ k^{21}\,m - 318624069960\ k^{22}\,m - 908851172040\ k^{23}\,m - 75740222880\ k^{24}\,m - 8499802919731200000\ m^2 - \\
&1786266684647420000\ k\,m^2 - 120191801423610009600 0\ k^2\,m^2 - 344535351351318720000 0\ k^3\,m^2 + \\
&95049258876884315520 0\ k^4\,m^2 + 937159178771398216608 0\ k^5\,m^2 + 294074971111232106912 0\ k^6\,m^2 - \\
&77581806485983002129 60\ k^7\,m^2 - 383925133436337779328 0\ k^8\,m^2 + 254644292164668462486 0\ k^9\,m^2 + \\
&16211342563660671307 20\ k^{10}\,m^2 - 290336692361457063150\ k^{11}\,m^2 - 281249143058931544440\ k^{12}\,m^2 - \\
&12718054737698368410\ k^{13}\,m^2 + 162256837607790064 20\ k^{14}\,m^2 + 4906689447227277960\ k^{15}\,m^2 + \\
&5830119601942716 00\ k^{16}\,m^2 - 315817644607237080\ k^{17}\,m^2 - 92495076879813480\ k^{18}\,m^2 + 3500322769682370\ k^{19}\,m^2 + \\
&26803825898612 40\ k^{20}\,m^2 + 272848485194550\ k^{21}\,m^2 - 1198342596780\ k^{22}\,m^2 - 6762915824220\ k^{23}\,m^2 - \\
&546560410320\ k^{24}\,m^2 - 1091569360237976 0000\ m^3 - 129221546059984896 0000\ k\,m^3 - 961966305136496056 3200\ k^2\,m^3 - \\
&225330027394738437273 60\ k^3\,m^3 + 137501679850251577228 80\ k^4\,m^3 + 670597976759386276252 80\ k^5\,m^3 + \\
&137158443264122691272 64\ k^6\,m^3 - 540035995576462612866 24\ k^7\,m^3 - 229817426679435947866 08\ k^8\,m^3 + \\
&164508013028865282437 40\ k^9\,m^3 + 970937159282746463567 4\ k^{10}\,m^3 - 166008828679144891007 4\ k^{11}\,m^3 - \\
&158277451282636710322 8\ k^{12}\,m^3 - 83961417829045838100\ k^{13}\,m^3 + 787913515390111022 64\ k^{14}\,m^3 + \\
&25597805330356754496\ k^{15}\,m^3 + 3761173887168808392\ k^{16}\,m^3 - 1454857173942822540\ k^{17}\,m^3 - \\
&459007798909033206\ k^{18}\,m^3 + 11197364095288326\ k^{19}\,m^3 + 11588283146464932\ k^{20}\,m^3 + 1230358419601620\ k^{21}\,m^3 + \\
&331982057720 4\ k^{22}\,m^3 - 26075102464764\ k^{23}\,m^3 - 2009836648368\ k^{24}\,m^3 + 4781442934315560960 00\ m^4 - \\
&254085627085573017600\ k\,m^4 - 465333212795790573 60560\ k^2\,m^4 - 101201014543895961500 160\ k^3\,m^4 + \\
&975727691035505558640 00\ k^4\,m^4 + 329840872991386639989408\ k^5\,m^4 + 330590233728670210092 48\ k^6\,m^4 - \\
&252572214513701815723952\ k^7\,m^4 - 882215792659927286988 20\ k^8\,m^4 + 688070367149855576115 26\ k^9\,m^4 + \\
&367445604476195968776 28\ k^{10}\,m^4 - 569315483673513933578 1\ k^{11}\,m^4 - 542610001939924495437 6\ k^{12}\,m^4 - \\
&380840370127787962151\ k^{13}\,m^4 + 204161287966994341738\ k^{14}\,m^4 + 820155159208059063 20\ k^{15}\,m^4 + \\
&158829693122756810 48\ k^{16}\,m^4 - 3794740033483367868\ k^{17}\,m^4 - 1369989527873843772\ k^{18}\,m^4 + \\
&66658372932695 11\ k^{19}\,m^4 + 27687413084294572\ k^{20}\,m^4 + 3226983486788285\ k^{21}\,m^4 + 41689348430918\ k^{22}\,m^4 - \\
&52046344807938\ k^{23}\,m^4 - 3659751480024\ k^{24}\,m^4 + 194330780726420183040 00\ m^5 + 629237576602285035724 80\ k\,m^5 - \\
&141319376488855204853760\ k^2\,m^5 - 364379913113477111790336\ k^3\,m^5 + 436624100364335260254912\ k^4\,m^5 + \\
&119824299171882188962448 0\ k^5\,m^5 + 495277410141693207851 2\ k^6\,m^5 - 833716854679384071268808\ k^7\,m^5 - \\
&214834572856263058876828\ k^8\,m^5 + 189609758195653578015356\ k^9\,m^5 + 853999085389208481436 33\ k^{10}\,m^5 - \\
&105335545491418694408 68\ k^{11}\,m^5 - 103227867123062809873 30\ k^{12}\,m^5 - 120380745880233570758 3\ k^{13}\,m^5 + \\
&11769132799357586702 6\ k^{14}\,m^5 + 152709301742882608040\ k^{15}\,m^5 + 44990797810400646964\ k^{16}\,m^5 - \\
&42668575320438206 62\ k^{17}\,m^5 - 22667019377831454 79\ k^{18}\,m^5 - 74871181344451784\ k^{19}\,m^5 + 25769234458596434\ k^{20}\,m^5 + \\
&42366419546793 29\ k^{21}\,m^5 + 144105035335948\ k^{22}\,m^5 - 24517555281724\ k^{23}\,m^5 - 684993499712\ k^{24}\,m^5 + \\
&184960957525279194193920\ m^6 + 536412775742670746096640\ k\,m^6 - 279566261854567264060416\ k^2\,m^6 - \\
&129001988419723358821555 2\ k^3\,m^6 + 136210115837366698317100 8\ k^4\,m^6 + 335054013359444673662964 8\ k^5\,m^6 - \\
&299360839484899534657600\ k^6\,m^6 - 192429986509813775676276 48\ k^7\,m^6 - 229999453055036481081340\ k^8\,m^6 + \\
&305202105216833148938123\ k^9\,m^6 + 754580848964444434863 64\ k^{10}\,m^6 - 262239756689949712400\ k^{11}\,m^6 - \\
&903147863243633424678\ k^{12}\,m^6 - 24064247370931919140 26\ k^{13}\,m^6 - 115925391880017117599 2\ k^{14}\,m^6 + \\
&677148761430055599 20\ k^{15}\,m^6 + 79514153998055586604\ k^{16}\,m^6 + 6591604314205536295\ k^{17}\,m^6 - \\
&38740326788552496 0\ k^{18}\,m^6 - 293222660636970168\ k^{19}\,m^6 - 54019355208672298\ k^{20}\,m^6 - 1824812958474160\ k^{21}\,m^6 + \\
&21326088105276 4\ k^{22}\,m^6 + 146507103807728\ k^{23}\,m^6 + 12731027621824\ k^{24}\,m^6 + 955597274967562956472320\ m^7 + \\
&247775451217409876869324 8\ k\,m^7 - 604694089908673155095040\ k^2\,m^7 - 488979263257136790344371 2\ k^3\,m^7 + \\
&319844496344345531697644 8\ k^4\,m^7 + 721471146835056084051041 6\ k^5\,m^7 - 156009884406518853319798 4\ k^6\,m^7 - \\
&262041374393541756947337 6\ k^7\,m^7 + 643274623980292488391544\ k^8\,m^7 + 155819090303711826404 81\ k^9\,m^7 - \\
&285080705416170760554609\ k^{10}\,m^7 + 574394908652605023372 23\ k^{11}\,m^7 + 625216221874879479392 32\ k^{12}\,m^7 - \\
&137398919796397399767 2\ k^{13}\,m^7 - 499792691569060619399 2\ k^{14}\,m^7 - 543043176759499054156\ k^{15}\,m^7 + \\
&470164164622122347362\ k^{16}\,m^7 + 38748191503266405151\ k^{17}\,m^7 + 876742762805825051 3\ k^{18}\,m^7 - \\
&51349581256820752 3\ k^{19}\,m^7 - 250000379535166378\ k^{20}\,m^7 - 20979637781895424\ k^{21}\,m^7 - 61260075731400\ k^{22}\,m^7 + \\
&44799162009385 6\ k^{23}\,m^7 + 33251233684736\ k^{24}\,m^7 + 28754309406282498914641 92\ m^8 + 713910536002499015061964 8\ k\,m^8 - \\
&348750515224973138048563 2\ k^2\,m^8 - 173520956681496289893102 08\ k^3\,m^8 + 652147670743331024743049 6\ k^4\,m^8 + \\
&106512853812453024274035 52\ k^5\,m^8 - 586488655282485055126324 0\ k^6\,m^8 + 946785055181835303074012\ k^7\,m^8 + \\
&438225019857052884542014 8\ k^8\,m^8 - 154002451850108431231055 1\ k^9\,m^8 - 155328475225816233627343 4\ k^{10}\,m^8 + \\
&170450226658821008147280\ k^{11}\,m^8 + 228701667992915903210790\ k^{12}\,m^8 + 962193936906027217600 8\ k^{13}\,m^8 - \\
&111818854583252608218 18\ k^{14}\,m^8 - 203542940466210690993 4\ k^{15}\,m^8 - 194205924850385747070\ k^{16}\,m^8 + \\
&865481588059786828 15\ k^{17}\,m^8 + 270748697376937611 08\ k^{18}\,m^8 - 33363234173831948 6\ k^{19}\,m^8 - \\
&49304612492926866 0\ k^{20}\,m^8 - 49834135252365936\ k^{21}\,m^8 - 896504895696192\ k^{22}\,m^8 + 694949282636896\ k^{23}\,m^8 +
\end{aligned}
$$

$$
\begin{aligned}
&46410104770688\,k^{24}\,m^{8}+3707564047252339600364544\,m^{9}+10762842857426543409523200\,k\,m^{9}-\\
&22045622938300249412365824\,k^{2}\,m^{9}-49846478641564347402946304\,k^{3}\,m^{9}+15420584714955085247507328\,k^{4}\,m^{9}+\\
&25298259025982266571190680\,k^{5}\,m^{9}-18990245578928519343338704\,k^{6}\,m^{9}+17879552528734479711868940\,k^{7}\,m^{9}+\\
&14381035322031703891308720\,k^{8}\,m^{9}-53164837761412550186973011\,k^{9}\,m^{9}-42923831975384123387511790\,k^{10}\,m^{9}+\\
&23815668527720069847414\,k^{11}\,m^{9}+49939962469076364870310 6\,k^{12}\,m^{9}+42025646221324240818588\,k^{13}\,m^{9}-\\
&15648482409125499173882\,k^{14}\,m^{9}-42823702306062411211 62\,k^{15}\,m^{9}-75325375519772067633 8\,k^{16}\,m^{9}+\\
&12168206295848250666 5\,k^{17}\,m^{9}+4891235017500704733 6\,k^{18}\,m^{9}+539573317076471128\,k^{19}\,m^{9}-\\
&627038246779043656\,k^{20}\,m^{9}-73078179455523456\,k^{21}\,m^{9}-1929389248407424\,k^{22}\,m^{9}+701222259494208\,k^{23}\,m^{9}+\\
&41904407491584\,k^{24}\,m^{9}-74988172132081326663874 56\,m^{10}-10699249183901067045623040\,k\,m^{10}-\\
&10494033177414837070525209 6\,k^{2}\,m^{10}-10713920767335163958286873 6\,k^{3}\,m^{10}+48280446350767500325242656\,k^{4}\,m^{10}-\\
&48142300340259750472774432\,k^{5}\,m^{10}-52071585231102316395967872\,k^{6}\,m^{10}+62580878037377788915559020\,k^{7}\,m^{10}+\\
&33968443327126482141902132\,k^{8}\,m^{10}-11383603395491854164443514\,k^{9}\,m^{10}-84317175499372222267956 86\,k^{10}\,m^{10}+\\
&24302828282017111687918\,k^{11}\,m^{10}+77349248758539578402688 6\,k^{12}\,m^{10}+10110667528354954457762 66\,k^{13}\,m^{10}-\\
&11804426060599833073206\,k^{14}\,m^{10}-64124777827660918786 74\,k^{15}\,m^{10}-15340315852478152193 62\,k^{16}\,m^{10}+\\
&11393163203632262201 36\,k^{17}\,m^{10}+62910881635131636852\,k^{18}\,m^{10}+1786609695746372576\,k^{19}\,m^{10}-\\
&5585412246682933136\,k^{20}\,m^{10}-75771621976066944\,k^{21}\,m^{10}-2396653662739520\,k^{22}\,m^{10}+489195116534784\,k^{23}\,m^{10}+\\
&25585130452992\,k^{24}\,m^{10}-44083901965463232176498688\,m^{11}-12647360792085648018142 6944\,k\,m^{11}-\\
&39225823543081932942836582 4\,k^{2}\,m^{11}-16024016018533966793616614 4\,k^{3}\,m^{11}+15652854384722007804372544 0\,k^{4}\,m^{11}-\\
&19657698054312436344129369 6\,k^{5}\,m^{11}-11511754843776840988930131 2\,k^{6}\,m^{11}+14873884651366784026518086 0\,k^{7}\,m^{11}+\\
&62088436574927607019291844\,k^{8}\,m^{11}-18079674433213122075248168\,k^{9}\,m^{11}-12670955694546555089546384\,k^{10}\,m^{11}-\\
&68914112124000624455837 6\,k^{11}\,m^{11}+87329705734567686707 0944\,k^{12}\,m^{11}+17347691455896762998729 6\,k^{13}\,m^{11}+\\
&34768132560945270829 20\,k^{14}\,m^{11}-72571879827157157318 92\,k^{15}\,m^{11}-21975786426446616726 76\,k^{16}\,m^{11}+\\
&62521645731427627912\,k^{17}\,m^{11}+61031980444223421576\,k^{18}\,m^{11}+2616932736803482208\,k^{19}\,m^{11}-\\
&352700439880949984\,k^{20}\,m^{11}-57823688640361728\,k^{21}\,m^{11}-1978665424996480\,k^{22}\,m^{11}+\\
&238803892771840\,k^{23}\,m^{11}+10525174956032\,k^{24}\,m^{11}-61270191792446786191885824\,m^{12}-\\
&46496056446889479786753331 2\,k\,m^{12}-12023883449771589368777180800\,k^{2}\,m^{12}-14208723930411717447446428352\,k^{3}\,m^{12}+\\
&43244645038881012359347491 2\,k^{4}\,m^{12}-49419906434029622470048782 4\,k^{5}\,m^{12}-19495086331536902591822703 2\,k^{6}\,m^{12}+\\
&27614599943033101290801821 6\,k^{7}\,m^{12}+88085710533943618245686952\,k^{8}\,m^{12}-22473397076017012352325920\,k^{9}\,m^{12}-\\
&14714006738090250039329376\,k^{10}\,m^{12}-18318005078592485469777 36\,k^{11}\,m^{12}+67387703580990833862960 8\,k^{12}\,m^{12}+\\
&22795526734415594422972 8\,k^{13}\,m^{12}+24856801899842226471592\,k^{14}\,m^{12}-62810772310628807111 84\,k^{15}\,m^{12}-\\
&23874948320704779597 76\,k^{16}\,m^{12}-292505042497328256\,k^{17}\,m^{12}+45575617541059471936\,k^{18}\,m^{12}+\\
&2532956794808898496\,k^{19}\,m^{12}-151750656800748736\,k^{20}\,m^{12}-32744155349599232\,k^{21}\,m^{12}-\\
&1122552428664832\,k^{22}\,m^{12}+80258400673792\,k^{23}\,m^{12}+2799584116736\,k^{24}\,m^{12}+18368412818142800251307520 0\,m^{13}-\\
&11648734286499809815262976 00\,k\,m^{13}-30767591580779781134127244 80\,k^{2}\,m^{13}-51064187968758450237955392\,k^{3}\,m^{13}+\\
&95907572110911171975952384 0\,k^{4}\,m^{13}-91557972361047868071151025 6\,k^{5}\,m^{13}-22788194611402220461419868 8\,k^{6}\,m^{13}+\\
&41817301108550038459918660 8\,k^{7}\,m^{13}+92106465592923379312939360\,k^{8}\,m^{13}-22295795280669697493823296\,k^{9}\,m^{13}-\\
&12655962458237795834989200\,k^{10}\,m^{13}-29501551062693228147326 4\,k^{11}\,m^{13}+22813288757859368784155 2\,k^{12}\,m^{13}+\\
&23607484154415748691788 8\,k^{13}\,m^{13}+41288304809854885825360\,k^{14}\,m^{13}-40653399083262906981 44\,k^{15}\,m^{13}-\\
&20249808851718108142 24\,k^{16}\,m^{13}-39362976463824234160\,k^{17}\,m^{13}+26266037732295046912\,k^{18}\,m^{13}+\\
&1765014477925398528\,k^{19}\,m^{13}-37752819304990592\,k^{20}\,m^{13}-13612834309074176\,k^{21}\,m^{13}-\\
&434143369211904\,k^{22}\,m^{13}+17710498758656\,k^{23}\,m^{13}+435095339008\,k^{24}\,m^{13}+12469235129383848079458134528\,m^{14}-\\
&22021678502272688085436377 60\,k\,m^{14}-65977846377290487491091737 6\,k^{2}\,m^{14}-17797353726934173190170643 2\,k^{3}\,m^{14}+\\
&16699658789042233520473213472\,k^{4}\,m^{14}-12902716932937168036914992 32\,k^{5}\,m^{14}-\\
&10322231662379121046699142 4\,k^{6}\,m^{14}+52506165642311773619627312\,k^{7}\,m^{14}+56487053335439410371872688\,k^{8}\,m^{14}-\\
&17770509280085617096928528\,k^{9}\,m^{14}-66272946114091523253020 96\,k^{10}\,m^{14}-34817756753054004303590 88\,k^{11}\,m^{14}-\\
&24990092514035732225576 0\,k^{12}\,m^{14}+19477280294246734055176 0\,k^{13}\,m^{14}+44804392659408040855920\,k^{14}\,m^{14}-\\
&17992482043547349517 12\,k^{15}\,m^{14}-13555560720744531263 04\,k^{16}\,m^{14}-44863973011097963776\,k^{17}\,m^{14}+\\
&11560874281539147904\,k^{18}\,m^{14}+907198024021279232\,k^{19}\,m^{14}-453416074391808\,k^{20}\,m^{14}-\\
&4036160486916096\,k^{21}\,m^{14}-109570390769664\,k^{22}\,m^{14}+2310506283008\,k^{23}\,m^{14}+30006575104\,k^{24}\,m^{14}+\\
&38268472787958523306469898 24\,m^{15}-31191293389625471732603146 8\,k\,m^{15}-11830917718849019569734212352\,k^{2}\,m^{15}-\\
&12428842330685065555224150464\,k^{3}\,m^{15}+22001811988237730277187962 88\,k^{4}\,m^{15}-\\
&13425860352229906611207326 08\,k^{5}\,m^{15}+27153586660444010235499059 2\,k^{6}\,m^{15}+\\
&54983428273329996860680464 0\,k^{7}\,m^{15}-17501284125906145055008192\,k^{8}\,m^{15}-11482305751343602373861664\,k^{9}\,m^{15}+\\
&88840293016125165586912 0\,k^{10}\,m^{15}-31848970684707929661762 88\,k^{11}\,m^{15}-54291728393438196500844 8\,k^{12}\,m^{15}+\\
&12786319574448361657340 8\,k^{13}\,m^{15}+36259003706798199148608\,k^{14}\,m^{15}-34083221547557603884 8\,k^{15}\,m^{15}-\\
&71653596275989443609 6\,k^{16}\,m^{15}-30793590166424501248\,k^{17}\,m^{15}+3788332156929971 20\,k^{18}\,m^{15}+\\
&342945126280275200\,k^{19}\,m^{15}+3468435123957760\,k^{20}\,m^{15}-807182459854848\,k^{21}\,m^{15}-16280442404864\,k^{22}\,m^{15}+\\
&13502958796 8\,k^{23}\,m^{15}+81527127180767787866031682 56\,m^{16}-28909292738993326825249443 84\,k\,m^{16}-\\
&17631910606713012615679641216\,k^{2}\,m^{16}-40904307076320060659706727 68\,k^{3}\,m^{16}+\\
&19423081863913493541328400 92\,k^{4}\,m^{16}-87695695522269974139620243 2\,k^{5}\,m^{16}+87219624162709806831088768\,k^{6}\,m^{16}+\\
&48040163852698983963962816 00\,k^{7}\,m^{16}-10363410468649238751696678 4\,k^{8}\,m^{16}-63266126594657396956154 88\,k^{9}\,m^{16}+\\
&66359482713014582985308 16\,k^{10}\,m^{16}-23001387323209772369423 36\,k^{11}\,m^{16}-57835102230184173726105 6\,k^{12}\,m^{16}+\\
&65829134160849439752256\,k^{13}\,m^{16}+22812401105288927977536\,k^{14}\,m^{16}+21359648302397514419 2\,k^{15}\,m^{16}-\\
&29671740022451935616 0\,k^{16}\,m^{16}-14867071159919658240\,k^{17}\,m^{16}+879817131495475200\,k^{18}\,m^{16}+\\
&93019912084898304\,k^{19}\,m^{16}+1296632184500224\,k^{20}\,m^{16}-97461355937792\,k^{21}\,m^{16}-1080236703744\,k^{22}\,m^{16}+\\
&13294723891641200666381168640\,m^{17}-29025550898261087561253273 6\,k\,m^{17}-21567946213267279820289156096\,k^{2}\,m^{17}-\\
&89130432064711803296167027 20\,k^{3}\,m^{17}+43790215078527215514823091 2\,k^{4}\,m^{17}+23671668713263533891850368\,k^{5}\,m^{17}+\\
&15166905578982189348265108 48\,k^{6}\,m^{17}+34912200574061732687797555 2\,k^{7}\,m^{17}-16524086965498022321939609 6\,k^{8}\,m^{17}-\\
&35498090365535119735275 52\,k^{9}\,m^{17}+87564308074082355346511 36\,k^{10}\,m^{17}-13093790175868619992212 48\,k^{11}\,m^{17}-
\end{aligned}
$$

$$438029637655765747314176\,k^{12}\,m^{17} + 25595578376986875527808\,k^{13}\,m^{17} + 11307750173938466699264\,k^{14}\,m^{17} +$$
$$243778279901350127616\,k^{15}\,m^{17} - 94647054457185854976\,k^{16}\,m^{17} - 5205282159247619328\,k^{17}\,m^{17} +$$
$$130252551246923776\,k^{18}\,m^{17} + 17160138353707008\,k^{19}\,m^{17} + 220573645062144\,k^{20}\,m^{17} - 5359924477952\,k^{21}\,m^{17} +$$
$$171907530414929972216505492 48\,m^{18} + 51945055091682426612864000 00\,k\,m^{18} - 21063978007677908719578938880\,k^{2}\,m^{18} -$$
$$14612579904129494077887205632\,k^{3}\,m^{18} - 21297795474535706308403207 68\,k^{4}\,m^{18} +$$
$$99400611287421311546538598 4\,k^{5}\,m^{18} + 19484567994525162086559877 12\,k^{6}\,m^{18} +$$
$$20982373225405525813805491 2\,k^{7}\,m^{18} - 17966322133920713881917465 6\,k^{8}\,m^{18} - 25607762352396428616829 44\,k^{9}\,m^{18} +$$
$$765273897935002199122355 2\,k^{10}\,m^{18} - 57434925736155094156518 4\,k^{11}\,m^{18} - 25524329131650041840716 8\,k^{12}\,m^{18} +$$
$$6780908625346030400512\,k^{13}\,m^{18} + 44121462780650550231 04\,k^{14}\,m^{18} + 13015286277774901401 6\,k^{15}\,m^{18} -$$
$$22596578053753330176\,k^{16}\,m^{18} - 1304945135889049600\,k^{17}\,m^{18} + 8817987719619584\,k^{18}\,m^{18} +$$
$$1929174287237120\,k^{19}\,m^{18} + 15374618918912\,k^{20}\,m^{18} + 17778571548641994343472188416\,m^{19} +$$
$$12508899331142944941158200320\,k\,m^{19} - 15269529731456768842388441088\,k^{2}\,m^{19} -$$
$$19087373659643213085000325632\,k^{3}\,m^{19} - 48403117285537980727166259 20\,k^{4}\,m^{19} +$$
$$16207051203545802128087 48032\,k^{5}\,m^{19} + 19970800052206574004886548 64\,k^{6}\,m^{19} +$$
$$10358775281315352124338 5344\,k^{7}\,m^{19} - 15109707918763847042133196 8\,k^{8}\,m^{19} - 21611876031420007762129 92\,k^{9}\,m^{19} +$$
$$50841698364415350073681 92\,k^{10}\,m^{19} - 18176081018254467761920 0\,k^{11}\,m^{19} - 11723895436868110471372 8\,k^{12}\,m^{19} +$$
$$736538220982278888448\,k^{13}\,m^{19} + 1339623556437329670144\,k^{14}\,m^{19} + 45848497936936307712\,k^{15}\,m^{19} -$$
$$3850269494191639552\,k^{16}\,m^{19} - 223177955120609280\,k^{17}\,m^{19} - 367220348909568\,k^{18}\,m^{19} +$$
$$99726852358144\,k^{19}\,m^{19} + 14492045408405933323 2283140096\,m^{20} + 19170332629059765227801287680\,k\,m^{20} -$$
$$60707113399080382916687063 04\,k^{2}\,m^{20} - 20467098309714215807107688448\,k^{3}\,m^{20} -$$
$$65598226224107071723334717 44\,k^{4}\,m^{20} + 17144831711385636498986792 96\,k^{5}\,m^{20} +$$
$$16830613851658525555728923648\,k^{6}\,m^{20} + 41941362402555477267085312\,k^{7}\,m^{20} - 10241727087726981888343449 6\,k^{8}\,m^{20} -$$
$$16613484072932282770165 76\,k^{9}\,m^{20} + 26736766551179643917947 68\,k^{10}\,m^{20} - 32569162250962116651008\,k^{11}\,m^{20} -$$
$$42594439200462377613312\,k^{12}\,m^{20} - 30005419090514491392\,k^{13}\,m^{20} + 30935109201723812659 2\,k^{14}\,m^{20} +$$
$$11167669863962771456\,k^{15}\,m^{20} - 431012455360008192\,k^{16}\,m^{20} - 23386065497726976\,k^{17}\,m^{20} - 77042819284992\,k^{18}\,m^{20} +$$
$$87758942934348680294554214 40\,m^{21} + 22603306359724085337181968384\,k\,m^{21} + 28838508133389980297728204 80\,k^{2}\,m^{21} -$$
$$18322427900949974607683727360\,k^{3}\,m^{21} - 66847331635773151051073080320\,k^{4}\,m^{21} +$$
$$13859198612697714836434350 08\,k^{5}\,m^{21} + 11821148568425027928143912 96\,k^{6}\,m^{21} + 14281550952547230000001024\,k^{7}\,m^{21} -$$
$$56843479356925456527159296\,k^{8}\,m^{21} - 10405442293259833876316 16\,k^{9}\,m^{21} + 11263700812240378566451 20\,k^{10}\,m^{21} +$$
$$2907394873739951181824\,k^{11}\,m^{21} - 12128540250183095889920\,k^{12}\,m^{21} - 188285127426651025408\,k^{13}\,m^{21} +$$
$$52277086938291556352\,k^{14}\,m^{21} + 1828446915124346880\,k^{15}\,m^{21} - 26886933787242496\,k^{16}\,m^{21} -$$
$$1134123120824320\,k^{17}\,m^{21} + 30907643146453970929845780 48\,m^{22} + 21711033786634211267033296896\,k\,m^{22} +$$
$$84677575583050964190169620 48\,k^{2}\,m^{22} - 13826669426266450216154578944\,k^{3}\,m^{22} -$$
$$54613045130718194175100272 64\,k^{4}\,m^{22} + 89990502378520694809742 9504\,k^{5}\,m^{22} + 69569277690723089415122 5344\,k^{6}\,m^{22} +$$
$$45422967422535421441679 36\,k^{7}\,m^{22} - 25946458535485960203481088\,k^{8}\,m^{22} - 51689035199301678514176\,k^{9}\,m^{22} +$$
$$37951646650174969340313 6\,k^{10}\,m^{22} + 4431050713784317849600\,k^{11}\,m^{22} - 26507090076482866667 52\,k^{12}\,m^{22} -$$
$$53421572437671882752\,k^{13}\,m^{22} + 605509168565943910 4\,k^{14}\,m^{22} + 182323089300971520\,k^{15}\,m^{22} -$$
$$600861505789952\,k^{16}\,m^{22} - 69127904573171435103569510 4\,m^{23} + 17407183768687093889175478272\,k\,m^{23} +$$
$$97723560406166649932307333 12\,k^{2}\,m^{23} - 88401133693345667389184409 60\,k^{3}\,m^{23} -$$
$$36701181068842057296961372 16\,k^{4}\,m^{23} + 47856915516547452927858278 4\,k^{5}\,m^{23} +$$
$$34319257315687126583894016 0\,k^{6}\,m^{23} + 16323997558965459270942 72\,k^{7}\,m^{23} - 97131686747175214695219 20\,k^{8}\,m^{23} -$$
$$20213728610023918451916 8\,k^{9}\,m^{23} + 10110782668866055970816 0\,k^{10}\,m^{23} + 16915816985559810785 28\,k^{11}\,m^{23} -$$
$$42884805659676959217 92\,k^{12}\,m^{23} - 9134839528890408960\,k^{13}\,m^{23} + 425010647666671616\,k^{14}\,m^{23} +$$
$$8401249336967168\,k^{15}\,m^{23} - 21556969096707109653529559 04\,m^{24} + 11785737862439636671929434112\,k\,m^{24} +$$
$$80176663563556862538474946 56\,k^{2}\,m^{24} - 47968759334934904901583152742 40\,k^{3}\,m^{24} -$$
$$20512850252262044894459330 56\,k^{4}\,m^{24} + 20982011410957571013496012 8\,k^{5}\,m^{24} +$$
$$14141322767373818983094681 6\,k^{6}\,m^{24} + 68405329706804271126937 6\,k^{7}\,m^{24} - 29560573331926058723901 44\,k^{8}\,m^{24} -$$
$$61659677720240413507584\,k^{9}\,m^{24} + 20839965095916498780160\,k^{10}\,m^{24} + 388481758008195219456\,k^{11}\,m^{24} -$$
$$48280814089141551104\,k^{12}\,m^{24} - 910334474721869824\,k^{13}\,m^{24} + 13382468332175360\,k^{14}\,m^{24} -$$
$$20307475346493298676797006 720\,m^{25} + 67730837939376456090705854144\,k\,m^{25} + 51877602778843839483085455 36\,k^{2}\,m^{25} -$$
$$22069094966213279954791956 48\,k^{3}\,m^{25} - 95681264690994873589766553 6\,k^{4}\,m^{25} + 75777732073498863155412992\,k^{5}\,m^{25} +$$
$$48319187686916850072682496\,k^{6}\,m^{25} + 27385617667599126914662 4\,k^{7}\,m^{25} - 72024193374764579645030 4\,k^{8}\,m^{25} -$$
$$14408820757133754040320\,k^{9}\,m^{25} + 3205426713701968183296\,k^{10}\,m^{25} + 57302881635546038272\,k^{11}\,m^{25} -$$
$$3365322744114839552\,k^{12}\,m^{25} - 41052683675598848\,k^{13}\,m^{25} - 13147355172775306447698001 92\,m^{26} +$$
$$33070824622114891408675307 52\,k\,m^{26} + 27418985103662887105126072 32\,k^{2}\,m^{26} - 85780231899664592320292454 4\,k^{3}\,m^{26} -$$
$$37183843107446117822142873 6\,k^{4}\,m^{26} + 22395761722837615760244736\,k^{5}\,m^{26} + 13533972013961217163264000\,k^{6}\,m^{26} +$$
$$89297564382204676472832\,k^{7}\,m^{26} - 13716388678892367033139 2\,k^{8}\,m^{26} - 24969655059806263460 96\,k^{9}\,m^{26} +$$
$$34614933484057922560\,k^{10}\,m^{26} + 5057229562053853184\,k^{11}\,m^{26} - 108809583835414528\,k^{12}\,m^{26} -$$
$$65759187242667491784681062 4\,m^{27} + 13688785261915987178898063 36\,k\,m^{27} + 11995329030113806059786731 52\,k^{2}\,m^{27} -$$
$$27992739282087895974412288 0\,k^{3}\,m^{27} - 11966732527658226968900403 2\,k^{4}\,m^{27} + 53499815640415161900072 96\,k^{5}\,m^{27} +$$
$$30549313654673849985269 76\,k^{6}\,m^{27} + 22114040393088505872384\,k^{7}\,m^{27} - 19667263273593299051520\,k^{8}\,m^{27} -$$
$$3028759979171836067 84\,k^{9}\,m^{27} + 2339032620259160883 2\,k^{10}\,m^{27} + 204956663036116992\,k^{11}\,m^{27} -$$
$$26334439156157368508979609 6\,m^{28} + 47783694924370147954891161 6\,k\,m^{28} + 43566123105076329458879692 8\,k^{2}\,m^{28} -$$

$$
\begin{aligned}
&759749695577735221283389 44\ k^3\ m^{28} - 3155344983130083286056960 0\ k^4\ m^{28} + 1013124586998978923200512\ k^5\ m^{28} + \\
&541933798210813732061184\ k^6\ m^{28} + 3983851766008655118336\ k^7\ m^{28} - 1996114963273645817856\ k^8\ m^{28} - \\
&22985309079385669632\ k^9\ m^{28} + 743197360972627968\ k^{10}\ m^{28} - 85313228171209651812040704\ m^{29} + \\
&139481681990655499993350144\ k\ m^{29} + 1308809226809971829965 2096\ k^2\ m^{29} - 1692125769236065858722201 6\ k^3\ m^{29} - \\
&670527946580218472300544 0\ k^4\ m^{29} + 1476676321729477623152 64\ k^5\ m^{29} + 7272675317657148850176 0\ k^6\ m^{29} + \\
&493265512916991344640\ k^7\ m^{29} - 127773628846939570176\ k^8\ m^{29} - 822334511671934976\ k^9\ m^{29} - \\
&223059049049384378915880 96\ m^{30} + 3362688987855731964562636 8\ k\ m^{30} + 3222018729804493893402624 0\ k^2\ m^{30} - \\
&303459739795272066347827 2\ k^3\ m^{30} - 1120393313358231738777600\ k^4\ m^{30} + 158247532703166360453 12\ k^5\ m^{30} + \\
&693815047460523343872 0\ k^6\ m^{30} + 37644363032193662976\ k^7\ m^{30} - 3876554718842978304\ k^8\ m^{30} - \\
&465190304573272373172623 6\ m^{31} + 6577734892619308046745600\ k\ m^{31} + 6398571591717348175773696\ k^2\ m^{31} - \\
&426445376233556271955968\ k^3\ m^{31} - 1417370754236233663166208\ k^4\ m^{31} + 1155733226728263254016\ k^5\ m^{31} + \\
&419137689454186070016\ k^6\ m^{31} + 1338542904343265280\ k^7\ m^{31} - 7571311708494943066521 60\ m^{32} + \\
&101750394448990475963596 8\ k\ m^{32} + 1000501829028142022393856\ k^2\ m^{32} - 4508757236549846827008 0\ k^3\ m^{32} - \\
&127603927242919378944 00\ k^4\ m^{32} + 49863607418785628160\ k^5\ m^{32} + 12047968536266539008\ k^6\ m^{32} - \\
&927694006226427834464 00\ m^{33} + 1197560393200096342179 84\ k\ m^{33} + 1186367638839931416084 48\ k^2\ m^{33} - \\
&335932412838898198118 4\ k^3\ m^{33} - 728387604547576528896\ k^4\ m^{33} + 894474879221366784\ k^5\ m^{33} - \\
&805308017154195456000 0\ m^{34} + 1007380156308085171814 4\ k\ m^{34} + 1002900027540824181964 8\ k^2\ m^{34} - \\
&156392599636156612608\ k^3\ m^{34} - 19812228356261726208\ k^4\ m^{34} - 4416224037921965998 08\ m^{35} + \\
&539329502537670721536\ k\ m^{35} + 538536643364697145344\ k^2\ m^{35} - 339496791252664320 0\ k^3\ m^{35} - \\
&115047801516824985 60\ m^{36} + 1380573618201899827 2\ k\ m^{36} + 1380573618201899827 2\ k^2\ m^{36})
\end{aligned}
$$

$$
\begin{aligned}
p_1(k,m) = -(-1+k+2\,m)\ &(6374852189798400000\ k^2 + 44133734093460480000\ k^3 + 54351707898071808000\ k^4 - 36635138687157696000\ k^5 - \\
&84598694332207574400\ k^6 - 7401450992613048000\ k^7 + 40330341826682140800\ k^8 + 12662137501854220800\ k^9 - \\
&71343007052017806 00\ k^{10} - 3518626888112420700\ k^{11} + 171557109422666100\ k^{12} + 323226563773377600\ k^{13} + \\
&73492421379892200\ k^{14} - 313311550343400\ k^{15} - 6819922732026600\ k^{16} - 1345203877323600\ k^{17} + 137180657565000\ k^{18} + \\
&55449999000900\ k^{19} + 4779506711700\ k^{20} - 228476278800\ k^{21} - 160680202200\ k^{22} - 12602368800\ k^{23} + \\
&21249507299328000 0\ k\,m + 2371175370060134400 00\ k^2\,m + 1351076001922446336000\ k^3\,m + 1489324895963715571200\ k^4\,m - \\
&10846856936640988800 00\ k^5\,m - 2249030643473278278720\ k^6\,m - 183465942664995806400\ k^7\,m + \\
&10114710618691575230 40\ k^8\,m + 3186121338173071490 40\ k^9\,m - 1656005299987284775 80\ k^{10}\,m - \\
&82873900251540525960\ k^{11}\,m + 297155578412605788 0\ k^{12}\,m + 696717919178435808 0\ k^{13}\,m + 1638923941472585760\ k^{14}\,m + \\
&28709406497418480\ k^{15}\,m - 139760743268299680\ k^{16}\,m - 28863175350383280\ k^{17}\,m + 2524914768716100\ k^{18}\,m + \\
&10869060567265 20\ k^{19}\,m + 95911693927560\ k^{20}\,m - 3760518363840\ k^{21}\,m - 2945372605560\ k^{22}\,m - \\
&22722066864 0\ k^{23}\,m + 4249901459865600000\ m^2 - 1297700096781312000 00\ k\,m^2 + 3585579949259094528000\ k^2\,m^2 + \\
&19047558502579194547200\ k^3\,m^2 + 1903023792707701956480 0\ k^4\,m^2 - 1430015803683071520096 0\ k^5\,m^2 - \\
&26936795133617319947520\ k^6\,m^2 - 2126734417978485901920\ k^7\,m^2 + 1117276228325943481680 0\ k^8\,m^2 + \\
&35487193729666202911 80\ k^9\,m^2 - 1648844600970507363900\ k^{10}\,m^2 - 8460282310618466056 50\ k^{11}\,m^2 + \\
&15737279402806312230\ k^{12}\,m^2 + 63051590366744894100\ k^{13}\,m^2 + 15739247046268402200\ k^{14}\,m^2 + \\
&723887377001928000\ k^{15}\,m^2 - 1196314501624785360\ k^{16}\,m^2 - 263474242120564200\ k^{17}\,m^2 + 18309934141327080\ k^{18}\,m^2 + \\
&8787202691331330\ k^{19}\,m^2 + 805365879499530\ k^{20}\,m^2 - 22830477792120\ k^{21}\,m^2 - 21776948257860\ k^{22}\,m^2 - \\
&16396812309 60\ k^{23}\,m^2 - 1516120328295014400 00\ m^3 - 5267335111548088320000\ k\,m^3 + 2913864320773621463040 0\ k^2\,m^3 + \\
&16500777437218564076544 0\ k^3\,m^3 + 1525789470393168979276 80\ k^4\,m^3 - 1116847075700422424995 20\ k^5\,m^3 - \\
&19371232878515807075683 2\ k^6\,m^3 - 1546558145755659624096 0\ k^7\,m^3 + 7187842110335858913974 4\ k^8\,m^3 + \\
&23134285439027310639564\ k^9\,m^3 - 9183894532962927221538\ k^{10}\,m^3 - 4916687470476152645376\ k^{11}\,m^3 - \\
&25162016614128247752\ k^{12}\,m^3 + 3083062953735074679 48\ k^{13}\,m^3 + 859247008661298794 76\ k^{14}\,m^3 + \\
&71827857172616570 28\ k^{15}\,m^3 - 5561742316619776788\ k^{16}\,m^3 - 1345073916497404248\ k^{17}\,m^3 + \\
&6370387092801045 0\ k^{18}\,m^3 + 37941832453526172\ k^{19}\,m^3 + 3700070885276316\ k^{20}\,m^3 - 54816720818544\ k^{21}\,m^3 - \\
&83161696518756\ k^{22}\,m^3 - 6029509945104\ k^{23}\,m^3 - 6966487333287475200000\ m^4 - 8323611335897622128640 0\ k\,m^4 + \\
&13288584593886324456960 0\ k^2\,m^4 + 9857742998789965481164 80\ k^3\,m^4 + 8635206482666890099113 60\ k^4\,m^4 - \\
&57733033755023571347337 6\ k^5\,m^4 - 9327120107615813395416 00\ k^6\,m^4 - 7742765087488314093187 2\ k^7\,m^4 + \\
&29601734768753519358015 6\ k^8\,m^4 + 9672982664236053780515 4\ k^9\,m^4 - 3035984146034128622518 2\ k^{10}\,m^4 - \\
&17536032795306029081559\ k^{11}\,m^4 - 715740930796852239615\ k^{12}\,m^4 + 8208875688556653208 62\ k^{13}\,m^4 + \\
&29058733436880 79744 84\ k^{14}\,m^4 + 39721127214452315232\ k^{15}\,m^4 - 14695868161778922048\ k^{16}\,m^4 - \\
&41496857995180010 52\ k^{17}\,m^4 + 77073256447875588\ k^{18}\,m^4 + 90441127048198431\ k^{19}\,m^4 + 9908302919925987\ k^{20}\,m^4 + \\
&2984551785625 2\ k^{21}\,m^4 - 163098615567150\ k^{22}\,m^4 - 10979254440072\ k^{23}\,m^4 - 1115729474146914281472 00\ m^5 - \\
&78698680503390972186624 0\ k\,m^5 + 2421482046284862341683 20\ k^2\,m^5 + 4301704989474469521675264\ k^3\,m^5 + \\
&36277501110498748925228 16\ k^4\,m^5 - 2075170485485284821367008\ k^5\,m^5 - 3122252867583403070997136\ k^6\,m^5 - \\
&26279572289246625366386 4\ k^7\,m^5 + 7807006287568288416718 20\ k^8\,m^5 + 2561892924596819238509 20\ k^9\,m^5 - \\
&51535463969201739455075\ k^{10}\,m^5 - 3554375700365569836265 7\ k^{11}\,m^5 - 4027218837848689846629\ k^{12}\,m^5 + \\
&63568911536804137519 6\ k^{13}\,m^5 + 5852172395083699319 74\ k^{14}\,m^5 + 1356475337733176899 10\ k^{15}\,m^5 - \\
&17125133824387976710\ k^{16}\,m^5 - 7205756007076173184\ k^{17}\,m^5 - 218934193893042143\ k^{18}\,m^5 + 83557511993011883\ k^{19}\,m^5 + \\
&13401025478385263\ k^{20}\,m^5 + 488638473852756\ k^{21}\,m^5 - 68288143384260\ k^{22}\,m^5 - 2054980499136\ k^{23}\,m^5 - \\
&10472399512917488271360 00\ m^6 - 5134309753634903318123520\ k\,m^6 - 9882348998801267881216 0\ k^2\,m^6 +
\end{aligned}
$$

$$
\begin{aligned}
&1408151490783727065719040\,0\,k^{3}\,m^{6}+1137647011943665583628000\,0\,k^{4}\,m^{6}-5234971757143690631599664\,k^{5}\,m^{6}-\\
&7004956009335727390906352\,k^{6}\,m^{6}-4806725127143189355093600\,k^{7}\,m^{6}+1072555786902497424289948\,k^{8}\,m^{6}+\\
&3189774906959459297702950\,k^{9}\,m^{6}+2111388928889093310910\,1\,k^{10}\,m^{6}-1065886764539613647232\,5\,k^{11}\,m^{6}-\\
&1193506688117474246117\,7\,k^{12}\,m^{6}-3775341834397969595987\,k^{13}\,m^{6}+3897198679461211784\,65\,k^{14}\,m^{6}+\\
&2825388659777447278\,05\,k^{15}\,m^{6}+2340046789556632883\,7\,k^{16}\,m^{6}-2127468584945585108\,k^{17}\,m^{6}-\\
&1124101543567273810\,k^{18}\,m^{6}-177348092458105348\,k^{19}\,m^{6}-4857116462417924\,k^{20}\,m^{6}+1145474058166748\,k^{21}\,m^{6}+\\
&479084381288080\,k^{22}\,m^{6}+38193082865472\,k^{23}\,m^{6}-67053724861821229729382\,40\,m^{7}-2485111785132033025532\,0064\,k\,m^{7}-\\
&9427389428977058496079872\,k^{2}\,m^{7}+3456133651315669391168486\,4\,k^{3}\,m^{7}+2527212496952514613578713\,6\,k^{4}\,m^{7}-\\
&8855606106282132351192016\,k^{5}\,m^{7}-7712454766363838659332240\,k^{6}\,m^{7}+5344492357316736892970\,56\,k^{7}\,m^{7}-\\
&8450112579904240927077\,88\,k^{8}\,m^{7}-5838074063407960719046\,85\,k^{9}\,m^{7}+3738510021664914201479\,20\,k^{10}\,m^{7}+\\
&1908690321697732255714\,05\,k^{11}\,m^{7}-1756617140410471837128\,5\,k^{12}\,m^{7}-1736841488124542961656\,3\,k^{13}\,m^{7}-\\
&1675345386720849024305\,k^{14}\,m^{7}+258386194175957386429\,k^{15}\,m^{7}+1454350933195996310\,21\,k^{16}\,m^{7}+\\
&2552996565322095254\,8\,k^{17}\,m^{7}-2188169428581358891\,k^{18}\,m^{7}-812328910373878790\,k^{19}\,m^{7}-\\
&6426938941824993\,6\,k^{20}\,m^{7}+80868184706653\,6\,k^{21}\,m^{7}+14182665060921\,28\,k^{22}\,m^{7}+9975370105420\,8\,k^{23}\,m^{7}-\\
&3150335650368183898830336\,0\,m^{8}-9319778791337867511218227\,2\,k\,m^{8}-3951809921625339081530112\,0\,k^{2}\,m^{8}+\\
&6054652972040405856361600\,0\,k^{3}\,m^{8}+3059621614082172275531260\,8\,k^{4}\,m^{8}-7889952945965471934572288\,k^{5}\,m^{8}+\\
&1304734490288967149520122\,4\,k^{6}\,m^{8}+7550535461425547407261788\,k^{7}\,m^{8}-8859024268853661825232004\,k^{8}\,m^{8}-\\
&4362255400980753413552565\,k^{9}\,m^{8}+1076664565813589661702869\,k^{10}\,m^{8}+7425791557526870850366\,95\,k^{11}\,m^{8}+\\
&8304666175756744552987\,k^{12}\,m^{8}-3937216898693015609419\,5\,k^{13}\,m^{8}-7007145881568229757415\,k^{14}\,m^{8}-\\
&4278856606681790351\,71\,k^{15}\,m^{8}+3268985992150987502\,69\,k^{16}\,m^{8}+8245756002782238142\,8\,k^{17}\,m^{8}-\\
&1886419424904123\,94\,k^{18}\,m^{8}-1587511084452715452\,k^{19}\,m^{8}-154205257241223328\,k^{20}\,m^{8}-\\
&1658586348295776\,k^{21}\,m^{8}+2157575694853280\,k^{22}\,m^{8}+139230314312064\,k^{23}\,m^{8}-\\
&1131518700550515260155115\,52\,m^{9}-2783429648121826170417945\,60\,k\,m^{9}-1127497486608297388578524\,16\,k^{2}\,m^{9}+\\
&5831337462478727987865062\,4\,k^{3}\,m^{9}-3067731443446408939955084\,8\,k^{4}\,m^{9}+4518408270837158865781600\,k^{5}\,m^{9}+\\
&9251477495527196165506092\,8\,k^{6}\,m^{9}+3083404705341981520912750\,0\,k^{7}\,m^{9}-2631717466795337059443413\,6\,k^{8}\,m^{9}-\\
&1320734696199099513366468\,3\,k^{9}\,m^{9}+1682118685472569312227441\,k^{10}\,m^{9}+1659859577650204519651621\,k^{11}\,m^{9}+\\
&1126578372604968105970\,61\,k^{12}\,m^{9}-5535447163132764578321\,3\,k^{13}\,m^{9}-1512393759000262355482\,1\,k^{14}\,m^{9}-\\
&2181730743145273909037\,k^{15}\,m^{9}+4580382263656660170\,87\,k^{16}\,m^{9}+1510346821697408742\,52\,k^{17}\,m^{9}+\\
&939793448012339624\,k^{18}\,m^{9}-1992973959948729944\,k^{19}\,m^{9}-225588511176249152\,k^{20}\,m^{9}-\\
&5239597697309056\,k^{21}\,m^{9}+2136559791416000\,k^{22}\,m^{9}+1257132224747\,52\,k^{23}\,m^{9}-\\
&3179991588443930104935413\,76\,m^{10}-6727819420820542135321612\,80\,k\,m^{10}-\\
&2481227470379508198602787\,84\,k^{2}\,m^{10}-5407259161967569635613363\,2\,k^{3}\,m^{10}-2783916220583769836739344\,32\,k^{4}\,m^{10}+\\
&3074077859186407181736108\,8\,k^{5}\,m^{10}+2759002205120346532900320\,96\,k^{6}\,m^{10}+8234406018006452248413758\,0\,k^{7}\,m^{10}-\\
&5118041174571749456046403\,6\,k^{8}\,m^{10}-2716963848034764565683081\,2\,k^{9}\,m^{10}+1149437231780538898035200\,k^{10}\,m^{10}+\\
&2590474037508082663827144\,k^{11}\,m^{10}+3169720813933955579834\,56\,k^{12}\,m^{10}-4241865003422837913939\,2\,k^{13}\,m^{10}-\\
&2259914671560161407064\,0\,k^{14}\,m^{10}-4694345971076161585076\,k^{15}\,m^{10}+4244333307133609887\,56\,k^{16}\,m^{10}+\\
&1945113148890931959\,16\,k^{17}\,m^{10}+517771823375424521\,6\,k^{18}\,m^{10}-1740537399358494960\,k^{19}\,m^{10}-\\
&231446644005562944\,k^{20}\,m^{10}-7179636817977152\,k^{21}\,m^{10}+1460531925980160\,k^{22}\,m^{10}+\\
&767553913589\,76\,k^{23}\,m^{10}-7061509363382418201900226\,56\,m^{11}-1322591113692790293332065536\,k\,m^{11}-\\
&4654363844987946959374656\,00\,k^{2}\,m^{11}-4162822702702806606564172\,80\,k^{3}\,m^{11}-8555121129971273247369887\,36\,k^{4}\,m^{11}+\\
&7297779282868606007588545\,92\,k^{5}\,m^{11}+5829662996200152500841959\,28\,k^{6}\,m^{11}+1617345627490639690024849\,40\,k^{7}\,m^{11}-\\
&7370768076774571984912045\,6\,k^{8}\,m^{11}-4173332216903692969817349\,6\,k^{9}\,m^{11}-1380736722623082248688680\,k^{10}\,m^{11}+\\
&2907031063542999785613024\,k^{11}\,m^{11}+5656514298848888007570\,24\,k^{12}\,m^{11}+9287557991444212721680\,k^{13}\,m^{11}-\\
&2499319369785946841200\,8\,k^{14}\,m^{11}-6815321442704714530140\,k^{15}\,m^{11}+2288362139900312612\,08\,k^{16}\,m^{11}+\\
&1871930325286095935\,28\,k^{17}\,m^{11}+8024091141381068112\,k^{18}\,m^{11}-1064318919325834272\,k^{19}\,m^{11}-\\
&1735684483555312\,64\,k^{20}\,m^{11}-6138185610378624\,k^{21}\,m^{11}+6968169422776\,32\,k^{22}\,m^{11}+3157552486809\,6\,k^{23}\,m^{11}-\\
&1232608609983078454956774912\,m^{12}-2090077801396444050947099136\,k\,m^{12}-8293851346068047033536832\,0\,k^{2}\,m^{12}-\\
&1233054438693478583632181952\,k^{3}\,m^{12}-1815453108787806370929809184\,k^{4}\,m^{12}+\\
&1658142861625409453435822\,56\,k^{5}\,m^{12}+9734451234440744061391794\,96\,k^{6}\,m^{12}+2403044636882495247402327\,60\,k^{7}\,m^{12}-\\
&8195894868436444115743656\,00\,k^{8}\,m^{12}-4877765348526650779190665\,6\,k^{9}\,m^{12}-5439643405194106275926192\,k^{10}\,m^{12}+\\
&2185845706897840434937032\,k^{11}\,m^{12}+7372344604801939335640\,16\,k^{12}\,m^{12}+8003083733547247501124\,8\,k^{13}\,m^{12}-\\
&2064408126033765838320\,8\,k^{14}\,m^{12}-7367624236789996656576\,k^{15}\,m^{12}-1589348933081594448\,k^{16}\,m^{12}+\\
&1374823322494537566\,08\,k^{17}\,m^{12}+7790434907230531\,20\,k^{18}\,m^{12}-430966932703024960\,k^{19}\,m^{12}-\\
&9600596104793267\,2\,k^{20}\,m^{12}-3517712173015040\,k^{21}\,m^{12}+228123604459520\,k^{22}\,m^{12}+8398752350208\,k^{23}\,m^{12}-\\
&1638676271372709235703546880\,m^{13}-2527399203629998062489169\,68\,k\,m^{13}-1486553170250680425567785472\,k^{2}\,m^{13}-\\
&2788719346208882897275293504\,k^{3}\,m^{13}-3022537230810573573870511104\,k^{4}\,m^{13}+\\
&4139333994821027676542509\,60\,k^{5}\,m^{13}+1350523579555716882437470528\,k^{6}\,m^{13}+\\
&2646977741038627919532628\,96\,k^{7}\,m^{13}-7219187759087537064963254\,4\,k^{8}\,m^{13}-4203616875034410107298766\,4\,k^{9}\,m^{13}-\\
&9085051211797487235026704\,k^{10}\,m^{13}+6533096621112842902399\,68\,k^{11}\,m^{13}+7344146960331668766880\,0\,k^{12}\,m^{13}+\\
&1320482040979120418102\,88\,k^{13}\,m^{13}-1225892581374244599494\,4\,k^{14}\,m^{13}-6149589645257599394688\,k^{15}\,m^{13}-\\
&1376068755200290590\,88\,k^{16}\,m^{13}+7722586932101657715\,2\,k^{17}\,m^{13}+5338636148208173824\,k^{18}\,m^{13}-\\
&8982793870420057\,6\,k^{19}\,m^{13}-38780403396818688\,k^{20}\,m^{13}-1357903736442880\,k^{21}\,m^{13}+4883761405952\,0\,k^{22}\,m^{13}+\\
&1305286017024\,k^{23}\,m^{13}-1487212402332710510664150528\,m^{14}-1919282505168341856114365184\,k\,m^{14}-\\
&2541702720598912622730832896\,k^{2}\,m^{14}-5409243998694516877493952192\,k^{3}\,m^{14}-\\
&4203833655567378164014430816\,k^{4}\,m^{14}+9610223713359093661989640\,0\,k^{5}\,m^{14}+\\
&1613457019996209848842692096\,k^{6}\,m^{14}+1919651750522598108123577\,28\,k^{7}\,m^{14}-5319734719232744797797276\,8\,k^{8}\,m^{14}-
\end{aligned}
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\begin{aligned}
&228322121477166967219795968\ k^{9}\ m^{14} - 10304932205839108302745952\ k^{10}\ m^{14} - 9169404315032202714499 52\ k^{11}\ m^{14} + \\
&566489976005400061966592\ k^{12}\ m^{14} + 14043740275694523265148 8\ k^{13}\ m^{14} - 4432901197658479767968\ k^{14}\ m^{14} - \\
&4018663850511038594496\ k^{15}\ m^{14} - 151291349103088615296\ k^{16}\ m^{14} + 32764249010563719040\ k^{17}\ m^{14} + \\
&2669254817772792832\ k^{18}\ m^{14} + 10296997252329216\ k^{19}\ m^{14} - 11119615736717312\ k^{20}\ m^{14} - \\
&339935102386176\ k^{21}\ m^{14} + 6151347896320\ k^{22}\ m^{14} + 90019725312\ k^{23}\ m^{14} - 4540558266334423707086960 64\ m^{15} + \\
&384208210939965863794364160\ k\ m^{15} - 37191955783900917575735705216\ k^{2}\ m^{15} - 9249016621217854128525347520\ k^{3}\ m^{15} - \\
&514946100029471486966724505 6\ k^{4}\ m^{15} + 18535384980013825954259566 72\ k^{5}\ m^{15} + \\
&171499100227805387810635968\ k^{6}\ m^{15} + 31681789475909375294268544\ k^{7}\ m^{15} - 384755392226717693856384 00\ k^{8}\ m^{15} - \\
&730061320703158456263 68\ k^{9}\ m^{15} - 865804971503226463521427 2\ k^{10}\ m^{15} - 18026899843763010342566 40\ k^{11}\ m^{15} + \\
&335100671566223341611456\ k^{12}\ m^{15} + 11067671494441849182144 0\ k^{13}\ m^{15} + 31909995285645130816\ k^{14}\ m^{15} - \\
&2059502075557803525248\ k^{15}\ m^{15} - 99712685829998174208\ k^{16}\ m^{15} + 10180076697981766400\ k^{17}\ m^{15} + \\
&974330062924446976\ k^{18}\ m^{15} + 13936599371509760\ k^{19}\ m^{15} - 2141308465094656\ k^{20}\ m^{15} - 49936567205888\ k^{21}\ m^{15} + \\
&345075613696\ k^{22}\ m^{15} + 12759725477755446651461 67136\ m^{16} + 4441128757355493962893518336\ k\ m^{16} - \\
&412910574058324545839965 04912\ k^{2}\ m^{16} - 13917495200933053896733800576\ k^{3}\ m^{16} - \\
&584667703940481674066472 5248\ k^{4}\ m^{16} + 29016918623809311358010966016\ k^{5}\ m^{16} + \\
&166897061686787650228145 3184\ k^{6}\ m^{16} - 14899947342854292492380 6720\ k^{7}\ m^{16} - \\
&341882585605049797375272 96\ k^{8}\ m^{16} + 16149602094243395043548 416\ k^{9}\ m^{16} - 5454436725299761547662080\ k^{10}\ m^{16} - \\
&182726944300910339369296\ k^{11}\ m^{16} + 14551937606080859167232 0\ k^{12}\ m^{16} + 67487290930840206271616\ k^{13}\ m^{16} + \\
&130171684169202489139 2\ k^{14}\ m^{16} - 821243487963521971968\ k^{15}\ m^{16} - 46009018708145250560\ k^{16}\ m^{16} + \\
&2169605317503607808\ k^{17}\ m^{16} + 253765654904075776\ k^{18}\ m^{16} + 4530690436141056\ k^{19}\ m^{16} - 247925575974912\ k^{20}\ m^{16} - \\
&3270716686336\ k^{21}\ m^{16} + 29105975595241614618009 35424\ m^{17} + 9366634413694521023816839680\ k\ m^{17} - \\
&264228097947412642664501 9904\ k^{2}\ m^{17} - 18274675874259660906052460544\ k^{3}\ m^{17} - \\
&636868180244263750483973 0432\ k^{4}\ m^{17} + 37121862103253464210049026 56\ k^{5}\ m^{17} + \\
&151162329398956286005150 5408\ k^{6}\ m^{17} - 26994731284495202486110 4128\ k^{7}\ m^{17} - \\
&357370285892307231544606 72\ k^{8}\ m^{17} + 21339827815057275570419456\ k^{9}\ m^{17} - 2466684758932135105811968\ k^{10}\ m^{17} - \\
&132231640980465633465395 2\ k^{11}\ m^{17} + 39915490845165750350080\ k^{12}\ m^{17} + 32285549327634070461696\ k^{13}\ m^{17} + \\
&100722849700990712 0896\ k^{14}\ m^{17} - 250260847872133387520\ k^{15}\ m^{17} - 15330266169632705280\ k^{16}\ m^{17} + \\
&267937726704778240\ k^{17}\ m^{17} + 44753486777415680\ k^{18}\ m^{17} + 723065627164672\ k^{19}\ m^{17} - 13019102773248\ k^{20}\ m^{17} + \\
&349993619948596748088202 9056\ m^{18} + 13558030570357984749964055040\ k\ m^{18} + 11434344429949521784743874 56\ k^{2}\ m^{18} - \\
&207939616362228686543323 10016\ k^{3}\ m^{18} - 66506698922642700916924093 44\ k^{4}\ m^{18} + \\
&392537464574736798807521 0752\ k^{5}\ m^{18} + 12708886496743087128675225 60\ k^{6}\ m^{18} - \\
&292793439219236586257869 312\ k^{7}\ m^{18} - 34983603468338887228120576\ k^{8}\ m^{18} + 17897312900587098376531968\ k^{9}\ m^{18} - \\
&653292411579298133316096\ k^{10}\ m^{18} - 73535704819982741715404 8\ k^{11}\ m^{18} + 13210737089438691717 12\ k^{12}\ m^{18} + \\
&121080685422520079820 80\ k^{13}\ m^{18} + 466136587811120241664\ k^{14}\ m^{18} - 56457651692243180544\ k^{15}\ m^{18} - \\
&3644260748291361792\ k^{16}\ m^{18} + 6075403405581312\ k^{17}\ m^{18} + 4792135375306752\ k^{18}\ m^{18} + \\
&48231818657792\ k^{19}\ m^{18} + 26445977011603535 7447928279 04\ m^{19} + 15505422769926742017055085568\ k\ m^{19} + \\
&619417109676366736730 6812416\ k^{2}\ m^{19} - 20421297495781177099674249216\ k^{3}\ m^{19} - \\
&646041262976932625419354 5216\ k^{4}\ m^{19} + 34634544753094981822770780 16\ k^{5}\ m^{19} + \\
&973699142320644438162442240\ k^{6}\ m^{19} - 23713915702716388256837120 0\ k^{7}\ m^{19} - 2848834120885444285528371 2\ k^{8}\ m^{19} + \\
&112572027119526107950305 28\ k^{9}\ m^{19} + 50284913712090333706240\ k^{10}\ m^{19} - 32158276921442822844006 4\ k^{11}\ m^{19} - \\
&509026845334222776524 8\ k^{12}\ m^{19} + 35179640791025303429 12\ k^{13}\ m^{19} + 14898208860843268505 6\ k^{14}\ m^{19} - \\
&892690968226907238 4\ k^{15}\ m^{19} - 589225812086990848\ k^{16}\ m^{19} - 3013156772702208\ k^{17}\ m^{19} + 235199037308928\ k^{18}\ m^{19} + \\
&822724618108912624397316096\ m^{20} + 14654483898258413641120800768\ k\ m^{20} + 10457745023168056100113446912\ k^{2}\ m^{20} - \\
&172757140124782702671430594 56\ k^{3}\ m^{20} - 56318658064030028594121728 00\ k^{4}\ m^{20} + \\
&256617643988888632820972 3392\ k^{5}\ m^{20} + 66365735039367276808975 5648\ k^{6}\ m^{20} - 15236790930698376840 20569088\ k^{7}\ m^{20} - \\
&187389661450768145620459 52\ k^{8}\ m^{20} + 55659561066182834867 11808\ k^{9}\ m^{20} + 15867220037257401734 3488\ k^{10}\ m^{20} - \\
&110931112196508902934528\ k^{11}\ m^{20} - 29673846787573721948 16\ k^{12}\ m^{20} + 77338616214385061888 0\ k^{13}\ m^{20} + \\
&334208829603395563 52\ k^{14}\ m^{20} - 893647938304090112\ k^{15}\ m^{20} - 58233317472923648\ k^{16}\ m^{20} - 305289708224512\ k^{17}\ m^{20} - \\
&982424491487671627567742976\ m^{21} + 11661296796681037887561043968\ k\ m^{21} + 12205604072426491720705511424\ k^{2}\ m^{21} - \\
&125752381146550386068760 69888\ k^{3}\ m^{21} - 42898803049425000836847996928\ k^{4}\ m^{21} + \\
&160210932024226844813166 5920\ k^{5}\ m^{21} + 39374357821321834625177 6000\ k^{6}\ m^{21} - \\
&794371687291236709073346 56\ k^{7}\ m^{21} - 99186182383719661033431 04\ k^{8}\ m^{21} + 21946113255339444292689 92\ k^{9}\ m^{21} + \\
&939713736357566614855 68\ k^{10}\ m^{21} - 29900589482749434093568\ k^{11}\ m^{21} - 98176573641290230169 6\ k^{12}\ m^{21} + \\
&123579044358185259008\ k^{13}\ m^{21} + 5069739073053614080\ k^{14}\ m^{21} - 44892515848970240\ k^{15}\ m^{21} - \\
&2658714536259584\ k^{16}\ m^{21} - 19949897817001156246967 25504\ m^{22} + 78762349094794091394832465 92\ k\ m^{22} + \\
&111231244449818275522046115 84\ k^{2}\ m^{22} - 78672815463608417621969 14176\ k^{3}\ m^{22} - \\
&280882354550230382620320 5632\ k^{4}\ m^{22} + 84336263773029195641439027 2\ k^{5}\ m^{22} + \\
&199861997244063073318821888\ k^{6}\ m^{22} - 33852971130179803395416064\ k^{7}\ m^{22} - 4219025028462131798958080\ k^{8}\ m^{22} + \\
&689633644158840069947392\ k^{9}\ m^{22} + 34854748164058799595520\ k^{10}\ m^{22} - 6164718935556429062144\ k^{11}\ m^{22} - \\
&215478749440871989248\ k^{12}\ m^{22} + 13399893759929081856\ k^{13}\ m^{22} + 469631443663314944\ k^{14}\ m^{22} - \\
&393852239233024\ k^{15}\ m^{22} - 20672671170077325601204469 76\ m^{23} + 45273756247236334857979330 56\ k\ m^{23} + \\
&825956829576064312840239 5136\ k^{2}\ m^{23} - 42225234829807980522144972 8\ k^{3}\ m^{23} - \\
&156505038815648396055599 5136\ k^{4}\ m^{23} + 37358474869829379997242163 2\ k^{5}\ m^{23} +
\end{aligned}
$$

$$
\begin{aligned}
&855793889841248618174218 24\,k^{6}\,m^{23}-1177795275276568280085299 2\,k^{7}\,m^{23}-14344818262963116710625 28\,k^{8}\,m^{23}+\\
&17080112214675476938752 0\,k^{9}\,m^{23}+9090165203662772387840\,k^{10}\,m^{23}-936922037569212186624\,k^{11}\,m^{23}-\\
&31272847384814632960\,k^{12}\,m^{23}+865876570824736768\,k^{13}\,m^{23}+20131453434019840\,k^{14}\,m^{23}-\\
&15560605391826303947485347 84\,m^{24}+2213081306369434381790085120\,k\,m^{24}+5091829288390253553562831 36\,k^{2}\,m^{24}-\\
&19385639181257967989809152 00\,k^{3}\,m^{24}-7365603603926418152889384 96\,k^{4}\,m^{24}+1385945845739293951152291 84\,k^{5}\,m^{24}+\\
&30505646916440603973681152\,k^{6}\,m^{24}-331873624451099866628096 0\,k^{7}\,m^{24}-3852267094386102585262 08\,k^{8}\,m^{24}+\\
&32618261544455902199808\,k^{9}\,m^{24}+168358707463715369779 2\,k^{10}\,m^{24}-985320746966798498 4\,k^{11}\,m^{24}-\\
&27446076504648581 12\,k^{12}\,m^{24}+24286892420071424\,k^{13}\,m^{24}-922329340038361115490779136\,m^{25}+\\
&91675962650726705313880473 6\,k\,m^{25}+26272613704997142600959262 72\,k^{2}\,m^{25}-7578843934029713102389248 00\,k^{3}\,m^{25}-\\
&29061826526910472511927091 2\,k^{4}\,m^{25}+427202679658029322641735 68\,k^{5}\,m^{25}+8920126438633464496586752\,k^{6}\,m^{25}-\\
&74611719872143933663600 80\,k^{7}\,m^{25}-8000820643323845895782 4\,k^{8}\,m^{25}+463005299341027599974 4\,k^{9}\,m^{25}+\\
&21285118661019847884 8\,k^{10}\,m^{25}-6363444942721974272\,k^{11}\,m^{25}-111021447253590016\,k^{12}\,m^{25}-\\
&44306857067794976524625510 4\,m^{26}+319862232050704299760287744\,k\,m^{26}+1136875468263358940560883712\,k^{2}\,m^{26}-\\
&25068238027520178839407820 8\,k^{3}\,m^{26}-9525421551223966341831065 6\,k^{4}\,m^{26}+1081117161057055775155814 4\,k^{5}\,m^{26}+\\
&21001160971116863910903808\,k^{6}\,m^{26}-1306689698456157207920 64\,k^{7}\,m^{26}-1241003486709233732812 8\,k^{8}\,m^{26}+\\
&45908117484311019520 0\,k^{9}\,m^{26}+1657156347342631731 2\,k^{10}\,m^{26}-188211553812480000\,k^{11}\,m^{26}-\\
&17422178758090508738337177 6\,m^{27}+9314760979137140259736780 8\,k\,m^{27}+4115466585881354706687090 88\,k^{2}\,m^{27}-\\
&69515259373275527030243328\,k^{3}\,m^{27}-2561598825789424380621619 2\,k^{4}\,m^{27}+2207664176584973002997760\,k^{5}\,m^{27}+\\
&38797101660763749561139 2\,k^{6}\,m^{27}-1716754033844010142924 8\,k^{7}\,m^{27}-1354380982995890995200\,k^{8}\,m^{27}+\\
&28286342503005683712\,k^{9}\,m^{27}+601781413612093440\,k^{10}\,m^{27}-5607928859454338982897254 4\,m^{28}+\\
&22351115645946339759489024\,k\,m^{28}+1238118188814965810213683 20\,k^{2}\,m^{28}-1595699046237753392234496 0\,k^{3}\,m^{28}-\\
&55546162808132855943659 52\,k^{4}\,m^{28}+3546542827953849630720 00\,k^{5}\,m^{28}+5412533404648990939545 6\,k^{6}\,m^{28}-\\
&15898902154051379527 68\,k^{7}\,m^{28}-92834045111687970816\,k^{8}\,m^{28}+811460308893696000\,k^{9}\,m^{28}-\\
&14670827168855359576080384\,m^{29}+434010115174848010282598 4\,k\,m^{29}+3061522530232821122963865 6\,k^{2}\,m^{29}-\\
&29784452449517289445785 60\,k^{3}\,m^{29}-9470868840554162647203 84\,k^{4}\,m^{29}+4313294352295000434278 4\,k^{5}\,m^{29}+\\
&53591630229699349708 80\,k^{6}\,m^{29}-92389595763009650688\,k^{7}\,m^{29}-3008240471994531840\,k^{8}\,m^{29}-\\
&30750144577310194485166 08\,m^{30}+6645500799965047459676 16\,k\,m^{30}+6118878248821068203556864\,k^{2}\,m^{30}-\\
&44061922523191702663987 2\,k^{3}\,m^{30}-1222350300160921593446 40\,k^{4}\,m^{30}+3730813500148791902208\,k^{5}\,m^{30}+\\
&33556029095842047590 4\,k^{6}\,m^{30}-2528090920612528128\,k^{7}\,m^{30}-5045156212230567493632 00\,m^{31}+\\
&77186178844898264875008\,k\,m^{31}+9641674563405285697781 76\,k^{2}\,m^{31}-4970985648449907798835 2\,k^{3}\,m^{31}-\\
&11223960700805940510720\,k^{4}\,m^{31}+204348394531039739904\,k^{5}\,m^{31}+9987576898907013120\,k^{6}\,m^{31}-\\
&62449229956637767237632\,m^{32}+6387061696026441154560\,k\,m^{32}+1153122481152049639587 84\,k^{2}\,m^{32}-\\
&40178729357775375622016\,k^{3}\,m^{32}-6532762922384653025 28\,k^{4}\,m^{32}+5323021911783899136\,k^{5}\,m^{32}-\\
&54853837101089119272 96\,m^{33}+335252109148972646400\,k\,m^{33}+9837472674516534558720\,k^{2}\,m^{33}-\\
&20719096396084622131 2\,k^{3}\,m^{33}-1811473989999840460 8\,k^{4}\,m^{33}-3047772749590545039 36\,m^{34}+\\
&83838985375234129 92\,k\,m^{34}+533303026163461914624\,k^{2}\,m^{34}-5120684935279017984\,k^{3}\,m^{34}-\\
&80533461061777489 92\,m^{35}+13805736182018998272\,k^{2}\,m^{35}\big)
\end{aligned}
$$

$$
\begin{aligned}
p_2(k,m) = (2+k-2\,m)\,(-1+k+2\,m)\,(k+2\,m)\,\big(&1062475364966400 00\,k+68243846664268800 00\,k^{2}+56464256497985280 00\,k^{3}-\\
&89290692727588800 00\,k^{4}-96352477523218224 00\,k^{5}+35840487107250340 0\,k^{6}+49296992824176552 00\,k^{7}-\\
&35449339089979080 0\,k^{8}-10118034220837347 00\,k^{9}-8053610364353610 0\,k^{10}+6886090339221240 0\,k^{11}+\\
&19440642266123400\,k^{12}+2528415763587000\,k^{13}-1316426473517400\,k^{14}-478440551912400\,k^{15}+15019629735600\,k^{16}+\\
&15353628059700\,k^{17}+1564852470300\,k^{18}+14158216800\,k^{19}-45158488200\,k^{20}-4200789600\,k^{21}-\\
&21249507299320800 00\,m+27512626696289280 000\,k\,m+20305236148994419 2000\,k^{2}\,m+16320940644683992 3200\,k^{3}\,m-\\
&24817487956073575 2000\,k^{4}\,m-25861018624933115 2320\,k^{5}\,m+90858331931403377 60\,k^{6}\,m+12486543429022523 4960\,k^{7}\,m-\\
&74829573511263995 40\,k^{8}\,m-23981048494057633 60\,k^{9}\,m-19990048441096414 80\,k^{10}\,m+14784859259405107 20\,k^{11}\,m+\\
&42593851003322592 0\,k^{12}\,m+6358568913748080 0\,k^{13}\,m-2666492199536832 0\,k^{14}\,m-1015473837489912 0\,k^{15}\,m+\\
&24525063449478 0\,k^{16}\,m+30066685549776 0\,k^{17}\,m+3134290730004 0\,k^{18}\,m+39661275024 0\,k^{19}\,m-8331109491 60\,k^{20}\,m-\\
&75740222880\,k^{21}\,m-90822594978017 2800 00\,m^{2}+28199958990423091 2000\,k\,m^{2}+28280822388244919 42400\,k^{2}\,m^{2}+\\
&21889227371484517 05600\,k^{3}\,m^{2}-31338126616216614 54480\,k^{4}\,m^{2}-31130859178275009 60240\,k^{5}\,m^{2}+\\
&10167153833061134 58600\,k^{6}\,m^{2}+13930569608855221 12500\,k^{7}\,m^{2}-63609686167572039 240\,k^{8}\,m^{2}-\\
&24582671276142232 0890\,k^{9}\,m^{2}-21809782802137799 430\,k^{10}\,m^{2}+13201390782452901 420\,k^{11}\,m^{2}+\\
&39798710414618893 20\,k^{12}\,m^{2}+703050078423426 480\,k^{13}\,m^{2}-223888655094708 240\,k^{14}\,m^{2}-9135206819398944 0\,k^{15}\,m^{2}+\\
&13538927160603 00\,k^{16}\,m^{2}+242464382278497 0\,k^{17}\,m^{2}+26241357453867 0\,k^{18}\,m^{2}+450560941092 0\,k^{19}\,m^{2}-\\
&62163554139 00\,k^{20}\,m^{2}-54656041032 0\,k^{21}\,m^{2}-173747485523016345600 0\,m^{3}+108315410287692119040 0\,k\,m^{3}+\\
&24759629740935930677760\,k^{2}\,m^{3}+181573572776497429017 60\,k^{3}\,m^{3}-24101718727685791831200\,k^{4}\,m^{3}-\\
&22278620828772269895312\,k^{5}\,m^{3}+668748146260582870449 6\,k^{6}\,m^{3}+90284703751133428259 16\,k^{7}\,m^{3}-\\
&26349628516538497985 4\,k^{8}\,m^{3}-1427471888906574335076\,k^{9}\,m^{3}-138148477656741437568\,k^{10}\,m^{3}+\\
&62148366198794228772\,k^{11}\,m^{3}+20803904634986919852\,k^{12}\,m^{3}+4523873191023960660\,k^{13}\,m^{3}-\\
&10056018361980866 52\,k^{14}\,m^{3}-457214972605059792\,k^{15}\,m^{3}+124782358003639 8\,k^{16}\,m^{3}+1041206072696649 6\,k^{17}\,m^{3}+\\
&12000963449267 64\,k^{18}\,m^{3}+2728345504214 4\,k^{19}\,m^{3}-2406526581634 0\,k^{20}\,m^{3}-200983664836 8\,k^{21}\,m^{3}-
\end{aligned}
$$

$$
\begin{aligned}
&200783437083670279680000\,m^4 - 5399936632204430791680\,k\,m^4 + 153759600615013597290240\,k^2\,m^4 + \\
&10409928317018850777120 0\,k^3\,m^4 - 127399918496846021280288\,k^4\,m^4 - 1048160610572583150514 32\,k^5\,m^4 + \\
&28959559015597314989344\,k^6\,m^4 + 3715257724579342949789 4\,k^7\,m^4 - 38987886830421569156 0\,k^8\,m^4 - \\
&503466144515775867836 3\,k^9\,m^4 - 55531185304400998086 5\,k^{10}\,m^4 + 145789116238537952170\,k^{11}\,m^4 + \\
&65295335959151256440\,k^{12}\,m^4 + 18711640297789591408\,k^{13}\,m^4 - 2464618138479236512\,k^{14}\,m^4 - \\
&1370372876624944912\,k^{15}\,m^4 - 19508356765249318\,k^{16}\,m^4 + 24495525834416651\,k^{17}\,m^4 + 3189594531981153\,k^{18}\,m^4 + \\
&95555769523808\,k^{19}\,m^4 - 48386593327914\,k^{20}\,m^4 - 3659751480024\,k^{21}\,m^4 - 1590555341547053748172 80\,m^5 - \\
&98466007266494862289920\,k\,m^5 + 7203044484831308911292 16\,k^2\,m^5 + 432536468577020658834624\,k^3\,m^5 - \\
&49261549749154387571428 8\,k^4\,m^5 - 3308488410017913313961 60\,k^5\,m^5 + 872614453481084259580 16\,k^6\,m^5 + \\
&95434591266873559794016\,k^7\,m^5 + 1064250707387694415805\,k^8\,m^5 - 9784539530362179992507\,k^9\,m^5 - \\
&13878377956965615213 67\,k^{10}\,m^5 - 12316846915540937020\,k^{11}\,m^5 + 110922469641587994386\,k^{12}\,m^5 + \\
&50709249845851592526\,k^{13}\,m^5 - 2043585698686520274\,k^{14}\,m^5 - 2249607654510707060\,k^{15}\,m^5 - \\
&105742918966806611\,k^{16}\,m^5 + 21239413713229653\,k^{17}\,m^5 + 4241093012038169\,k^{18}\,m^5 + 183786602068520\,k^{19}\,m^5 - \\
&23832561782012\,k^{20}\,m^5 - 684993499712\,k^{21}\,m^5 - 92725504963937953507584 0\,m^6 - 67686988889175854593996 8\,k\,m^6 + \\
&26214090471453631513570 56\,k^2\,m^6 + 13046731272877968543 11424\,k^3\,m^6 - 14306334368347322461171 28\,k^4\,m^6 - \\
&63452402405048226613306 8\,k^5\,m^6 + 18722375777427159300399 0\,k^6\,m^6 + 11027187758574572487946 1\,k^7\,m^6 + \\
&49055409856452810786 03\,k^8\,m^6 - 7046012420386169046 11\,k^9\,m^6 - 15103991132278262003 49\,k^{10}\,m^6 - \\
&12955294115101239956 83\,k^{11}\,m^6 - 3376091807586011295\,k^{12}\,m^6 + 81555710990598997657\,k^{13}\,m^6 + \\
&71370103732180836 35\,k^{14}\,m^6 - 247996424891972786\,k^{15}\,m^6 - 257414596050712648\,k^{16}\,m^6 - 53168790146242838\,k^{17}\,m^6 - \\
&17665271866768 56\,k^{18}\,m^6 + 88468299228060\,k^{19}\,m^6 + 133776076185904\,k^{20}\,m^6 + 12731027621824\,k^{21}\,m^6 - \\
&41644329634533283350497 28\,m^7 - 2948405201386879574029824\,k\,m^7 + 7436207565053782192971 20\,k^2\,m^7 + \\
&26356705665357118252178 880\,k^3\,m^7 - 3060240345576518658127920\,k^4\,m^7 - 2081779457291003939839 548\,k^5\,m^7 + \\
&28130184963777986153487 2\,k^6\,m^7 - 2290217097718430562550 43\,k^7\,m^7 - 3827908127793656088498\,k^8\,m^7 + \\
&61300189683690343320755\,k^9\,m^7 + 3523221556849540967785\,k^{10}\,m^7 - 4768653938350786871861\,k^{11}\,m^7 - \\
&58967586400773949954 31\,k^{12}\,m^7 + 23304852103707841967\,k^{13}\,m^7 + 30576316599066269923\,k^{14}\,m^7 + \\
&91677079254346942 12\,k^{15}\,m^7 - 283864497981806875\,k^{16}\,m^7 - 230051727670419850\,k^{17}\,m^7 - \\
&20871415168409376\,k^{18}\,m^7 - 515456176743912\,k^{19}\,m^7 + 414740386409120\,k^{20}\,m^7 + 33251233684736\,k^{21}\,m^7 - \\
&14879348597707865356810752\,m^8 - 8807034349849513874504832\,k\,m^8 + 1589606067135834910950 7776\,k^2\,m^8 + \\
&19652992637928107378924 32\,k^3\,m^8 - 4114271022827675783403896\,k^4\,m^8 + 3400693603241391429368648\,k^5\,m^8 + \\
&24425250653717836004146 4\,k^6\,m^8 - 1527911134646169054211153\,k^7\,m^8 - 8442658494276795930900 5\,k^8\,m^8 + \\
&22137586443527338375575 3\,k^9\,m^8 + 2189028189029502195836 5\,k^{10}\,m^8 - 9711189393563517349849\,k^{11}\,m^8 - \\
&18635682337822222022209\,k^{12}\,m^8 - 273346566205065987489\,k^{13}\,m^8 + 60041405784056253687\,k^{14}\,m^8 + \\
&27562236691503456126\,k^{15}\,m^8 + 166830501064313194\,k^{16}\,m^8 - 442213780782589732\,k^{17}\,m^8 - \\
&49385813827476048\,k^{18}\,m^8 - 1641847567767968\,k^{19}\,m^8 + 648539177866208\,k^{20}\,m^8 + 46410104770688\,k^{21}\,m^8 - \\
&43319972199399962978098944\,m^9 - 1702051003607465325411 3024\,k\,m^9 + 2211944543920866073870 3296\,k^2\,m^9 - \\
&95956728783453172918359 68\,k^3\,m^9 + 5668562169781910598984 80\,k^4\,m^9 + 1446378525166250080236 9800\,k^5\,m^9 - \\
&16915697246128471797558 0\,k^6\,m^9 - 4328854448814311677300 717\,k^7\,m^9 - 3416283973729582341289 73\,k^8\,m^9 + \\
&46860446292644954562239\,k^9\,m^9 + 6008559060924386416290 3\,k^{10}\,m^9 - 1173620953229747080968 7\,k^{11}\,m^9 - \\
&35362309015955011883 1\,k^{12}\,m^9 - 8919851021534245090 7\,k^{13}\,m^9 + 71796774555774282281\,k^{14}\,m^9 + \\
&48716711702475640160\,k^{15}\,m^9 + 1211672853730065832\,k^{16}\,m^9 - 549544308666315400\,k^{17}\,m^9 - \\
&71655828840431616\,k^{18}\,m^9 - 2676671721483968\,k^{19}\,m^9 + 659317852002624\,k^{20}\,m^9 + 41904407491584\,k^{21}\,m^9 - \\
&10469862918658059285069 2736\,m^{10} - 1148789538423534825384710 4\,k\,m^{10} + 2482361376932020813091040\,k^2\,m^{10} - \\
&51519751855985776435709456\,k^3\,m^{10} + 2316462002051391799413130 4\,k^4\,m^{10} + 3699357161632148475692726 8\,k^5\,m^{10} - \\
&15397083820096835194420 38\,k^6\,m^{10} - 8327746500706964240339546\,k^7\,m^{10} - 8391383785366895193083 10\,k^8\,m^{10} + \\
&68213529081850987125889 0\,k^9\,m^{10} + 1128421315383839644611 34\,k^{10}\,m^{10} - 4774271655248121697346\,k^{11}\,m^{10} - \\
&47566999057237860355 62\,k^{12}\,m^{10} - 1683141307256057730422\,k^{13}\,m^{10} + 4744829623745702332 8\,k^{14}\,m^{10} + \\
&61156755642706752004\,k^{15}\,m^{10} + 2383343866811206272\,k^{16}\,m^{10} - 477120108279390992\,k^{17}\,m^{10} - \\
&73144538731073280\,k^{18}\,m^{10} - 2865868019223872\,k^{19}\,m^{10} + 463609986081792\,k^{20}\,m^{10} + 25585130452992\,k^{21}\,m^{10} - \\
&21317953402611719833634150 4\,m^{11} + 5645291653979280382671993 6\,k\,m^{11} - 9488459110240645675526880 0\,k^2\,m^{11} - \\
&15213046388293661693754862 4\,k^3\,m^{11} + 8342265537802206821840646 4\,k^4\,m^{11} + 7049365924087774923935780\,k^5\,m^{11} - \\
&51156295675862733745197 72\,k^6\,m^{11} - 1188955701467798566069601 6\,k^7\,m^{11} - 1458869117910620337780028\,k^8\,m^{11} + \\
&68008824399323209423757 6\,k^9\,m^{11} + 1591355281673207210670 60\,k^{10}\,m^{11} + 1271803703550842193754 4\,k^{11}\,m^{11} - \\
&47121294317467451635 88\,k^{12}\,m^{11} - 2277876868851651919292\,k^{13}\,m^{11} - 3183929249339463928\,k^{14}\,m^{11} + \\
&57747456192005900104\,k^{15}\,m^{11} + 2938336984704804128\,k^{16}\,m^{11} - 292353360444877792\,k^{17}\,m^{11} - \\
&54756322015283968\,k^{18}\,m^{11} - 2139148391574656\,k^{19}\,m^{11} + 228278717815808\,k^{20}\,m^{11} + 10525174956032\,k^{21}\,m^{11} - \\
&36996985459582400434079577 6\,m^{12} + 2739581011618460238565313 28\,k\,m^{12} - 3443224591472784563694635 52\,k^2\,m^{12} - \\
&33457589039750326974801214 4\,k^3\,m^{12} + 2014851613191927494395961 52\,k^4\,m^{12} + 1068318869713393680886865 04\,k^5\,m^{12} - \\
&12577253645246935440376096\,k^6\,m^{12} - 1277408880294486893898458 4\,k^7\,m^{12} - 1878868127247998359934232\,k^8\,m^{12} + \\
&37424539853917250253802 4\,k^9\,m^{12} + 1739942261244860663609 6\,k^{10}\,m^{12} + 3376523705346095462349 6\,k^{11}\,m^{12} - \\
&33720887103484614791 36\,k^{12}\,m^{12} - 2355018003311930637792\,k^{13}\,m^{12} - 4825867313045794812 8\,k^{14}\,m^{12} + \\
&41903718458143322944\,k^{15}\,m^{12} + 2575080136692771648\,k^{16}\,m^{12} - 121283723563233984\,k^{17}\,m^{12} - \\
&3035339066673408 0\,k^{18}\,m^{12} - 1125102693918720\,k^{19}\,m^{12} + 77458816557056\,k^{20}\,m^{12} + 2799584116736\,k^{21}\,m^{12} - \\
&55212704357117065789083724 8\,m^{13} + 7335445641860474909870296 32\,k\,m^{13} - 8059319121814306066963163 52\,k^2\,m^{13} - \\
&59294148112151904692684273 6\,k^3\,m^{13} + 3814392455986417828000571 04\,k^4\,m^{13} + 1324945946800464141702759280\,k^5\,m^{13} - \\
&24559555566321883867372688\,k^6\,m^{13} - 9859955480942003755700032\,k^7\,m^{13} - 1787927245009394482074992\,k^8\,m^{13} - \\
&11818199521146663334364 8\,k^9\,m^{13} + 1485063138508273181963 04\,k^{10}\,m^{13} + 4720022742041478591017 6\,k^{11}\,m^{13} -
\end{aligned}
$$

$$153809905855584736139 2\,k^{12}\,m^{13} - 1908124430273842834112\,k^{13}\,m^{13} - 63837984956487663536\,k^{14}\,m^{13} +$$
$$23475462392376943232\,k^{15}\,m^{13} + 166937719107067852 8\,k^{16}\,m^{13} - 28697554770406016\,k^{17}\,m^{13} -$$
$$12344972829412608\,k^{18}\,m^{13} - 409794205151232\,k^{19}\,m^{13} + 17275403419648\,k^{20}\,m^{13} + 435095339008\,k^{21}\,m^{13} -$$
$$71272859876882939410267737 6\,m^{14} + 14564745892513307626553255 04\,k\,m^{14} - 14664220171785141988222930208\,k^{2}\,m^{14} -$$
$$87535983143334865031824150 4\,k^{3}\,m^{14} + 59454532905875592327916758 4\,k^{4}\,m^{14} + 13631612336659199734781110 4\,k^{5}\,m^{14} -$$
$$38782020653390706741476672\,k^{6}\,m^{14} - 42847698889005261771087 68\,k^{7}\,m^{14} - 11710547034624654810301 12\,k^{8}\,m^{14} -$$
$$54619637468430549474556 8\,k^{9}\,m^{14} + 97541032848963032908800\,k^{10}\,m^{14} + 46775490278730357430176\,k^{11}\,m^{14} -$$
$$13463934251403316355 2\,k^{12}\,m^{14} - 12237144725440080863 04\,k^{13}\,m^{14} - 51836718991300727808\,k^{14}\,m^{14} +$$
$$10079744766846916480\,k^{15}\,m^{14} + 808401596118431744\,k^{16}\,m^{14} - 25561599278336\,k^{17}\,m^{14} -$$
$$3583061454618624\,k^{18}\,m^{14} - 98474135339008\,k^{19}\,m^{14} + 2280499707904\,k^{20}\,m^{14} + 30006575104\,k^{21}\,m^{14} -$$
$$79790690327951665541368627 2\,m^{15} + 2321067520803657343515028416\,k\,m^{15} - 21990447847357351831311439168\,k^{2}\,m^{15} -$$
$$10950636285152605799874901 76\,k^{3}\,m^{15} + 77951984999332510184800544 0\,k^{4}\,m^{15} +$$
$$11690285462077924548779801 6\,k^{5}\,m^{15} - 50084651768762345999799232\,k^{6}\,m^{15} + 12780227797866052845489 28\,k^{7}\,m^{15} -$$
$$35247620627046084089683 2\,k^{8}\,m^{15} - 72070336393115524953824 0\,k^{9}\,m^{15} + 46792002984732602482112\,k^{10}\,m^{15} +$$
$$35410805515248735933120\,k^{11}\,m^{15} + 465507293062149421120\,k^{12}\,m^{15} - 6215990719722089552 64\,k^{13}\,m^{15} -$$
$$30033390927561415680\,k^{14}\,m^{15} + 3252616049522963200\,k^{15}\,m^{15} + 289878604096116992\,k^{16}\,m^{15} +$$
$$2595870943891456\,k^{17}\,m^{15} - 702613367808000\,k^{18}\,m^{15} - 14044952559616\,k^{19}\,m^{15} + 135029587968\,k^{20}\,m^{15} -$$
$$77369580478375165785404736 0\,m^{16} + 30725549913472133368655243 52\,k\,m^{16} - 27902623894199936400719431 68\,k^{2}\,m^{16} -$$
$$11719718687913469241067311 36\,k^{3}\,m^{16} + 86913060762333781517142080 0\,k^{4}\,m^{16} + 83333578411176238042535296\,k^{5}\,m^{16} -$$
$$53384633704041751302298240\,k^{6}\,m^{16} + 45749604315973486242168576\,k^{7}\,m^{16} + 26779875764116997332851 2\,k^{8}\,m^{16} -$$
$$63690553785897965099737 6\,k^{9}\,m^{16} + 13542779291689137626496\,k^{10}\,m^{16} + 21046053829658603948672\,k^{11}\,m^{16} +$$
$$47395248839293114393 6\,k^{12}\,m^{16} - 248402705594825068288\,k^{13}\,m^{16} - 12895503150929662208\,k^{14}\,m^{16} +$$
$$760068870393321984\,k^{15}\,m^{16} + 74905108441985536\,k^{16}\,m^{16} + 916330988728320\,k^{17}\,m^{16} - 83380770570240\,k^{18}\,m^{16} -$$
$$90019725312 0\,k^{19}\,m^{16} - 64582356738572130500484 7104\,m^{17} + 34409796710031673473659719 68\,k\,m^{17} -$$
$$30373697451308903017004805 12\,k^{2}\,m^{17} - 10787364478427144704836125952\,k^{3}\,m^{17} +$$
$$82888863710513523701103308 8\,k^{4}\,m^{17} + 48762275447918401150373120\,k^{5}\,m^{17} - 47288737347473870414837248\,k^{6}\,m^{17} +$$
$$51121745748544903698342 40\,k^{7}\,m^{17} + 50565050315811768752768 0\,k^{8}\,m^{17} - 42581938517727022936140 8\,k^{9}\,m^{17} -$$
$$60907933960444495377 92\,k^{10}\,m^{17} + 9906371819895488144640\,k^{11}\,m^{17} + 27746787941671707008 0\,k^{12}\,m^{17} -$$
$$76962962905916395264\,k^{13}\,m^{17} - 4107987004472206592\,k^{14}\,m^{17} + 119958652671201 28\,k^{15}\,m^{17} +$$
$$1320911197015449 6\,k^{16}\,m^{17} + 148756658372608\,k^{17}\,m^{17} - 4519740375040\,k^{18}\,m^{17} - 458035258659993287724358656\,m^{18} +$$
$$32950608760526841039545763 84\,k\,m^{18} - 28582474404019821909055503 36\,k^{2}\,m^{18} - 85606950667685661804321075 2\,k^{3}\,m^{18} +$$
$$67812027774371060647151923 2\,k^{4}\,m^{18} + 22708691647714504809216000\,k^{5}\,m^{18} - 34964594008783143683442176\,k^{6}\,m^{18} +$$
$$38992859077614263337210 88\,k^{7}\,m^{18} + 44395336247398125879808\,k^{8}\,m^{18} - 22387279922713227299379 2\,k^{9}\,m^{18} -$$
$$32867519831561660584 96\,k^{10}\,m^{18} + 3684984807856828498432\,k^{11}\,m^{18} + 1137031469226017 0240\,k^{12}\,m^{18} -$$
$$18037303059619402240\,k^{13}\,m^{18} - 948588366960537600\,k^{14}\,m^{18} + 11000547767452672\,k^{15}\,m^{18} +$$
$$1424066106867712\,k^{16}\,m^{18} + 9928425537536\,k^{17}\,m^{18} - 26906154163463425682130739 2\,m^{19} +$$
$$27149146961886263547876003 84\,k\,m^{19} - 23342680978368348148569354 24\,k^{2}\,m^{19} - 58594132905760275373025792 0\,k^{3}\,m^{19} +$$
$$47625809066615847927418060 8\,k^{4}\,m^{19} + 77558252010173520832819 20\,k^{5}\,m^{19} - 21617275048669707664794624\,k^{6}\,m^{19} +$$
$$22801630200194751392010 24\,k^{7}\,m^{19} + 27264072518263307854745 6\,k^{8}\,m^{19} - 93715644896222827773952\,k^{9}\,m^{19} -$$
$$21166887574356715571 20\,k^{10}\,m^{19} + 1071212027969240137728\,k^{11}\,m^{19} + 33867033314970645504\,k^{12}\,m^{19} -$$
$$30722237926946396 16\,k^{13}\,m^{19} - 150607575307532288\,k^{14}\,m^{19} + 335394859616256\,k^{15}\,m^{19} + 70830520532992\,k^{16}\,m^{19} -$$
$$12384915583088779183613952 0\,m^{20} + 19309850573854992393808650 24\,k\,m^{20} - 16566966056691793524894535 68\,k^{2}\,m^{20} -$$
$$34540052790479067316810956 8\,k^{3}\,m^{20} + 28684781611215719381119590 4\,k^{4}\,m^{20} + 13710161249921746741683 68\,k^{5}\,m^{20} -$$
$$11165855073636103491624960\,k^{6}\,m^{20} + 10600651604201375057571 84\,k^{7}\,m^{20} + 12713701868538448332645 12\,k^{8}\,m^{20} -$$
$$31214333166111229544448\,k^{9}\,m^{20} - 840553888773083533312\,k^{10}\,m^{20} + 2382077643865605959 68\,k^{11}\,m^{20} +$$
$$72556005330417254 4\,k^{12}\,m^{20} - 355564270234804224\,k^{13}\,m^{20} - 14737365532606464\,k^{14}\,m^{20} -$$
$$17029669076992\,k^{15}\,m^{20} - 37751700416268820180967424\,m^{21} + 11867312066973762470561464 32\,k\,m^{21} -$$
$$10211955450460894266633088 00\,k^{2}\,m^{21} - 17479950789911838870960128 0\,k^{3}\,m^{21} + 14775106765878885337820364 8\,k^{4}\,m^{21} -$$
$$40032799176809314360115 2\,k^{5}\,m^{21} - 48004514170913810637209 60\,k^{6}\,m^{21} + 39615736779116845702348 8\,k^{7}\,m^{21} +$$
$$46088116577417605885952\,k^{8}\,m^{21} - 81858136330469387466 80\,k^{9}\,m^{21} - 233740052520256950272\,k^{10}\,m^{21} +$$
$$39093025980418195456\,k^{11}\,m^{21} + 1066389750196170752\,k^{12}\,m^{21} - 24645250162925568\,k^{13}\,m^{21} -$$
$$670776591355904\,k^{14}\,m^{21} - 47543008506367498975641 6\,m^{22} + 62955509495097889563166310 4\,k\,m^{22} -$$
$$54542241740288870282327654 4\,k^{2}\,m^{22} - 75568686124927019314479104\,k^{3}\,m^{22} + 64784018341285715457081344\,k^{4}\,m^{22} -$$
$$47594848473096508370124 8\,k^{5}\,m^{22} - 17058262016795183470018 56\,k^{6}\,m^{22} + 11879929173055121998643 2\,k^{7}\,m^{22} +$$
$$12995856235901594001408\,k^{8}\,m^{22} - 16552117031945652551 68\,k^{9}\,m^{22} - 46085908520596127744\,k^{10}\,m^{22} +$$
$$44543672096394690 56\,k^{11}\,m^{22} + 96644281117384704\,k^{12}\,m^{22} - 752044945924096\,k^{13}\,m^{22} +$$
$$85720325372207572861255 68\,m^{23} + 28744395273114364510352179 2\,k\,m^{23} - 25140658480930306447147008 0\,k^{2}\,m^{23} -$$
$$27711407212764666960035840\,k^{3}\,m^{23} + 24016468292459132552544256\,k^{4}\,m^{23} - 23876685201307706399129 6\,k^{5}\,m^{23} -$$
$$49551362220801199513600 0\,k^{6}\,m^{23} + 28265572157804876218368\,k^{7}\,m^{23} + 28087479602161851824128\,k^{8}\,m^{23} -$$
$$24909280569970961612 8\,k^{9}\,m^{23} - 6198559770973487104\,k^{10}\,m^{23} + 313897123361964032\,k^{11}\,m^{23} +$$
$$4082168555880448\,k^{12}\,m^{23} + 67816984029862990911897 60\,m^{24} + 11240196863100120692726169 6\,k\,m^{24} -$$

$$
\begin{aligned}
&9943279507184961482067148 8\ k^2\ m^{24} - 8536950076500262674759680\ k^3\ m^{24} + 7456294235454414159577088\ k^4\ m^{24} - \\
&8246726895982901379072 0\ k^5\ m^{24} - 1157389000420218701414 40\ k^6\ m^{24} + 5222153710109305896960\ k^7\ m^{24} + \\
&4508879906367177031 68\ k^8\ m^{24} - 2626410853512134656 0\ k^9\ m^{24} - 513269809387765760\ k^{10}\ m^{24} + \\
&10267531718197248\ k^{11}\ m^{24} + 3399468189161011072598016\ m^{25} + 3737437199528890673568153 6\ k\ m^{25} - \\
&3347975736094245969749606 4\ k^2\ m^{25} - 2180795908310801661493248\ k^3\ m^{25} + 1913431288687660802506752\ k^4\ m^{25} - \\
&2123572596860385656832 0\ k^5\ m^{25} - 2121056589508104801484 8\ k^6\ m^{25} + 7228087474402551398 40\ k^7\ m^{25} + \\
&5075999889998531788 8\ k^8\ m^{25} - 1730173876847640576\ k^9\ m^{25} - 19821713375625216\ k^{10}\ m^{25} + \\
&1273404315084777264513024\ m^{26} + 1046224842731411854668595 2\ k\ m^{26} - 9497174566948263880163328\ k^2\ m^{26} - \\
&4538136443875121768693 76\ k^3\ m^{26} + 3985342996382891059445 76\ k^4\ m^{26} - 4119561047003905130496\ k^5\ m^{26} - \\
&2937337762729925541888\ k^6\ m^{26} + 705387715734357934 08\ k^7\ m^{26} + 3582294594944237568\ k^8\ m^{26} - \\
&53534775652909056\ k^9\ m^{26} + 371031050291161468502016\ m^{27} + 2432297805856211260145664\ k\ m^{27} - \\
&2238245853471494959005696\ k^2\ m^{27} - 750415038008954357022 72\ k^3\ m^{27} + 6564534845619756859392 0\ k^4\ m^{27} - \\
&5892683792205781401 60\ k^5\ m^{27} - 289015191452123136000\ k^6\ m^{27} + 4327541119934005248\ k^7\ m^{27} + \\
&119415576592908288\ k^8\ m^{27} + 845036587426985772318 72\ m^{28} + 4609083376581586609766 40\ k\ m^{28} - \\
&4300247532338151334871 04\ k^2\ m^{28} - 950942992559739450163 2\ k^3\ m^{28} + 8225897272553751183360\ k^4\ m^{28} - \\
&5893143702330566246 4\ k^5\ m^{28} - 180071629075010027 52\ k^6\ m^{28} + 125485156572069888\ k^7\ m^{28} + \\
&1483613597289390538752 0\ m^{29} + 693296264878855909539 84\ k\ m^{29} - 6558431790505907139379 2\ k^2\ m^{29} - \\
&8727904489479811891 20\ k^3\ m^{29} + 736541088467156729856\ k^4\ m^{29} - 3689739663534194688\ k^5\ m^{29} - \\
&534112775917535232\ k^6\ m^{29} + 1945180947224199168000\ m^{30} + 7959175277275804336128\ k\ m^{30} - \\
&7633914074219340103680\ k^2\ m^{30} - 525954086987020369 92\ k^3\ m^{30} + 4196239644877966540 8\ k^4\ m^{30} - \\
&109033691692400640\ k^5\ m^{30} + 179688186585192333312\ m^{31} + 654706636211179487232\ k\ m^{31} - \\
&6366874454081006469 12\ k^2\ m^{31} - 1690659020769067008\ k^3\ m^{31} + 1142758643683295232\ k^4\ m^{31} + \\
&104450290567882997 76\ m^{32} + 34345680938018537472\ k\ m^{32} - 33867196571515355136\ k^2\ m^{32} - \\
&14116533244526592\ k^3\ m^{32} + 287619503792062464\ m^{33} + 862858511376187392\ k\ m^{33} - 862858511376187392\ k^2\ m^{33})
\end{aligned}
$$

Appendix D. The certificate polynomials of $A_0(m)$

$$
\begin{aligned}
p_0(k,m) = {} & 10894441510823270400000000 - 889001339444172460800000\,k - 469030476737116804608000\,k^2 - \\
& 154635707232024380308800\,k^3 + 12862799312736337111072000\,k^4 - 54879970304989115496000\,k^5 - \\
& 427982450572990886277600\,k^6 + 49852741313854543507380\,k^7 + 25115770688724233480340\,k^8 - \\
& 5638710471894723628440\,k^9 - 57111445894533944180400\,k^{10} - 1764802243003052710800\,k^{11} + \\
& 545348999530452526920\,k^{12} + 56104349497368451200\,k^{13} - 189008718749027755200\,k^{14} - 33669234551435351400\,k^{15} - \\
& 2175654522204009000\,k^{16} + 925366890732580800\,k^{17} + 254992462143552000\,k^{18} - 8583930975690000\,k^{19} - \\
& 3740281940833200\,k^{20} - 135388148187600\,k^{21} - 2030481658800\,k^{22} + 2033182166400\,k^{23} + 145227297600\,k^{24} + \\
& 350389909882708762752000\,m - 2341260441454236051456000\,k\,m - 129139394799275879995872000\,k^2\,m - \\
& 382156510465656535752864000\,k^3\,m + 29542927184520462943221600\,k^4\,m - 19156170222278741904089040\,k^5\,m - \\
& 7894240440588132814191960\,k^6\,m + 1426802836959455315725638\,0\,k^7\,m + 5517845268816203099084700\,k^8\,m - \\
& 1559886305890444056995400\,k^9\,m - 1283056454535415192430280\,k^{10}\,m - 39779881613690680294560\,k^{11}\,m + \\
& 11775923257480354467432\,0\,k^{12}\,m + 1449347716403028282204\,0\,k^{13}\,m - 3178019256389487266400\,k^{14}\,m - \\
& 934409961612530153100\,k^{15}\,m - 140479490203681836060\,k^{16}\,m + 26285582694052997280\,k^{17}\,m + 9432044926150200480\,k^{18}\,m - \\
& 70662336373646640\,k^{19}\,m - 144872661863870640\,k^{20}\,m - 12996621951374880\,k^{21}\,m - 199896062919840\,k^{22}\,m + \\
& 192586506124320\,k^{23}\,m + 13709498806080\,k^{24}\,m + 53891607769610121975040000\,m^2 - 26670108932086241986963200\,k\,m^2 - \\
& 16986077139805593312056256\,00\,k^2\,m^2 - 492751897328905148948912640\,k^3\,m^2 + 32356706183268918655145760\,0\,k^4\,m^2 - \\
& 25329189813258185438165568\,0\,k^5\,m^2 - 57943860780798206936381520\,k^6\,m^2 + 17091057163849918054593984\,0\,k^7\,m^2 + \\
& 52840761988857226623910440\,k^8\,m^2 - 16719292987488906431672160\,k^9\,m^2 - 12245315963441212157050440\,k^{10}\,m^2 - \\
& 57544532828637033883536\,0\,k^{11}\,m^2 + 10390941212476984092183\,60\,k^{12}\,m^2 + 15463865143545768134388\,0\,k^{13}\,m^2 - \\
& 19825150845058671573720\,k^{14}\,m^2 - 907685577836237858112\,0\,k^{15}\,m^2 - 176756425108889266620\,0\,k^{16}\,m^2 + \\
& 227744374617701731440\,k^{17}\,m^2 + 94375826607414786840\,k^{18}\,m^2 + 363148976538855360\,k^{19}\,m^2 - \\
& 1320410505856395240\,k^{20}\,m^2 - 13568018114208348\,0\,k^{21}\,m^2 - 2508122290219560\,k^{22}\,m^2 + 1791113061517920\,k^{23}\,m^2 + \\
& 123530022635040\,k^{24}\,m^2 + 52470904622552231563733760\,00\,m^3 - 16507933980912230716935360\,00\,k\,m^3 - \\
& 1417490436286742177199339072\,0\,k^2\,m^3 - 437583023303012434790113982\,4\,k^3\,m^3 + 21897909892803707685012517\,20\,k^4\,m^3 - \\
& 19062599309275991162207690\,52\,k^5\,m^3 - 14764190416652785463146346\,6\,k^6\,m^3 + 12217296637771250957224939\,26\,k^7\,m^3 + \\
& 29649459533278610815933537\,8\,k^8\,m^3 - 10317520153219726878015898\,8\,k^9\,m^3 - 68770327668875486110176588\,k^{10}\,m^3 - \\
& 50748468913959512484551\,40\,k^{11}\,m^3 + 52445494363389947642499\,12\,k^{12}\,m^3 + 96832653759709239134358\,0\,k^{13}\,m^3 - \\
& 51235172735193713577942\,k^{14}\,m^3 - 48964507460194358074554\,k^{15}\,m^3 - 11340476003888478118026\,k^{16}\,m^3 + \\
& 1032043572391324676652\,k^{17}\,m^3 + 497399304012100755372\,k^{18}\,m^3 + 6509593278190426104\,k^{19}\,m^3 - \\
& 6125324048331557256\,k^{20}\,m^3 - 684589335214891392\,k^{21}\,m^3 - 14804465940860256\,k^{22}\,m^3 + 7887396752334288\,k^{23}\,m^3 + \\
& 523679403605472\,k^{24}\,m^3 + 3621238458538531101867106560\,0\,m^4 - 4852384576620644099371564800\,k\,m^4 - \\
& 84098862447459445513499713632\,k^2\,m^4 - 295139929892575903724764514\,88\,k^3\,m^4 + 98661869388798052135695490\,08\,k^4\,m^4 - \\
& 94611651390852527877506447632\,k^5\,m^4 + 88524176041829250929191946734\,k^6\,m^4 + 60204884799435749866069869\,60\,k^7\,m^4 + \\
& 10691647235592765508130276\,28\,k^8\,m^4 - 42851927978917603943675772\,0\,k^9\,m^4 - 25794198029093474266135826\,4\,k^{10}\,m^4 - \\
& 29033307404094402013003620\,k^{11}\,m^4 + 17122476286485805677064\,64\,k^{12}\,m^4 + 41044448364040491711415660\,k^{13}\,m^4 + \\
& 4418950845607137620607\,0\,k^{14}\,m^4 - 17357533849770559174812\,0\,k^{15}\,m^4 - 46587114378584438654592\,k^{16}\,m^4 + \\
& 2888545218813685780536\,k^{17}\,m^4 + 1687498042263155752632\,k^{18}\,m^4 + 36543979932366115692\,k^{19}\,m^4 - \\
& 1783181321457946335\,6\,k^{20}\,m^4 - 215026948574084420\,4\,k^{21}\,m^4 - 52892106799619700\,k^{22}\,m^4 + \\
& 212011928985315\,36\,k^{23}\,m^4 + 1346045398736208\,k^{24}\,m^4 + 1883391951135441555016806400\,m^5 + \\
& 897590429696753831844250617\,6\,k\,m^5 - 37662380884047983607865298534\,4\,k^2\,m^5 - 15772023617559452102648718273\,6\,k^3\,m^5 + \\
& 28685030140387031346699779520\,k^4\,m^5 - 32874243807515280784027439184\,k^5\,m^5 + \\
& 11136915906320493665227428570\,k^6\,m^5 + 22081126666165949978593536183\,k^7\,m^5 + 24021363406379421953053755\,67\,k^8\,m^5 - \\
& 13004673175078832828940466\,66\,k^9\,m^5 - 68201146550832113987557683\,0\,k^{10}\,m^5 - 11634306823230476458082639\,2\,k^{11}\,m^5 + \\
& 37443025515458182460791144\,k^{12}\,m^5 + 12703022071527504261497766\,k^{13}\,m^5 + 83079814930461753403188\,4\,k^{14}\,m^5 - \\
& 43938275627175169487905\,9\,k^{15}\,m^5 - 13572456902638625104627\,9\,k^{16}\,m^5 + 52500424868926924160\,72\,k^{17}\,m^5 + \\
& 40515049524816034734\,28\,k^{18}\,m^5 + 12109856078043454930\,8\,k^{19}\,m^5 - 35799696954219395216\,k^{20}\,m^5 - \\
& 4678465557938349556\,k^{21}\,m^5 - 127542415305927580\,k^{22}\,m^5 + 38650146454707944\,k^{23}\,m^5 + 2328067297171072\,k^{24}\,m^5 + \\
& 7668660240524783130819890215\,68\,m^6 + 1807631787100127038688290556\,16\,k\,m^6 - 1317890052782797217414209738848\,k^2\,m^6 - \\
& 68281367194553421922869017995\,2\,k^3\,m^6 + 36705049239469579068366590184\,k^4\,m^6 - \\
& 79936754713983531635220166812\,k^5\,m^6 + 63003144783281922617084128142\,k^6\,m^6 + 63139113023089678047522361800\,k^7\,m^6 + \\
& 20501358972030420230434274\,70\,k^8\,m^6 - 30337355934807138914534676\,36\,k^9\,m^6 - 12731654856122054164028294\,88\,k^{10}\,m^6 - \\
& 34576124091461508240401286\,8\,k^{11}\,m^6 + 51064880282828641295358400\,k^{12}\,m^6 + 30050473510629290979853680\,k^{13}\,m^6 + \\
& 33968696485664378516315\,70\,k^{14}\,m^6 - 83138426014321166172126\,0\,k^{15}\,m^6 - 29683485203134185850890\,54\,k^{16}\,m^6 + \\
& 5820125579253072180736\,k^{17}\,m^6 + 726148373675852577920\,k^{18}\,m^6 + 27507428100437966916\,0\,k^{19}\,m^6 - \\
& 5208821194060790733\,6\,k^{20}\,m^6 - 7464034737340690224\,k^{21}\,m^6 - 220076649817696704\,k^{22}\,m^6 + 50469917812250176\,k^{23}\,m^6 + \\
& 2856686783927168\,k^{24}\,m^6 + 25073316593582780738371781226\,24\,m^7 + 11015611434100638115876972362\,24\,k\,m^7 - \\
& 36721513056601757775984255924\,48\,k^2\,m^7 - 24314112775679520374986546573\,92\,k^3\,m^7 - \\
& 11946770531203526462510930244\,0\,k^4\,m^7 - 12065851428624531826625116\,1308\,k^5\,m^7 + \\
& 24522492436780768144393786583\,8\,k^6\,m^7 + 14508886317214616047659798862\,3\,k^7\,m^7 - \\
& 79912770012462696477607384\,37\,k^8\,m^7 - 56373289949921235246630880\,14\,k^9\,m^7 - 15201408688209290436850729\,46\,k^{10}\,m^7 -
\end{aligned}
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\begin{aligned}
&792993942607363668201382240\ k^{11}\ m^7 + 20757738126122549665841328\ k^{12}\ m^7 + 55969775314528630236830018\ k^{13}\ m^7 + \\
&87038502052887512664625 92\ k^{14}\ m^7 - 12045051203096722137818 07\ k^{15}\ m^7 - 50483446580485604796 1015\ k^{16}\ m^7 + \\
&19692952597461100307080\ k^{17}\ m^7 + 10038918605027429347188\ k^{18}\ m^7 + 458353875948169170288\ k^{19}\ m^7 - \\
&56355485395971996748\ k^{20}\ m^7 - 9026817173890631712\ k^{21}\ m^7 - 28113286693098897 6\ k^{22}\ m^7 + 48657297875323840\ k^{23}\ m^7 + \\
&2557300524832256\ k^{24}\ m^7 + 669670386041105023492170715 9680\ m^8 + 450222778899500980193178905 15744\ k\ m^8 - \\
&81930958107382235203178398963 20\ k^2\ m^8 - 72101024433322103852051159957 28\ k^3\ m^8 - \\
&93896786075439019484467495708 0\ k^4\ m^8 - 20763373701419540644878389900\ k^5\ m^8 + \\
&73061207843668886615231806540 6\ k^6\ m^8 + 27355839392566507371469242869 2\ k^7\ m^8 - \\
&43262043946176385679488015718\ k^8\ m^8 - 85923177313319895899800361 48\ k^9\ m^8 - 44834968197146938872337308 8\ k^{10}\ m^8 - \\
&14427557295199123173399966 20\ k^{11}\ m^8 - 93169426935558674938868728\ k^{12}\ m^8 + 83709952475877073513583208\ k^{13}\ m^8 + \\
&16289449482341502255710930\ k^{14}\ m^8 - 13450986052972527760753 04\ k^{15}\ m^8 - 68326513916765573263974 6\ k^{16}\ m^8 - \\
&61986545941902768148 24\ k^{17}\ m^8 + 10927615372385595779072\ k^{18}\ m^8 + 581807709546427330496\ k^{19}\ m^8 - \\
&45712999231561802232\ k^{20}\ m^8 - 8438724036286890512\ k^{21}\ m^8 - 271109566395461184\ k^{22}\ m^8 + \\
&35179429762281728\ k^{23}\ m^8 + 1690826957181184\ k^{24}\ m^8 + 14770379458030126351530587647104\ m^9 + \\
&14121648265854324424980868255296\ k\ m^9 - 14459727931616213518298867303808\ k^2\ m^9 - \\
&17999035952418647149696047729168\ k^3\ m^9 - 34910902019981062577553040048 24\ k^4\ m^9 + \\
&50251840144266762836615292 1492\ k^5\ m^9 + 17485852406871854671236214786 42\ k^6\ m^9 + \\
&42944805063084780379289671591 6\ k^7\ m^9 - 12117302360316333022633893602 0\ k^8\ m^9 - \\
&11061897278088768708446633260\ k^9\ m^9 + 28061704277867669198790842 76\ k^{10}\ m^9 - \\
&21223346687391787089459392 72\ k^{11}\ m^9 - 29127930383593205107021996 4\ k^{12}\ m^9 + 10183715039951092417364552 4\ k^{13}\ m^9 + \\
&23644815910568866702798862\ k^{14}\ m^9 - 11399129252742181033107 80\ k^{15}\ m^9 - 74733445648517686306087 2\ k^{16}\ m^9 - \\
&14800206978346504618172\ k^{17}\ m^9 + 9482251645016055417388\ k^{18}\ m^9 + 5753943810753744002 96\ k^{19}\ m^9 - \\
&27496859201915922288\ k^{20}\ m^9 - 6161311773700766304\ k^{21}\ m^9 - 19923001191222713 6\ k^{22}\ m^9 + 19168812169329792\ k^{23}\ m^9 + \\
&825720775858176\ k^{24}\ m^9 + 27039011119634240740010133328896\ m^{10} + 35883481524801575574888571597056\ k\ m^{10} - \\
&19172465669608055188376846775840\ k^2\ m^{10} - 38185591895037102353854732556768\ k^3\ m^{10} - \\
&93296058116357209249606384619 04\ k^4\ m^{10} + 17985645427115707793761736376 40\ k^5\ m^{10} + \\
&34560269355658088964799077554 62\ k^6\ m^{10} + 56726901223048432140183535337 6\ k^7\ m^{10} - \\
&24638264694435571222412090070 4\ k^8\ m^{10} - 12448924462101809562213445728\ k^9\ m^{10} + \\
&80696289835304880718481773 04\ k^{10}\ m^{10} - 25565469550348639289097991 60\ k^{11}\ m^{10} - \\
&51138363155012404172991810 8\ k^{12}\ m^{10} + 10152432529290069390315988 0\ k^{13}\ m^{10} + \\
&27466890930467341084103874\ k^{14}\ m^{10} - 68641353789892172520256 8\ k^{15}\ m^{10} - 66720730132989185557478 8\ k^{16}\ m^{10} - \\
&19139371807716855189504\ k^{17}\ m^{10} + 6599470628856760465840\ k^{18}\ m^{10} + 449213517989158523088\ k^{19}\ m^{10} - \\
&11712872333560102112\ k^{20}\ m^{10} - 3524019688865244544\ k^{21}\ m^{10} - 111680067959059968\ k^{22}\ m^{10} + \\
&7834506401491968\ k^{23}\ m^{10} + 294197350723584\ k^{24}\ m^{10} + 40998336892724169681648199769088\ m^{11} + \\
&76055133475152938151605996608320\ k\ m^{11} - 15446608335724349741624592722400\ k^2\ m^{11} - \\
&69415017968541993706638180364448\ k^3\ m^{11} - 19747528472322546471601897987912\ k^4\ m^{11} + \\
&40301658987286755380414972 2356\ k^5\ m^{11} + 57429893466378784616059383178 32\ k^6\ m^{11} + \\
&63537156272467199317228333 6656\ k^7\ m^{11} - 39673780700154828990322870133 6\ k^8\ m^{11} - \\
&12744294588311996577188944920\ k^9\ m^{11} + 13731571451945574219225303488\ k^{10}\ m^{11} - \\
&25408686987515852251432568 80\ k^{11}\ m^{11} - 65853300873347881394958696 8\ k^{12}\ m^{11} + 83137500316665388680124500\ k^{13}\ m^{11} + \\
&26008832782886428102303368\ k^{14}\ m^{11} - 21721941907084682202854 4\ k^{15}\ m^{11} - 48905389285957802203748 0\ k^{16}\ m^{11} - \\
&17619034949259455259008\ k^{17}\ m^{11} + 3686538729876074099376\ k^{18}\ m^{11} + 278584232577800151264\ k^{19}\ m^{11} - \\
&30012953611561052 80\ k^{20}\ m^{11} - 1573173792014770432\ k^{21}\ m^{11} - 474032120587266 56\ k^{22}\ m^{11} + \\
&2364713540417536\ k^{23}\ m^{11} + 74344675868672\ k^{24}\ m^{11} + 50788131893256750157848133279872\ m^{12} + \\
&13693670305665666934773658830144\ k\ m^{12} + 48632958728944380251379067361 28\ k^2\ m^{12} - \\
&10887794165862028486887296397593 6\ k^3\ m^{12} - 34494047019873756681584894756816\ k^4\ m^{12} + \\
&68913944301845998752252853559 20\ k^5\ m^{12} + 81237713724847183570851459570 08\ k^6\ m^{12} + \\
&60696648450318990862102327191 04\ k^7\ m^{12} - 52537422713403130886787959831 60\ k^8\ m^{12} - \\
&12301189401123961354568746912\ k^9\ m^{12} + 17513536958370608292653627144\ k^{10}\ m^{12} - \\
&20889919847404292978599253 12\ k^{11}\ m^{12} - 67000004233418579703555637 6\ k^{12}\ m^{12} + 55728098830274581210346880\ k^{13}\ m^{12} + \\
&20299609427439830260058376\ k^{14}\ m^{12} + 81760721664954733067904\ k^{15}\ m^{12} - 29498747657634664302009 6\ k^{16}\ m^{12} - \\
&12408842721794727478864\ k^{17}\ m^{12} + 1644893493619904589760\ k^{18}\ m^{12} + 137295011948438691904\ k^{19}\ m^{12} - \\
&31174020599088256\ k^{20}\ m^{12} - 542162872948029696\ k^{21}\ m^{12} - 14972075610759168\ k^{22}\ m^{12} + \\
&511240936718336\ k^{23}\ m^{12} + 12620706807808\ k^{24}\ m^{12} + 49373365013080703250444213895296\ m^{13} + \\
&21206657535891185514251210130 3168\ k\ m^{13} + 45643543046283160925460285293280\ k^2\ m^{13} - \\
&14821451426713708092188260815 0560\ k^3\ m^{13} - 50841988545130220908321165184160\ k^4\ m^{13} + \\
&95747115468296777569335820403 84\ k^5\ m^{13} + 98695282317750883771467088585 28\ k^6\ m^{13} + \\
&49699684560731001513336837681 6\ k^7\ m^{13} - 58390876467787317861208207996 8\ k^8\ m^{13} - \\
&11358282079740188765156739856\ k^9\ m^{13} + 17923777163186923782979269664\ k^{10}\ m^{13} - \\
&14164183303504463386246209 12\ k^{11}\ m^{13} - 55602450402962492096204572 8\ k^{12}\ m^{13} + 30208913061489874588199216\ k^{13}\ m^{13} + \\
&13140322750337658873409184\ k^{14}\ m^{13} + 17659103021858035750204 8\ k^{15}\ m^{13} - 14627480961362112177078 4\ k^{16}\ m^{13} - \\
&68716003963378238597 44\ k^{17}\ m^{13} + 5795399744225826301 44\ k^{18}\ m^{13} + 53480978009735390848\ k^{19}\ m^{13} + \\
&355621654023412480\ k^{20}\ m^{13} - 141348161940834816\ k^{21}\ m^{13} - 3408228573978624\ k^{22}\ m^{13} + \\
&74857138847744\ k^{23}\ m^{13} + 1290282729472\ k^{24}\ m^{13} + 32986381491920851786543289784192\ m^{14} + \\
&28500379683129011814415627780 5696\ k\ m^{14} + 10119042750065999683291584571747 2\ k^2\ m^{14} -
\end{aligned}
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\begin{aligned}
&1759466039808869636898484862457600\ k^{3}\ m^{14} - 641088172913449151987316700771 84\ k^{4}\ m^{14} + \\
&1112839555729029174967956714 2976\ k^{5}\ m^{14} + 10364401980731538807561201229024\ k^{6}\ m^{14} + \\
&3506839686751951506781983239 04\ k^{7}\ m^{14} - 55135743947169362081152325456 0\ k^{8}\ m^{14} - \\
&9892663592520093880690770944\ k^{9}\ m^{14} + 15160272201781645166723523424\ k^{10}\ m^{14} - \\
&7830582070806354256536338 08\ k^{11}\ m^{14} - 382463247647759464136697376\ k^{12}\ m^{14} + 12865651651361977164318272\ k^{13}\ m^{14} + \\
&7071990058981862816306912\ k^{14}\ m^{14} + 145982180570019222787712\ k^{15}\ m^{14} - 59365984425675465142464\ k^{16}\ m^{14} - \\
&3020277060326571978752\ k^{17}\ m^{14} + 157898904098748661504\ k^{18}\ m^{14} + 16267376842290255616\ k^{19}\ m^{14} + \\
&172295937177644544\ k^{20}\ m^{14} - 26938926389174272\ k^{21}\ m^{14} - 528161619836928\ k^{22}\ m^{14} + 6646456385536\ k^{23}\ m^{14} + \\
&60013150208\ k^{24}\ m^{14} + 472269180768515014228707678 4512\ m^{15} + 33457254116612067987491721616 4160\ k\ m^{15} + \\
&1560730914647874572635529281332 48\ k^{2}\ m^{15} - 18283839574177781331275098768384\ k^{3}\ m^{15} - \\
&6978878037659667192724374737216 0\ k^{4}\ m^{15} + 10996268574148370047701024704704\ k^{5}\ m^{15} + \\
&9451135790323925657611303720832\ k^{6}\ m^{15} + 21487999912955153081059536 7104\ k^{7}\ m^{15} - \\
&4456873635983253310511514838656\ k^{8}\ m^{15} - 7889648602736883443545287104\ k^{9}\ m^{15} + \\
&1075700126248749542220610726 4\ k^{10}\ m^{15} - 34339219139352299719481772 8\ k^{11}\ m^{15} - \\
&2198134650693598325276819 20\ k^{12}\ m^{15} + 39973543378198412848407 68\ k^{13}\ m^{15} + 31618106612653114225637 12\ k^{14}\ m^{15} + \\
&8251767163648035666112 0\ k^{15}\ m^{15} - 19556553258013383603200\ k^{16}\ m^{15} - 10523423194471382435 84\ k^{17}\ m^{15} + \\
&320582136704381235 20\ k^{18}\ m^{15} + 3782967468534893056\ k^{19}\ m^{15} + 45818415323602944\ k^{20}\ m^{15} - 3538221479215104\ k^{21}\ m^{15} - \\
&49844672069632\ k^{22}\ m^{15} + 270059175936\ k^{23}\ m^{15} - 2606623011119635696317344445388 8\ m^{16} + \\
&3447331823831789150904933095846 40\ k\ m^{16} + 19199370548065363684344485761612 8\ k^{2}\ m^{16} - \\
&1668103936090294128193979274318592\ k^{3}\ m^{16} - 65997186939915648022802208658560\ k^{4}\ m^{16} + \\
&9328329599261972357448257030656\ k^{5}\ m^{16} + 75066896978887151946480590110 40\ k^{6}\ m^{16} + \\
&1158418033848438969745582307 84\ k^{7}\ m^{16} - 30982800958376787180400670553 6\ k^{8}\ m^{16} - \\
&5600883782194663230638695168\ k^{9}\ m^{16} + 64524103778710879780952116 48\ k^{10}\ m^{16} - \\
&1113000714625249809273157 12\ k^{11}\ m^{16} - 10588061942608294707192025 6\ k^{12}\ m^{16} + 67765442204343456523033 6\ k^{13}\ m^{16} + \\
&1169570282469809887526272\ k^{14}\ m^{16} + 351810744683163350159 36\ k^{15}\ m^{16} - 51604287915413045478 40\ k^{16}\ m^{16} - \\
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&7519174682353664\ k^{20}\ m^{16} - 286041428000768\ k^{21}\ m^{16} - 2160473407488\ k^{22}\ m^{16} - 4875789377076088935248892750182 4\ m^{17} + \\
&3128593627308474377606186503464 96\ k\ m^{17} + 19752977895642938284413851266 4064\ k^{2}\ m^{17} - \\
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&3469483586903547694698210048\ k^{9}\ m^{17} + 32826239243242530092001326 08\ k^{10}\ m^{17} - \\
&2032534083253141710873853 44\ k^{11}\ m^{17} - 427138865294504097272816 64\ k^{12}\ m^{17} - 11216659092005968541721 6\ k^{13}\ m^{17} + \\
&3552258349959411730896 64\ k^{14}\ m^{17} + 11627630966270853464064\ k^{15}\ m^{17} - 10691728356833419991 04\ k^{16}\ m^{17} - \\
&60494786453929107456\ k^{17}\ m^{17} + 365874338119009280\ k^{18}\ m^{17} + 77572059754303488\ k^{19}\ m^{17} + 717890430664704\ k^{20}\ m^{17} - \\
&10719848955904\ k^{21}\ m^{17} - 5722510601386073756215015 0289840128\ m^{18} + 2506932057113121879882267778160 64\ k\ m^{18} + \\
&1737965239343949135981457665315 84\ k^{2}\ m^{18} - 946808357777699849009598157137 92\ k^{3}\ m^{18} - \\
&3928855275124090241607238716876 8\ k^{4}\ m^{18} + 43404580605949864273667009474 56\ k^{5}\ m^{18} + \\
&3148166657778301888044154297344\ k^{6}\ m^{18} + 25256587850938302800509003776\ k^{7}\ m^{18} - \\
&9594553372135544767341711513 6\ k^{8}\ m^{18} - 18519422930525417884540364 80\ k^{9}\ m^{18} + \\
&1416481869033053362533949952\ k^{10}\ m^{18} + 30095069005721753956229 12\ k^{11}\ m^{18} - 1437255430644700396689356 8\ k^{12}\ m^{18} - \\
&1357497447218125143695 36\ k^{13}\ m^{18} + 875343620269559485848448\ k^{14}\ m^{18} + 2984694449925897848832\ k^{15}\ m^{18} - \\
&168783777979963407360\ k^{16}\ m^{18} - 9449043579027824640\ k^{17}\ m^{18} + 3339720977883136\ k^{18}\ m^{18} + 5753158769278976\ k^{19}\ m^{18} + \\
&30749237837824\ k^{20}\ m^{18} - 5223993722308482575215108464 2304\ m^{19} + 17762996093164215344291227450060 8\ k\ m^{19} + \\
&1323563803439499392653926060257 28\ k^{2}\ m^{19} - 5901529791443982340912923108454 4\ k^{3}\ m^{19} - \\
&2483401507123095099192907619942 4\ k^{4}\ m^{19} + 23930194230487163248015645122 56\ k^{5}\ m^{19} + \\
&1662527402014876206299556679680\ k^{6}\ m^{19} + 10864269740166309779608425472\ k^{7}\ m^{19} - \\
&4270857426090815644846015283 2\ k^{8}\ m^{19} - 84466235546413960011712000 0\ k^{9}\ m^{19} + \\
&5170232046492627552400445 44\ k^{10}\ m^{19} + 42733667142637849818327 04\ k^{11}\ m^{19} - 4003068878800605183117312\ k^{12}\ m^{19} - \\
&5762188569000412555776 0\ k^{13}\ m^{19} + 17192557568479390329 28\ k^{14}\ m^{19} + 5865308414819140362 24\ k^{15}\ m^{19} - \\
&193752478229730365 44\ k^{16}\ m^{19} - 1033160147882311680\ k^{17}\ m^{19} - 2275297083518976\ k^{18}\ m^{19} + \\
&199453704716288\ k^{19}\ m^{19} - 3944957770317008854480400850124 8\ m^{20} + 1113676882785961076125543188234 24\ k\ m^{20} + \\
&8784791535409493228593102272921 6\ k^{2}\ m^{20} - 3241777173409334088379431470368\ k^{3}\ m^{20} - \\
&1375424107371945988458886680985 6\ k^{4}\ m^{20} + 11454533094511355160715053834 24\ k^{5}\ m^{20} + \\
&7649824427702551103060747509 76\ k^{6}\ m^{20} + 45168067455884494122007592 96\ k^{7}\ m^{20} - \\
&1632734525577581103197563904 0\ k^{8}\ m^{20} - 32687473655560077288886681 6\ k^{9}\ m^{20} + 1587585052864973635921776 64\ k^{10}\ m^{20} + \\
&2002162454817585692516352\ k^{11}\ m^{20} - 91200440820054895050137 6\ k^{12}\ m^{20} - 1611421066588764055961 6\ k^{13}\ m^{20} + \\
&2621832827248894932992\ k^{14}\ m^{20} + 85510694903741251584\ k^{15}\ m^{20} - 1496050024170848256\ k^{16}\ m^{20} - \\
&70588716532764672\ k^{17}\ m^{20} - 154085638569984\ k^{18}\ m^{20} - 25287742959771896026144681918464\ m^{21} + \\
&6177512065455573857301499232256 0\ k\ m^{21} + 5100980995125085065993781321728 0\ k^{2}\ m^{21} - \\
&1567788683632155496786818849996 8\ k^{3}\ m^{21} - 6669614052150214479180763824128\ k^{4}\ m^{21} + \\
&4753508836850364554291892551 68\ k^{5}\ m^{21} + 30583414810225877740364011929 6\ k^{6}\ m^{21} + \\
&1778308552163809510845599744\ k^{7}\ m^{21} - 5335146391862502978870444032\ k^{8}\ m^{21} - 1065269521281565293506355 20\ k^{9}\ m^{21} + \\
&4065906972656770282838272\ k^{10}\ m^{21} + 627152389675455640416256\ k^{11}\ m^{21} - 1670208097480155208335 36\ k^{12}\ m^{21} - \\
&3232945849158027014144\ k^{13}\ m^{21} + 298301566867005464576\ k^{14}\ m^{21} + 8730759835996413952\ k^{15}\ m^{21} - \\
&66992820701863936\ k^{16}\ m^{21} - 2268246241648640\ k^{17}\ m^{21} - 1393224512060282419568915664076 8\ m^{22} + \\
&3028825936915878959882335479398 4\ k\ m^{22} + 2595688842585232267521121542144 0\ k^{2}\ m^{22} - \\
&6663170913928126934446014414848\ k^{3}\ m^{22} - 2826398764303841813774582267904\ k^{4}\ m^{22} +
\end{aligned}
$$

$$
\begin{aligned}
&170530261912808301883784986624\ k^5\ m^{22} + 10579465563290662891586632089 6\ k^6\ m^{22} + \\
&63770890223290428628843 2956\ k^7\ m^{22} - 14797232788368493080847892 48\ k^8\ m^{22} - 28962112084202482255167488\ k^9\ m^{22} + \\
&85771836300174859368939 52\ k^{10}\ m^{22} + 14481295224694585873203 2\ k^{11}\ m^{22} - 23966074149003200028672\ k^{12}\ m^{22} - \\
&46617590466491054489 6\ k^{13}\ m^{22} + 2373220889749252505 6\ k^{14}\ m^{22} + 557874913352187904\ k^{15}\ m^{22} - \\
&1201723011579904\ k^{16}\ m^{22} - 663731601862263068160834 6599424\ m^{23} + 13103063311390869928637204692992\ k\ m^{23} + \\
&11576086320367290602873739214848\ k^2\ m^{23} - 24817688383288441571047537049 60\ k^3\ m^{23} - \\
&10435705745601760181373292380 16\ k^4\ m^{23} + 52648628679640243639582392320\ k^5\ m^{23} + \\
&31483279815375730950570573824\ k^6\ m^{23} + 20038956477300909237038284 8\ k^7\ m^{23} - \\
&34499075523331965523483033 6\ k^8\ m^{23} - 64862860532613666755870 72\ k^9\ m^{23} + 14640635082776987164672 00\ k^{10}\ m^{23} + \\
&25038980729511039385600\ k^{11}\ m^{23} - 25919602924450016460 80\ k^{12}\ m^{23} - 46275609002445520896\ k^{13}\ m^{23} + \\
&11712005353055518 72\ k^{14}\ m^{23} + 16802498673934336\ k^{15}\ m^{23} - 27402683260114617568887012066 28\ m^{24} + \\
&49882001989672809667797669642 24\ k\ m^{24} + 45185337794759326512921170411 52\ k^2\ m^{24} - \\
&80697980648633677653548367 8720\ k^3\ m^{24} - 33425468780158015471254909747 2\ k^4\ m^{24} + \\
&13899905132591063003623653376\ k^5\ m^{24} + 79977503967317725796441784 32\ k^6\ m^{24} + 53507042658943487046909952\ k^7\ m^{24} - \\
&66718078736849477992480768\ k^8\ m^{24} - 11757063889260197244436 48\ k^9\ m^{24} + 19707116172736080822272 0\ k^{10}\ m^{24} + \\
&31812099205839530885 12\ k^{11}\ m^{24} - 1983762541544089518 08\ k^{12}\ m^{24} - 2846986041333710848\ k^{13}\ m^{24} + \\
&26764936664350720\ k^{14}\ m^{24} - 98001673952976785646999856 7424\ m^{25} + 16648358637333585292292954370 2528\ k\ m^{25} + \\
&15394974449770946634163381862 40\ k^2\ m^{25} - 22791341419872633624025235456 0\ k^3\ m^{25} - \\
&92326728187703117752899207168\ k^4\ m^{25} + 31111793933148288903323320 32\ k^5\ m^{25} + \\
&17164954950665392362596597 76\ k^6\ m^{25} + 11843763828070243876667392\ k^7\ m^{25} - 10507038734918652487925760\ k^8\ m^{25} - \\
&16815985881358639779020 8\ k^9\ m^{25} + 20124417020450413608960\ k^{10}\ m^{25} + 28206362624733079142 4\ k^{11}\ m^{25} - \\
&95596946679504568 32\ k^{12}\ m^{25} - 82105367351197696\ k^{13}\ m^{25} - 30280548842123523975892962524 16\ m^{26} + \\
&48473906365900549824948115660 8\ k\ m^{26} + 45592415014365873164639561318 4\ k^2\ m^{26} - \\
&55538654876562595303400407040\ k^3\ m^{26} - 21819570684670002504385167360\ k^4\ m^{26} + \\
&58357063123286657993277440 0\ k^5\ m^{26} + 30697215639780063425999667 2\ k^6\ m^{26} + 21223902930290066231132 16\ k^7\ m^{26} - \\
&13126104427276832495370 24\ k^8\ m^{26} - 18267008948141414416384\ k^9\ m^{26} + 14643017778374959169536\ k^{10}\ m^{26} + \\
&15648289026082865152\ k^{11}\ m^{26} - 217619167670829056\ k^{12}\ m^{26} - 80446909981003930062708277248\ m^{27} + \\
&12233657462743976668704355614 72\ k\ m^{27} + 11668189430247832936097356185 6\ k^2\ m^{27} - \\
&11577062066493205497179537408\ k^3\ m^{27} - 43661500618188858525142220 80\ k^4\ m^{27} + \\
&90310291192164830229037056\ k^5\ m^{27} + 44889919364237824847511552\ k^6\ m^{27} + 29938251439489027119513 6\ k^7\ m^{27} - \\
&12509849521393503672729 6\ k^8\ m^{27} - 14162745559265687633 92\ k^9\ m^{27} + 67590178512416800768\ k^{10}\ m^{27} + \\
&409913326072233984\ k^{11}\ m^{27} - 18246860522851297037446545408\ m^{28} + 26539617987565953820372500480\ k\ m^{28} + \\
&25603558453441745540805033984\ k^2\ m^{28} - 20413381287086377461790801 92\ k^3\ m^{28} - 72951776425861284940165939 2\ k^4\ m^{28} + \\
&11286654665311568667869184\ k^5\ m^{28} + 52271527638634985926164488\ k^6\ m^{28} + 31964239954867925286912\ k^7\ m^{28} - \\
&85408158144568869519 36\ k^8\ m^{28} - 6981831899725745356 8\ k^9\ m^{28} + 1486394721945255936\ k^{10}\ m^{28} - \\
&34986816565663963791470100 48\ m^{29} + 48957014696837963516812984 32\ k\ m^{29} + 47670163331772044658868224 00\ k^2\ m^{29} - \\
&30001254915145793676509184 0\ k^3\ m^{29} - 99869372328810858160324608\ k^4\ m^{29} + 11051919498762272825671 68\ k^5\ m^{29} + \\
&46598701437436874981376 0\ k^6\ m^{29} + 2427073142059601756160\ k^7\ m^{29} - 37184389925916848947 2\ k^8\ m^{29} - \\
&16446690233438699 52\ k^9\ m^{29} - 55947107927438382634264166 4\ m^{30} + 75703266350015701111442124 8\ k\ m^{30} + \\
&74264618175095836197991219 2\ k^2\ m^{30} - 36028327618463886521401344\ k^3\ m^{30} - 10907585514770831945760768\ k^4\ m^{30} + \\
&81063474944326213042176\ k^5\ m^{30} + 29857465519284336721920\ k^6\ m^{30} + 11678355609902854963 2\ k^7\ m^{30} - \\
&77531094376859566 08\ k^8\ m^{30} - 73232027522407594010345472\ m^{31} + 96256799877796852515471360\ k\ m^{31} + \\
&94986704876462834668535808\ k^2\ m^{31} - 34387380321009319437926 40\ k^3\ m^{31} - 91374013251411648498892 8\ k^4\ m^{31} + \\
&41432432589721074401 28\ k^5\ m^{31} + 12238103720689013888 28\ k^6\ m^{31} + 2677085808686530560\ k^7\ m^{31} - \\
&76418573429230576179609 60\ m^{32} + 97967897847367987101696 00\ k\ m^{32} + 97119853874107838889984 00\ k^2\ m^{32} - \\
&25045598141264699169177 6\ k^3\ m^{32} - 55114905190335315443712\ k^4\ m^{32} + 1292448858518763601 92\ k^5\ m^{32} + \\
&24095937072533078016\ k^6\ m^{32} - 61126222477664377110528 0\ m^{33} + 76711509784629178425344\ k\ m^{33} + \\
&76311742761776781774028 8\ k^2\ m^{33} - 13045999598383965143040\ k^3\ m^{33} - 21303908202080517488 64\ k^4\ m^{33} + \\
&17889497584427335 68\ k^5\ m^{33} - 35186950223590777159680\ m^{34} + 43372942612316163145728\ k\ m^{34} + \\
&43255589965916868182016\ k^2\ m^{34} - 43160907621074573721 6\ k^3\ m^{34} - 39624447712523452416\ k^4\ m^{34} - \\
&12974168930449631477 76\ m^{35} + 15756655076280253808 64\ k\ m^{35} + 15740797892820782284 80\ k^2\ m^{35} - \\
&67899358250532864 00\ k^3\ m^{35} - 23009560303364997120\ m^{36} + 27611472364037996544\ k\ m^{36} + 27611472364037996544\ k^2\ m^{36}
\end{aligned}
$$

$$
\begin{aligned}
p_1(k,m) = &-(k+2\,m)\,(-161815037129093174400000\,-319693227466239793152000\,k-\\
&161244991735950875256000\,k^2-94240955138688980496000\,k^3-14725760806491813689180 0\,k^4+\\
&77508551457214207092000\,k^5+10259326116982420817094 00\,k^6+32772047057543669151780 0\,k^7-\\
&14084717092315344393660 0\,k^8-89250868915173180041400\,k^9+25974512448072343596 00\,k^{10}+\\
&87995616624641236392 00\,k^{11}+66331418420504641320 0\,k^{12}-31526601863272094880 0\,k^{13}-\\
&52941408188972965200\,k^{14}-2469231171861350400\,k^{15}+164467384242109200 0\,k^{16}+375678709779088800\,k^{17}-\\
&16552049450391000\,k^{18}-5743434312693000\,k^{19}-206020374550200\,k^{20}-1085153970600\,k^{21}+\\
&3195000547200\,k^{22}+217840946400\,k^{23}-340686610801718819097600 00\,m-6163494582860649681696000 0\,k\,m-\\
&35925497051765198620176000\,k^2\,m-41975058850497920784319200\,k^3\,m-45555081377808560872088880\,k^4\,m+\\
&69995110180851190596070 80\,k^5\,m+27751050340050049492690980\,k^6\,m+71323677466907556630403860\,k^7\,m-\\
&35728407931504175467083 00\,k^8\,m-20116789710481716186702 60\,k^9\,m+49282199177529524364900\,k^{10}\,m+\\
&19127391813000220229550 0\,k^{11}\,m+18977840496310989072780\,k^{12}\,m-55600992206641320450 00\,k^{13}\,m-\\
&15461339904385459392 80\,k^{14}\,m-19063022807765168316 0\,k^{15}\,m+48546729680648769360\,k^{16}\,m+\\
&14156082836168938560\,k^{17}\,m-240724731541525560\,k^{18}\,m-229928922954816120\,k^{19}\,m-19782198274224960\,k^{20}\,m-\\
&116435974420080\,k^{21}\,m+302444030694960\,k^{22}\,m+20564248209120\,k^{23}\,m-331195732599234540139200000\,m^2-\\
&52927679229209047699914240 0\,k\,m^2-37747625823376775394006240 0\,k^2\,m^2-66566844468795634321822128 0\,k^3\,m^2-\\
&59415559053356603481018264 0\,k^4\,m^2+12134731508615334723636838 60\,k^5\,m^2+32103716520808638987709428 0\,k^6\,m^2+\\
&70138509693366680792432640\,k^7\,m^2-36445313733038167022631720\,k^8\,m^2-19447182663901727774259780\,k^9\,m^2+\\
&49598969022059983094520\,k^{10}\,m^2+16968473252879997447282 60\,k^{11}\,m^2+21654680003878037051754 0\,k^{12}\,m^2-\\
&36250605193838342112540\,k^{13}\,m^2-15279130621634794845840\,k^{14}\,m^2-25149753808836110823 60\,k^{15}\,m^2+\\
&42502887927687553884 0\,k^{16}\,m^2+14277828174517222938 0\,k^{17}\,m^2-543615258174710400\,k^{18}\,m^2-\\
&2099645803947329820\,k^{19}\,m^2-206942952758254020\,k^{20}\,m^2-2252259979675020\,k^{21}\,m^2+2796490116105840\,k^{22}\,m^2+\\
&185295033952560\,k^{23}\,m^2-19516400731494762245612352 00\,m^3-25039699240935113698383950 40\,k\,m^3-\\
&24739649668530995852697840 0\,k^2\,m^3-60866444581057478426220706 32\,k^3\,m^3-46555816217023747696154574 60\,k^4\,m^3+\\
&11277351871184904751917856306\,k^5\,m^3+22385653746196989523676324 94\,k^6\,m^3+41504887085428697857978716 0\,k^7\,m^3-\\
&21653449687387721929832348 4\,k^8\,m^3-11106333427674841731183868 2\,k^9\,m^3-32054174249583358626928 74\,k^{10}\,m^3+\\
&85968322199797243776236 64\,k^{11}\,m^3+14156892082995838541046 54\,k^{12}\,m^3-10438970016998150770170 6\,k^{13}\,m^3-\\
&82902444279609889376280\,k^{14}\,m^3-16603089692465429358372\,k^{15}\,m^3+19365565365499368976 74\,k^{16}\,m^3+\\
&75616625804331690023 4\,k^{17}\,m^3+547226795005677471 6\,k^{18}\,m^3-971083576845684622 8\,k^{19}\,m^3-\\
&1044823977190788984\,k^{20}\,m^3-16564476972322512\,k^{21}\,m^3+12231244509471864\,k^{22}\,m^3+785519105408208\,k^{23}\,m^3-\\
&76279946321147622013785715 20\,m^4-54397169886131871618854188 80\,k\,m^4-11121518545748734192379054640\,k^2\,m^4-\\
&38231324135689664098905507864\,k^3\,m^4-25396827977008135075515252948\,k^4\,m^4+\\
&70866914612882013206694707 22\,k^5\,m^4+10857089368944602991182267550\,k^6\,m^4+16410329665836850849095729 06\,k^7\,m^4-\\
&87217111898395305463014780 6\,k^8\,m^4-42525669469534959308300002 8\,k^9\,m^4-29187226262662154844060720\,k^{10}\,m^4+\\
&28127653049153489573911734\,k^{11}\,m^4+61497680556343119780493 08\,k^{12}\,m^4-87993569246680915561 2\,k^{13}\,m^4-\\
&29328065766591509574734 4\,k^{14}\,m^4-69451986800363215068126\,k^{15}\,m^4+54476255232504868206 48\,k^{16}\,m^4+\\
&25713854167796087879 88\,k^{17}\,m^4+44769199049902760634\,k^{18}\,m^4-28104612478233731562\,k^{19}\,m^4-\\
&3278003866062272802\,k^{20}\,m^4-67250625469271190\,k^{21}\,m^4+32624155342928040\,k^{22}\,m^4+2019068098104312\,k^{23}\,m^4-\\
&19880374549764717709314955200\,m^5+11910513110810886097880091552\,k\,m^5-34563590697081447416968185312\,k^2\,m^5-\\
&18018360079449956084183350068 0\,k^3\,m^5-10466473052502699047628514938 0\,k^4\,m^5+\\
&33288747997659062098507952622\,k^5\,m^5+39641227193413342546900924521\,k^6\,m^5+\\
&44683641594528496564541497 75\,k^7\,m^5-25924482715952618337846250 11\,k^8\,m^5-11573828981253421070102014 47\,k^9\,m^5-\\
&13906845845883996423674857 1\,k^{10}\,m^5+61571074886276778374765025\,k^{11}\,m^5+19210969622445636417473889\,k^{12}\,m^5+\\
&11606292203741450170335 01\,k^{13}\,m^5-73527622575343472069274 0\,k^{14}\,m^5-20445468393985941375295 2\,k^{15}\,m^5+\\
&99951057449141975286 76\,k^{16}\,m^5+61720382492084726160 42\,k^{17}\,m^5+16676628441713456515 8\,k^{18}\,m^5-\\
&55936509608219339036\,k^{19}\,m^5-7109427325544375466\,k^{20}\,m^5-175626076234351026\,k^{21}\,m^5+58957241580496780\,k^{22}\,m^5+\\
&3492100945756608\,k^{23}\,m^5-30044260345288243106309444928\,m^6+15769167985830361951295597808 0\,k\,m^6-\\
&64306529153697754623744362304\,k^2\,m^6-67097481070180896235373335312\,k^3\,m^6-\\
&34347591342698511253983004759 2\,k^4\,m^6+12319523760803130221266978438 4\,k^5\,m^6+\\
&11462353344591389559478450543 8\,k^6\,m^6+79228251628696155772028995 42\,k^7\,m^6-\\
&60304943003128155747100823 78\,k^8\,m^6-22749981754973846916203093 50\,k^9\,m^6-44394556371585222621354810 2\,k^{10}\,m^6+\\
&84194469148632633959262386\,k^{11}\,m^6+45264403602445669089479586\,k^{12}\,m^6+51024240627121837173269 86\,k^{13}\,m^6-\\
&13659010530876068706131 6\,k^{14}\,m^6-44903456818815984124787 2\,k^{15}\,m^6+11416382345342863787380\,k^{16}\,m^6+\\
&11028522581274082481228\,k^{17}\,m^6+39884459504247688909 2\,k^{18}\,m^6-80399258324999304896\,k^{19}\,m^6-\\
&11280921629958165020\,k^{20}\,m^6-318995499760363720\,k^{21}\,m^6+76233496205131360\,k^{22}\,m^6+4285030175890752\,k^{23}\,m^6+\\
&64370466923603944866823238 4\,m^7+76436015590870394968503090432 0\,k\,m^7+94929543359449488971601924 96\,k^2\,m^7-\\
&20401535236746965791096100 66664\,k^3\,m^7-93132645150610405986354933609 2\,k^4\,m^7+\\
&37036944843724263850314258 1950\,k^5\,m^7+27197458222378313737875538278 37\,k^6\,m^7+\\
&57081248136545161082412234 31\,k^7\,m^7-11531605463964656512215657851\,k^8\,m^7-30982663185467094421551217 39\,k^9\,m^7-\\
&10394411983969121846736606 961\,k^{10}\,m^7+36009592687727016578386667\,k^{11}\,m^7+82932252075465624770508655\,k^{12}\,m^7+\\
&13311847924213901853412547\,k^{13}\,m^7-19203341645424561068225 76\,k^{14}\,m^7-76292260155523628771299 6\,k^{15}\,m^7+\\
&49861344934905342626436\,k^{16}\,m^7+15154510152663340114006\,k^{17}\,m^7+68050064743852491440 8\,k^{18}\,m^7-\\
&85491597472898111994\,k^{19}\,m^7-13536640042313149024\,k^{20}\,m^7-421191674974995976\,k^{21}\,m^7+72686560553890848\,k^{22}\,m^7+\\
&3835950787248384\,k^{23}\,m^7+13654923858200965256319 8878272\,m^8+25429592365591467847419569586 24\,k\,m^8+\\
&59306767188900479662182961 3888\,k^2\,m^8-51778253435036328597787801555 20\,k^3\,m^8-\\
&21430514887114172497274490948 24\,k^4\,m^8+92236344858259881120247589716 0\,k^5\,m^8+\\
&54363700615196313352981095059 4\,k^6\,m^8-16023015494069328231556370330\,k^7\,m^8-
\end{aligned}
$$

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\begin{aligned}
&189862083707985170291892127 78\, k^{8}\, m^{8} - 225742357665747641412268159 0\, k^{9}\, m^{8} - \\
&186871781995632701163838124 2\, k^{10}\, m^{8} - 144668647265261900128759422\, k^{11}\, m^{8} + 120479836212560633619345378\, k^{12}\, m^{8} + \\
&249664508197267871112996 54\, k^{13}\, m^{8} - 204305456248828256292910 0\, k^{14}\, m^{8} - 102692911022980300091428 4\, k^{15}\, m^{8} - \\
&903287810654643829508 8\, k^{16}\, m^{8} + 16341591602597961869456\, k^{17}\, m^{8} + 871144122858507801228\, k^{18}\, m^{8} - \\
&675863633133138721 00\, k^{19}\, m^{8} - 125253476999512228 24\, k^{20}\, m^{8} - 41467548509878032 0\, k^{21}\, m^{8} + 51902671075771520\, k^{22}\, m^{8} + \\
&253624043577177 6\, k^{23}\, m^{8} + 381583868626777498743963656448\, m^{9} + 655961834224334752558818013900 8\, k\, m^{9} + \\
&264296583276456754352686798104 0\, k^{2}\, m^{9} - 111382816854383680493454865715 92\, k^{3}\, m^{9} - \\
&426517215857930292115287235132 4\, k^{4}\, m^{9} + 192878365723773737341278203927 4\, k^{5}\, m^{9} + \\
&933543053712361591286618918060\, k^{6}\, m^{9} - 735558551575649784894703678 42\, k^{7}\, m^{9} - \\
&280484929218263311196612491 82\, k^{8}\, m^{9} + 147380671149622886875840260 4\, k^{9}\, m^{9} - \\
&264575261230535063833553 20\, k^{10}\, m^{9} - 453842868626287423097692046\, k^{11}\, m^{9} + 140349159086973033671563716\, k^{12}\, m^{9} + \\
&360412428320134776426216 16\, k^{13}\, m^{9} - 158931957893567332903839 2\, k^{14}\, m^{9} - 111266750996735121135637 6\, k^{15}\, m^{9} - \\
&237013374654862212425210\, k^{16}\, m^{9} + 13994849921330157589826\, k^{17}\, m^{9} + 8605642326314005409 62\, k^{18}\, m^{9} - \\
&38983557658047866328\, k^{19}\, m^{9} - 9029073729351348944\, k^{20}\, m^{9} - 308420476642550 72\, k^{21}\, m^{9} + \\
&27888112072671680\, k^{22}\, m^{9} + 1238581163787264\, k^{23}\, m^{9} + 505218696082367932722742970112\, m^{10} + \\
&137948025869639740694677907232 00\, k\, m^{10} + 771353444552670570134200491566 4\, k^{2}\, m^{10} - \\
&205334913596424958047813996212 08\, k^{3}\, m^{10} - 743468085902172484964530851218 8\, k^{4}\, m^{10} + \\
&342112150864016914648070122300 6\, k^{5}\, m^{10} + 139692091298446236608769141545 4\, k^{6}\, m^{10} - \\
&171753911276049550933230797522\, k^{7}\, m^{10} - 381667895967890616145200641 42\, k^{8}\, m^{10} + \\
&787303337625213036204618751 2\, k^{9}\, m^{10} - 298624917233873606122299682 6\, k^{10}\, m^{10} - \\
&788079547889708255178728476\, k^{11}\, m^{10} + 131572846394841317696218470\, k^{12}\, m^{10} + \\
&414394821909561996717656 2\, k^{13}\, m^{10} - 783834511496162550988060\, k^{14}\, m^{10} - 980493362002886837075730\, k^{15}\, m^{10} - \\
&309115242869291666482846 4\, k^{16}\, m^{10} + 9571127592698935744128\, k^{17}\, m^{10} + 666552393656197357896\, k^{18}\, m^{10} - \\
&152795183418164852 32\, k^{19}\, m^{10} - 5086190042109216768\, k^{20}\, m^{10} - 173889640104781184\, k^{21}\, m^{10} + \\
&112202361771033 60\, k^{22}\, m^{10} + 441296026085376\, k^{23}\, m^{10} - 722444897708833771376456584 32\, m^{11} + \\
&243020383025052377314926230273 28\, k\, m^{11} + 174050281270978082302220217536 16\, k^{2}\, m^{11} - \\
&327054725337524161675278063496 96\, k^{3}\, m^{11} - 114331805895391026783157091396 80\, k^{4}\, m^{11} + \\
&518716883073338383425128266456\, k^{5}\, m^{11} + 183878890655933709852132477730 4\, k^{6}\, m^{11} - \\
&291289412396595796467028693508\, k^{7}\, m^{11} - 479950669102130650739221604 68\, k^{8}\, m^{11} + \\
&147921959391384927575183874 56\, k^{9}\, m^{11} - 268779420690520705660365294 0\, k^{10}\, m^{11} - \\
&998803642423695245576517056\, k^{11}\, m^{11} + 986843367717748050301089 96\, k^{12}\, m^{11} + \\
&387018573215420752442726 08\, k^{13}\, m^{11} - 416669578671571555554556\, k^{14}\, m^{11} - 706963609791350662711948\, k^{15}\, m^{11} - \\
&281992878167245636016 32\, k^{16}\, m^{11} + 5225736921643259908472\, k^{17}\, m^{11} + 4079978355888388194672\, k^{18}\, m^{11} - \\
&29740321153129734 08\, k^{19}\, m^{11} - 2230804385449469568\, k^{20}\, m^{11} - 73897294885113216\, k^{21}\, m^{11} + \\
&33272176357713 92\, k^{22}\, m^{11} + 111517013803008\, k^{23}\, m^{11} - 209047203985317398047114812806 4\, m^{12} + \\
&364742165828440639447713640428 48\, k\, m^{12} + 321293492382417724023405094403 52\, k^{2}\, m^{12} - \\
&452861676532049487519336360405 76\, k^{3}\, m^{12} - 155633622177433903946929955272 32\, k^{4}\, m^{12} + \\
&676810602158705334071714938159 2\, k^{5}\, m^{12} + 214068397169472861752878277211 2\, k^{6}\, m^{12} - \\
&391783799588141560488944014496\, k^{7}\, m^{12} - 549508406649039372944927865 60\, k^{8}\, m^{12} + \\
&193125632552384242228885751 2\, k^{9}\, m^{12} - 189720069831666747006572095 2\, k^{10}\, m^{12} - \\
&996735444250799211573983408\, k^{11}\, m^{12} + 579259153469693976375628 24\, k^{12}\, m^{12} + \\
&297054503245436284133333 44\, k^{13}\, m^{12} + 357004584127208577223144\, k^{14}\, m^{12} - 418065517994952672826496\, k^{15}\, m^{12} - \\
&195262843509528649436 24\, k^{16}\, m^{12} + 2262859947494069266880\, k^{17}\, m^{12} + 197634794393487075744\, k^{18}\, m^{12} + \\
&681231337128217920\, k^{19}\, m^{12} - 753555863355754624\, k^{20}\, m^{12} - 23289796322084864\, k^{21}\, m^{12} + \\
&70513743601664 0\, k^{22}\, m^{12} + 18931060211712\, k^{23}\, m^{12} - 589982214960526025771711958770 56\, m^{13} + \\
&471685462272453392834827878400 32\, k\, m^{13} + 498714412413441106767817135824 00\, k^{2}\, m^{13} - \\
&547703845334226589528268419095 52\, k^{3}\, m^{13} - 187683534348408082570834931917 92\, k^{4}\, m^{13} + \\
&763481641662457530791336067785 6\, k^{5}\, m^{13} + 220881994003312216218362122363 2\, k^{6}\, m^{13} - \\
&433722147477993564202722241744\, k^{7}\, m^{13} - 561308622085581879613763828 16\, k^{8}\, m^{13} + \\
&196877660433447824323190023 20\, k^{9}\, m^{13} - 993670399213314049081786656\, k^{10}\, m^{13} - \\
&809026634598666509994972608\, k^{11}\, m^{13} + 249905050493741278978779 2\, k^{12}\, m^{13} + \\
&188609200345937759830709 44\, k^{13}\, m^{13} + 411210107159486601590864\, k^{14}\, m^{13} - 202533781833253619593248\, k^{15}\, m^{13} - \\
&105780393770315307268 32\, k^{16}\, m^{13} + 765695142822545772096\, k^{17}\, m^{13} + 754030064964427880000\, k^{18}\, m^{13} + \\
&7857051875804778 24\, k^{19}\, m^{13} - 192118442618339072\, k^{20}\, m^{13} - 5277234510344192\, k^{21}\, m^{13} + 100955284193280\, k^{22}\, m^{13} + \\
&193542409420 8\, k^{23}\, m^{13} - 109135282982160943363477869924 48\, m^{14} + 529734275646639943763140041775 68\, k\, m^{14} + \\
&661389689949331529310521884600 92\, k^{2}\, m^{14} - 580669035210614081045601357064 96\, k^{3}\, m^{14} - \\
&200400845663824913549285392581 44\, k^{4}\, m^{14} + 747628470383879714728234093120 0\, k^{5}\, m^{14} + \\
&201965540948004275173403111264\, k^{6}\, m^{14} - 403013403037489986085303989792\, k^{7}\, m^{14} - \\
&503111526208064672540665670 08\, k^{8}\, m^{14} + 163607281521530810780312751 36\, k^{9}\, m^{14} - \\
&311974469171218691839841024\, k^{10}\, m^{14} - 542888507406994112692794144\, k^{11}\, m^{14} + 622755992741209110019875 2\, k^{12}\, m^{14} + \\
&993222367204486175469219 2\, k^{13}\, m^{14} + 289222152126476452879296\, k^{14}\, m^{14} - 799982373561169781592 32\, k^{15}\, m^{14} - \\
&453123730495011832972 8\, k^{16}\, m^{14} + 196859814519934642688\, k^{17}\, m^{14} + 22395707653618299264\, k^{18}\, m^{14} + \\
&321427119250183936\, k^{19}\, m^{14} - 35723820239900672\, k^{20}\, m^{14} - 812580857020416\, k^{21}\, m^{14} + 8739414999040\, k^{22}\, m^{14} + \\
&9001972531 2\, k^{23}\, m^{14} - 156172789224948042510502264542 72\, m^{15} + 519525101786243209473505246358 40\, k\, m^{15} + \\
&757158233406403579479850812120 96\, k^{2}\, m^{15} - 541117368407714739412841000207 872\, k^{3}\, m^{15} - \\
&189249487882860492369490922712 32\, k^{4}\, m^{15} + 637389040119017452083534814944 0\, k^{5}\, m^{15} + \\
&163377988434196913448266189299 2\, k^{6}\, m^{15} - 317918858738229627625429697664\, k^{7}\, m^{15} -
\end{aligned}
$$

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\begin{aligned}
&391463799179891663179899266 56\,k^{8}\,m^{15}+1131064053148599557058698188 8\,k^{9}\,m^{15}+\\
&34831348979875744048379072\,k^{10}\,m^{15}-30366837461291245223866393 6\,k^{11}\,m^{15}-\\
&86645654063400290334790 4\,k^{12}\,m^{15}+43345927960633967834092800\,k^{13}\,m^{15}+14948346210746093258105 6\,k^{14}\,m^{15}-\\
&25531103576918027079040\,k^{15}\,m^{15}-153393673446952719616 0\,k^{16}\,m^{15}+3638864300152049689 6\,k^{17}\,m^{15}+\\
&5071848627658562816\,k^{18}\,m^{15}+79685999687331840\,k^{19}\,m^{15}-4567362567184384\,k^{20}\,m^{15}-76102300696576\,k^{21}\,m^{15}+\\
&345075613696\,k^{22}\,m^{15}-18309355395569947570512684783744\,m^{16}+4466426796647464762683710394854 4\,k\,m^{16}+\\
&75349350164210570617409916362880\,k^{2}\,m^{16}-4440933174275588003488917687616 0\,k^{3}\,m^{16}-\\
&15785563446955089588112391802496\,k^{4}\,m^{16}+4740507849725393992564198551040\,k^{5}\,m^{16}+\\
&11663771337572392832412178140 16\,k^{6}\,m^{16}-21435616453368543235220692992 0\,k^{7}\,m^{16}-\\
&26274224878724171317589666944\,k^{8}\,m^{16}+65725273335604825142507555328\,k^{9}\,m^{16}+\\
&12630685876269295693209292 8\,k^{10}\,m^{16}-14204230106484802891483456 0\,k^{11}\,m^{16}-\\
&19235345903856274803712 0\,k^{12}\,m^{16}+15612873165107794706865 92\,k^{13}\,m^{16}+59668546014581267183872\,k^{14}\,m^{16}-\\
&64888521691116824207 36\,k^{15}\,m^{16}-406403700899047613696\,k^{16}\,m^{16}+4222046020892418048\,k^{17}\,m^{16}+\\
&845957954011636224\,k^{18}\,m^{16}+12521615330701312\,k^{19}\,m^{16}-358587949547520\,k^{20}\,m^{16}-\\
&3270716686336\,k^{21}\,m^{16}-18076889329550428859736134160384\,m^{17}+3374431733937508125620082921446 4\,k\,m^{17}+\\
&65499512050416295320836617909248\,k^{2}\,m^{17}-32137243351057591629126491358208\,k^{3}\,m^{17}-\\
&11613676923382046343753506947072\,k^{4}\,m^{17}+3079089011879098811409946805760\,k^{5}\,m^{17}+\\
&73271426324911204021856924083 2\,k^{6}\,m^{17}-12398855469941294670342504448 0\,k^{7}\,m^{17}-\\
&15151108875453899809827492352\,k^{8}\,m^{17}+32254634824440640527346682 88\,k^{9}\,m^{17}+\\
&10014416533456470773092326 4\,k^{10}\,m^{17}-55526006651836581490623232\,k^{11}\,m^{17}-11904402947966011364725 76\,k^{12}\,m^{17}+\\
&46054363469480001025075 2\,k^{13}\,m^{17}+18648547773943357633024\,k^{14}\,m^{17}-12844666557145369548 80\,k^{15}\,m^{17}-\\
&82610789517085040640\,k^{16}\,m^{17}+150300121899976 96\,k^{17}\,m^{17}+97937464000133120\,k^{18}\,m^{17}+\\
&1157151995084800\,k^{19}\,m^{17}-13019102773248\,k^{20}\,m^{17}-152626224967074715049062414986 24\,m^{18}+\\
&22434944476404527713034273714688\,k\,m^{18}+49897747753440507980843502211584\,k^{2}\,m^{18}-\\
&20517750657515473066911759161344\,k^{3}\,m^{18}-75248868581265063028543116702 72\,k^{4}\,m^{18}+\\
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&61609656194882008800111353856\,k^{7}\,m^{18}-74819785718230015883288212 48\,k^{8}\,m^{18}+\\
&13380295845994894237675 05216\,k^{9}\,m^{18}+52924516342533224165030912\,k^{10}\,m^{18}-18066494842951280672069120\,k^{11}\,m^{18}-\\
&48879338858168588740147 2\,k^{12}\,m^{18}+10989742317403006250137 6\,k^{13}\,m^{18}+45476809691461364930 56\,k^{14}\,m^{18}-\\
&19135501582375929907 2\,k^{15}\,m^{18}-12449999834734803968\,k^{16}\,m^{18}-37050847075753984\,k^{17}\,m^{18}+\\
&7026526229741568\,k^{18}\,m^{18}+48231818657792\,k^{19}\,m^{18}-11121602331166013516256107034624\,m^{19}+\\
&13131103221470632175185666285888\,k\,m^{19}+33379665075678562679128030015488\,k^{2}\,m^{19}-\\
&11555126153368942284163949310976\,k^{3}\,m^{19}-42861822251190404215517180436 48\,k^{4}\,m^{19}+\\
&86508162944329104997430462566 4\,k^{5}\,m^{19}+19432323941213490278189332582 4\,k^{6}\,m^{19}-\\
&26282802982455995095546506240\,k^{7}\,m^{19}-31525714413857027412166574 08\,k^{8}\,m^{19}+\\
&46816900896622935741509222 4\,k^{9}\,m^{19}+21226486084545168578724864\,k^{10}\,m^{19}-48553623287422539336171 52\,k^{11}\,m^{19}-\\
&14904678441729651832115 2\,k^{12}\,m^{19}+20830672707654886010880\,k^{13}\,m^{19}+85089777371222146969 6\,k^{14}\,m^{19}-\\
&2029606270343326822 4\,k^{15}\,m^{19}-1311141499969855488\,k^{16}\,m^{19}-6066053854947328\,k^{17}\,m^{19}+\\
&235199037308928\,k^{18}\,m^{19}-70324580359836966229120514211 84\,m^{20}+67622826751615904893926822789 12\,k\,m^{20}+\\
&19626599758062677904558380064768\,k^{2}\,m^{20}-57354693342381759686930584883 20\,k^{3}\,m^{20}-\\
&21414647036322347494788578816 00\,k^{4}\,m^{20}+37324343370819826190443446476 8\,k^{5}\,m^{20}+\\
&81373596399417135359648337920\,k^{6}\,m^{20}-96020042922146913298761871 36\,k^{7}\,m^{20}-\\
&11279943221201523888487538688\,k^{8}\,m^{20}+13745432209289765163758592 0\,k^{9}\,m^{20}+\\
&67058690135837169056419 84\,k^{10}\,m^{20}-10649909193068267839488 00\,k^{11}\,m^{20}-34649942133624743180288\,k^{12}\,m^{20}+\\
&3052622950835506352128\,k^{13}\,m^{20}+118230959602915084288\,k^{14}\,m^{20}-1387764321533984768\,k^{15}\,m^{20}-\\
&86149820103649280\,k^{16}\,m^{20}-305289708224512\,k^{17}\,m^{20}-38701167727615205241521338122 24\,m^{21}+\\
&30595853333597726394843021803 52\,k\,m^{21}+10142368681268735805904201678848\,k^{2}\,m^{21}-\\
&25049584325663387510229646336 00\,k^{3}\,m^{21}-93578664343900003123756285952 0\,k^{4}\,m^{21}+\\
&13990025876646418262549368012 8\,k^{5}\,m^{21}+29487341636727389522196508672\,k^{6}\,m^{21}-\\
&29906501475797862610762096 64\,k^{7}\,m^{21}-34046149701946765528421990 4\,k^{8}\,m^{21}+33578841961018666075394048\,k^{9}\,m^{21}+\\
&16834078051003256510218 24\,k^{10}\,m^{21}-18731942258152816673585 72\,k^{11}\,m^{21}-61122016446544984268 80\,k^{12}\,m^{21}+\\
&331889152604347830272\,k^{13}\,m^{21}+11508999383051833344\,k^{14}\,m^{21}-49224890480533504\,k^{15}\,m^{21}-\\
&2658714536259584\,k^{16}\,m^{21}-18554962034148288303985347133 44\,m^{22}+12132379622406585679562230824 96\,k\,m^{22}+\\
&46015690290743877055657896837 12\,k^{2}\,m^{22}-96023865955779652595011116236 8\,k^{3}\,m^{22}-\\
&35633527231238650208943026176 0\,k^{4}\,m^{22}+45363221214706001814938484736\,k^{5}\,m^{22}+\\
&91865498592254585812082769 92\,k^{6}\,m^{22}-78874857638189954737849139 2\,k^{7}\,m^{22}-\\
&85894198605189185041342464\,k^{8}\,m^{22}+67407386540338968205670 4\,k^{9}\,m^{22}+33375610859587324592128\,k^{10}\,m^{22}-\\
&25743386113299382460416\,k^{11}\,m^{22}-79641308833372260761 6\,k^{12}\,m^{22}+25033269901398482944\,k^{13}\,m^{22}+\\
&701143158154543104\,k^{14}\,m^{22}-393852239233024\,k^{15}\,m^{22}-77454980249129637912131036774 4\,m^{23}+\\
&42015609711023993771511929241 6\,k\,m^{23}+18291085363630618472527631281 28\,k^{2}\,m^{23}-\\
&32194172061183638321122208972 8\,k^{3}\,m^{23}-11767177921795209601893420376 8\,k^{4}\,m^{23}+\\
&12651201135016812795094056960\,k^{5}\,m^{23}+24404039723366671092229242 88\,k^{6}\,m^{23}-\\
&17446853032248643523392307 2\,k^{7}\,m^{23}-17884110080458650356875264\,k^{8}\,m^{23}+10921943964166812323676 16\,k^{9}\,m^{23}+\\
&51302886211992820793344\,k^{10}\,m^{23}-26577340596225744568 32\,k^{11}\,m^{23}-72534747734412181504\,k^{12}\,m^{23}+\\
&1157319279865593856\,k^{13}\,m^{23}+20131453434019840\,k^{14}\,m^{23}-28090655219794748060333762150 4\,m^{24}+\\
&12646739381552274489328926720 0\,k\,m^{24}+63495861005310474048196647321 6\,k^{2}\,m^{24}-\\
&93960340450181043660143165440\,k^{3}\,m^{24}-33490290573694110104807571456\,k^{4}\,m^{24}+
\end{aligned}
$$

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\begin{aligned}
&301113013836993729928668774 4\,k^5\,m^{24} + 5469638526281921975407083 52\,k^6\,m^{24} -\\
&319398210071921160883404 80\,k^7\,m^{24} - 3018802599231195205828608\,k^8\,m^{24} + 139174879009785910263808\,k^9\,m^{24} +\\
&591069276883244462899 2\,k^{10}\,m^{24} - 193367325227968528384\,k^{11}\,m^{24} - 4132375741134733312\,k^{12}\,m^{24} +\\
&24286892420071424\,k^{13}\,m^{24} - 881830377857537252055412899 84\,m^{25} + 3287848514264763949382094028 8\,k\,m^{25} +\\
&191631492020788843717011701760\,k^2\,m^{25} - 2372393402867957574950590873 6\,k^3\,m^{25} -\\
&814886914806881460873920512 0\,k^4\,m^{25} + 6053524730572056566560194 56\,k^5\,m^{25} +\\
&101995015084691459619094528\,k^6\,m^{25} - 4750730474079319516250112\,k^7\,m^{25} - 4026666774022108859596 80\,k^8\,m^{25} +\\
&134125278175014257950 72\,k^9\,m^{25} + 481087830809201803264\,k^{10}\,m^{25} - 881019514228421427 2\,k^{11}\,m^{25} -\\
&111021447253590016\,k^{12}\,m^{25} - 2383125225367944718635997593 6\,m^{26} + 7322182955451748918364995584\,k\,m^{26} +\\
&499772170189415800925352099 84\,k^2\,m^{26} - 5140290636036050152276492288\,k^3\,m^{26} -\\
&167737948033281651558750617 6\,k^4\,m^{26} + 1013810261710940874753703936\,k^5\,m^{26} +\\
&155281496846391973309317 12\,k^6\,m^{26} - 5592044833650589304094 72\,k^7\,m^{26} - 4084100923617488142336 0\,k^8\,m^{26} +\\
&917629797824141197312\,k^9\,m^{26} + 24695612557189578752\,k^{10}\,m^{26} - 18821155381248000\,k^{11}\,m^{26} -\\
&550262346181305584091188 4288\,m^{27} + 1381982035194696188207038464\,k\,m^{27} + 1117337163395019386660270899 2\,k^2\,m^{27} -\\
&945682397493201698031403008\,k^3\,m^{27} - 2880413897368315057888296 96\,k^4\,m^{27} + 1387956742113610373293670 4\,k^5\,m^{27} +\\
&187961878194659984120217 6\,k^6\,m^{27} - 5008655346626837033779 2\,k^7\,m^{27} - 2959394022466967568384\,k^8\,m^{27} +\\
&396467868275174277 12\,k^9\,m^{27} + 601781413612093440\,k^{10}\,m^{27} - 1074554896262900781611483136\,m^{28} +\\
&217938970318601308512190464\,k\,m^{28} + 2118960723072817149379608576\,k^2\,m^{28} - 1456669725888086919217152 00\,k^3\,m^{28} -\\
&404894894965914455205098 84\,k^4\,m^{28} + 1512589677943203412574208\,k^5\,m^{28} + 1739371492913555712245 76\,k^6\,m^{28} -\\
&320446924158539032166 4\,k^7\,m^{28} - 136453531955608682496\,k^8\,m^{28} + 811460308893696000\,k^9\,m^{28} -\\
&175047815165403031595384832\,m^{29} + 2817153335283340048072704 0\,k\,m^{29} + 3361232718500695214684897 28\,k^2\,m^{29} -\\
&184320034661201583978577 92\,k^3\,m^{29} - 4536757514247583646613504\,k^4\,m^{29} + 1261509204747212701040 64\,k^5\,m^{29} +\\
&115536232018441823846400\,k^6\,m^{29} - 130310959572197572608\,k^7\,m^{29} - 30082404719945318 40\,k^8\,m^{29} -\\
&233448389059852263342735 36\,m^{30} + 2905874170750317512097792\,k\,m^{30} + 4375718351587228472731238 4\,k^2\,m^{30} -\\
&186548478748427075715072 0\,k^3\,m^{30} - 3895669337279528467169 28\,k^4\,m^{30} + 7558268332441111363584\,k^5\,m^{30} +\\
&490367732891479179264\,k^6\,m^{30} - 2528090920612528128\,k^7\,m^{30} - 2482101927065476852088832\,m^{31} +\\
&229903600495053225590784\,k\,m^{31} + 4551799754445917238853632\,k^2\,m^{31} - 1451753030313600866058 24\,k^3\,m^{31} -\\
&240675270434211747594 24\,k^4\,m^{31} + 289516745119582126080\,k^5\,m^{31} + 9987576898907013120\,k^6\,m^{31} -\\
&202291717769558879109120\,m^{32} + 1309456326687214849228 8\,k\,m^{32} + 3637211145780676106649 60\,k^2\,m^{32} -\\
&815469990330438254592 0\,k^3\,m^{32} - 952169500588438978560\,k^4\,m^{32} + 5323021911783899136\,k^5\,m^{32} -\\
&118645326177067786567 68\,m^{33} + 4777783842868706672 64\,k\,m^{33} + 2095722737637071310028 8\,k^2\,m^{33} -\\
&294242607860589527040\,k^3\,m^{33} - 18114739899998404608\,k^4\,m^{33} - 4457108318171651112 96\,m^{34} +\\
&8383898537523412992\,k\,m^{34} + 7749034093487943843 84\,k^2\,m^{34} - 5120684935279017984\,k^3\,m^{34} -\\
&8053346106177748992\,m^{35} + 1380573618201899827 2\,k^2\,m^{35}\Big)
\end{aligned}
$$

$$
\begin{aligned}
p_2(k,m) = {} & (1+k-2\,m)\,(k+2\,m)\,(1+k+2\,m)\,\Big(- 6472601485163726976000 00 + 3215378374476915225600 00\,k -\\
&225920648995993286496000\,k^2 - 3435968966901065400960 00\,k^3 + 1541176868240940184128 00\,k^4 +\\
&135837221219064175088 00\,k^5 - 3906230502543346580400\,k^6 - 2599442553974915995920 0\,k^7 -\\
&327503405586550648140 0\,k^8 + 2380363475501834285400\,k^9 + 3609842693807923146 00\,k^{10} - 7784897565743944380 0\,k^{11} -\\
&154119423184028442 00\,k^{12} - 1485761766687815400\,k^{13} + 331830618073038000\,k^{14} + 130811834187997200\,k^{15} -\\
&245908792163160 0\,k^{16} - 1802418991077600\,k^{17} - 66116077486200\,k^{18} - 1995525088200\,k^{19} +\\
&94397743440 0\,k^{20} + 72613648800\,k^{21} - 1398014127322201428480000 0\,m + 1044709189528892339289600 0\,k\,m -\\
&946912438009663530491520 0\,k^2\,m - 9489479421499534208001600\,k^3\,m + 5172019515439979087380320\,k^4\,m +\\
&332673148708835148073944 0\,k^5\,m - 2709819101161749529234 80\,k^6\,m - 5937203022626421320166 00\,k^7\,m -\\
&676368678521203513486 20\,k^8\,m + 5036027560481429039178 0\,k^9\,m + 8363494403163203585400\,k^{10}\,m -\\
&113687854954505892396 0\,k^{11}\,m - 38624776476441857730 0\,k^{12}\,m - 81242941104505201140\,k^{13}\,m +\\
&83670399160175192 40\,k^{14}\,m + 4714394889987445320\,k^{15}\,m + 33635701890501480\,k^{16}\,m - 65928072709374120\,k^{17}\,m -\\
&634647911952096 0\,k^{18}\,m - 192959137118160\,k^{19}\,m + 89438503659120\,k^{20}\,m + 6854749403040\,k^{21}\,m -\\
&144637965894684727382400000\,m^2 + 1504867805700586618233600 00\,k\,m^2 - 1462538581495453337282668 80\,k^2\,m^2 -\\
&117353975995329436879084800\,k^3\,m^2 + 7020642993677870641695072 0\,k^4\,m^2 + 3562757030771490805001280 0\,k^5\,m^2 -\\
&419434785995616240287004 0\,k^6\,m^2 - 5693877418788131312048400\,k^7\,m^2 - 6490280094992622308469 60\,k^8\,m^2 +\\
&430152459948693732024180\,k^9\,m^2 + 8122834386544007285568 0\,k^{10}\,m^2 - 5384296365746215478400\,k^{11}\,m^2 -\\
&353856638186186821686 0\,k^{12}\,m^2 - 9821236835179505017 80\,k^{13}\,m^2 + 6604014093513892902 0\,k^{14}\,m^2 +\\
&467193042715739268 60\,k^{15}\,m^2 + 814366548089817660\,k^{16}\,m^2 - 591653709231444720\,k^{17}\,m^2 -\\
&662194099426897 80\,k^{18}\,m^2 - 2132008987123260\,k^{19}\,m^2 + 833791519441440\,k^{20}\,m^2 + 61765011317520\,k^{21}\,m^2 -\\
&955134107749212800247360 00\,m^3 + 1319090393288249287101319680\,k\,m^3 - 1299212224946104581844799904\,k^2\,m^3 -\\
&890806197347489856358740528\,k^3\,m^3 + 5638495079462442001060729 68\,k^4\,m^3 + 2304834176935526355839120 04\,k^5\,m^3 -\\
&348190436764262016014605092 88\,k^6\,m^3 - 3192424463748559040639721 0\,k^7\,m^3 - 3815822168158190627259612\,k^8\,m^3 +\\
&205883926058908237263696 0\,k^9\,m^3 + 4642937720429331472209 90\,k^{10}\,m^3 - 126085704009205043 56\,k^{11}\,m^3 -\\
&180721571962298936497 56\,k^{12}\,m^3 - 6123679369819833237912\,k^{13}\,m^3 + 263227478241063419406\,k^{14}\,m^3 +\\
&244113661898476778358\,k^{15}\,m^3 + 6198211022881683132\,k^{16}\,m^3 - 2714316842607456708\,k^{17}\,m^3 -\\
&333440290604213064\,k^{18}\,m^3 - 11297413021912704\,k^{19}\,m^3 + 3681858674364408\,k^{20}\,m^3 + 261839701802736\,k^{21}\,m^3 -\\
&452530233367463896530888600\,m^4 + 8033744666076186644985392640\,k\,m^4 - 7847942413370662302207217056\,k^2\,m^4 -\\
&473290762143117054969304560 0\,k^3\,m^4 + 3118452715963078062958103760\,k^4\,m^4 + 1029918584456344362818296080\,k^5\,m^4 -
\end{aligned}
$$

$$193215961003916212878287898\,k^{6}\,m^{4} - 11920993783549215287791890 6\,k^{7}\,m^{4} - 153416549044689761148472 32\,k^{8}\,m^{4} +$$
$$61232860993012620734514 18\,k^{9}\,m^{4} + 17938769783730640778130 84\,k^{10}\,m^{4} + 11641223334973432527277 2\,k^{11}\,m^{4} -$$
$$60255786876072420538188\,k^{12}\,m^{4} - 24515598540307345779006\,k^{13}\,m^{4} + 585552970339097330748\,k^{14}\,m^{4} +$$
$$820345737413773517340\,k^{15}\,m^{4} + 26736777205913401848\,k^{16}\,m^{4} - 7818486815535150348\,k^{17}\,m^{4} -$$
$$1043466440632209306\,k^{18}\,m^{4} - 36863227835091918\,k^{19}\,m^{4} + 9927573749897664\,k^{20}\,m^{4} + 673022699368104\,k^{21}\,m^{4} -$$
$$16389634707820560510914442624\,m^{5} + 365701223349563701211715231424\,k\,m^{5} - 3513619059669529312926792988 8\,k^{2}\,m^{5} -$$
$$18910600958853225312870784464\,k^{3}\,m^{5} + 12866150375946255695339461824\,k^{4}\,m^{5} +$$
$$34185948672010074398885829260\,k^{5}\,m^{5} - 790766739354589403025167406\,k^{6}\,m^{5} - 3147554244026854575445546 40\,k^{7}\,m^{5} -$$
$$44581382276347476664722775\,k^{8}\,m^{5} + 10983222403531238841494045\,k^{9}\,m^{5} + 5040724342731493362747474\,k^{10}\,m^{5} +$$
$$66242903034163999083049 2\,k^{11}\,m^{5} - 14154437373034955467848 3\,k^{12}\,m^{5} - 6965783069288579302170 5\,k^{13}\,m^{5} +$$
$$571135750842791960072\,k^{14}\,m^{5} + 1948255165712535028640\,k^{15}\,m^{5} + 76839589643217264348\,k^{16}\,m^{5} -$$
$$15521956383624485802\,k^{17}\,m^{5} - 225803872578789563 4\,k^{18}\,m^{5} - 82289271243944786\,k^{19}\,m^{5} + 18161039578768436\,k^{20}\,m^{5} +$$
$$1164033648585536\,k^{21}\,m^{5} - 4719511967388105658986462297 6\,m^{6} + 13026914914092801707623684032 0\,k\,m^{6} -$$
$$12278974931803825518566594995 2\,k^{2}\,m^{6} - 5942536718420555863119987134 4\,k^{3}\,m^{6} +$$
$$41567172164384172772436452848\,k^{4}\,m^{6} + 8814844457066078189800273292\,k^{5}\,m^{6} -$$
$$25183930295688790507688255 42\,k^{6}\,m^{6} - 5969584001165796736081372 66\,k^{7}\,m^{6} - 9608157367013376404152620 2\,k^{8}\,m^{6} +$$
$$68535841077359003195612 62\,k^{9}\,m^{6} + 10755978480779171035031826\,k^{10}\,m^{6} + 2172209798353678536960062\,k^{11}\,m^{6} -$$
$$24300950491284014745019 4\,k^{12}\,m^{6} - 14860718626709153174522 2\,k^{13}\,m^{6} - 7196240264667732394 52\,k^{14}\,m^{6} +$$
$$34491700388541732804 28\,k^{15}\,m^{6} + 158889772677166953804\,k^{16}\,m^{6} - 22309689231297801868\,k^{17}\,m^{6} -$$
$$3576775406462856992\,k^{18}\,m^{6} - 132849043219954096\,k^{19}\,m^{6} + 23806615514161504\,k^{20}\,m^{6} +$$
$$1428343391963584\,k^{21}\,m^{6} - 11091276499495252759562632 8832\,m^{7} + 3746773370745548001787741 00992\,k\,m^{7} -$$
$$34657914031996915634593763 9808\,k^{2}\,m^{7} - 1513798928577243496339916 2896\,k^{3}\,m^{7} +$$
$$10857556857392674220173692625 6\,k^{4}\,m^{7} + 1818540353507664796486862118 8\,k^{5}\,m^{7} -$$
$$64484544698997750783436265 22\,k^{6}\,m^{7} - 7771909834725258905382768 44\,k^{7}\,m^{7} - 1533429969471592803234971 55\,k^{8}\,m^{7} -$$
$$24556929294868161598118155\,k^{9}\,m^{7} + 17890395194290164137375052\,k^{10}\,m^{7} + 5018367553318608424374142\,k^{11}\,m^{7} -$$
$$30689655062352588661084 3\,k^{12}\,m^{7} - 24654392757374355445183 7\,k^{13}\,m^{7} - 3870792935396826687352\,k^{14}\,m^{7} +$$
$$4704272473039744083648\,k^{15}\,m^{7} + 247044676494329638004\,k^{16}\,m^{7} - 23814600586886354166\,k^{17}\,m^{7} -$$
$$428776146428144792 0\,k^{18}\,m^{7} - 160128242164362296\,k^{19}\,m^{7} + 23049998675245792\,k^{20}\,m^{7} +$$
$$1278650262416128\,k^{21}\,m^{7} - 21656956579616461118626359 0144\,m^{8} + 88990093696158344386389262867 2\,k\,m^{8} -$$
$$80920285243899333799397796 3552\,k^{2}\,m^{8} - 31939817584392394020966962299 2\,k^{3}\,m^{8} +$$
$$23446266554717056812983246020 8\,k^{4}\,m^{8} + 3062537214893057858793995902 8\,k^{5}\,m^{8} -$$
$$13563741231414678720759555606\,k^{6}\,m^{8} - 5197170818595787763318523174\,k^{7}\,m^{8} - 1719943949084704441652481 70\,k^{8}\,m^{8} -$$
$$97467189738742647003920074\,k^{9}\,m^{8} + 2352950721133798564573670 6\,k^{10}\,m^{8} + 8797844725559262876639410\,k^{11}\,m^{8} -$$
$$27421465604950050310370 2\,k^{12}\,m^{8} - 3255114945570870897088 22\,k^{13}\,m^{8} - 8041306838098387882060\,k^{14}\,m^{8} +$$
$$5047263649776776477396\,k^{15}\,m^{8} + 2968032752391630152 56\,k^{16}\,m^{8} - 19036089058040939588\,k^{17}\,m^{8} -$$
$$3967472833907941736\,k^{18}\,m^{8} - 146709031512626912\,k^{19}\,m^{8} + 16744301402550272\,k^{20}\,m^{8} +$$
$$845413478590592\,k^{21}\,m^{8} - 3556322654170166228278885155 84\,m^{9} + 17748789462813515385670465397 76\,k\,m^{9} -$$
$$15903748227258469062386030883 84\,k^{2}\,m^{9} - 5670566113633532151643770869 92\,k^{3}\,m^{9} +$$
$$42538420747747121312724115 1184\,k^{4}\,m^{9} + 4267129942780602694646488464 4\,k^{5}\,m^{9} -$$
$$23790786069873091903785883160\,k^{6}\,m^{9} + 4250722969764001868988006 18\,k^{7}\,m^{9} - 1036041128817363594080990 54\,k^{8}\,m^{9} -$$
$$20262624098144824749049 05198\,k^{9}\,m^{9} + 2455596297156805869595943 8\,k^{10}\,m^{9} + 1216360073550357351673206 4\,k^{11}\,m^{9} -$$
$$14235864484931557069319 0\,k^{12}\,m^{9} - 3473608657392354545248 578\,k^{13}\,m^{9} - 1115391024046040104835 8\,k^{14}\,m^{9} +$$
$$4315318325077746658042\,k^{15}\,m^{9} + 2802811744797980441 00\,k^{16}\,m^{9} - 112743559441646657 36\,k^{17}\,m^{9} -$$
$$2863655365292438960\,k^{18}\,m^{9} - 103011907229498336\,k^{19}\,m^{9} + 9171545696735808\,k^{20}\,m^{9} +$$
$$412860387929088\,k^{21}\,m^{9} - 4948022598494646201186141065152\,m^{10} + 3010844115572071129891854112512\,k\,m^{10} -$$
$$26656499177264796358928318814 72\,k^{2}\,m^{10} - 8572319133243344012514135308 80\,k^{3}\,m^{10} +$$
$$65622480241723687850020587904 8\,k^{4}\,m^{10} + 4959743963235223088398493929 6\,k^{5}\,m^{10} -$$
$$35179218741151821354678766242\,k^{6}\,m^{10} + 1924856192659604937061379970\,k^{7}\,m^{10} +$$
$$58666965869618100122676692\,k^{8}\,m^{10} - 3023393077671074730633085 18\,k^{9}\,m^{10} + 2011981227072210789571619 8\,k^{10}\,m^{10} +$$
$$13571328179156836626354438\,k^{11}\,m^{10} + 2247705368510573209400 0\,k^{12}\,m^{10} - 3026574179962525860582 42\,k^{13}\,m^{10} -$$
$$11539817179978504649912\,k^{14}\,m^{10} + 2960551869826293379184\,k^{15}\,m^{10} + 210127830198311021288\,k^{16}\,m^{10} -$$
$$4731220853805014512\,k^{17}\,m^{10} - 1617688926928391744\,k^{18}\,m^{10} - 55436977746909696\,k^{19}\,m^{10} + 3770154525384192\,k^{20}\,m^{10} +$$
$$147098675361792\,k^{21}\,m^{10} - 5850976808889636954252778818 28\,m^{11} + 4387295211245360933964265516032\,k\,m^{11} -$$
$$38484867965018756951141520948 48\,k^{2}\,m^{11} - 1113395448855398801847260945216\,k^{3}\,m^{11} +$$
$$86855265556657897896222613343 2\,k^{4}\,m^{11} + 4824812598981613711061851087 2\,k^{5}\,m^{11} -$$
$$44213148703631696817765665032\,k^{6}\,m^{11} + 3425243444444871296323153132\,k^{7}\,m^{11} +$$
$$25739492643472167050088646 4\,k^{8}\,m^{11} - 3520457541978320846513345 2\,k^{9}\,m^{11} + 1244340305372471611293581 6\,k^{10}\,m^{11} +$$
$$12399110299335466201858780\,k^{11}\,m^{11} + 1380503267163716868974 16\,k^{12}\,m^{11} - 2166322372454831954960 76\,k^{13}\,m^{11} -$$
$$92983633072920578922 24\,k^{14}\,m^{11} + 1632840525834385822248\,k^{15}\,m^{11} + 125608043989047850992\,k^{16}\,m^{11} -$$
$$1203047379140134816\,k^{17}\,m^{11} - 712830115293362816\,k^{18}\,m^{11} - 22682411653753984\,k^{19}\,m^{11} + 1145184432274432\,k^{20}\,m^{11} +$$
$$37172337934336\,k^{21}\,m^{11} - 5868525751511481286524173617 92\,m^{12} + 5534029190735328373970840168256\,k\,m^{12} -$$
$$48225072525161419206536806991 04\,k^{2}\,m^{12} - 1250929380517011535943147455040\,k^{3}\,m^{12} +$$
$$99301878404570113797018442589 6\,k^{4}\,m^{12} + 3918268810161390558235079072\,k^{5}\,m^{12} -$$
$$47518079142276095403939078872\,k^{6}\,m^{12} + 4282635976264015876525047224\,k^{7}\,m^{12} +$$
$$40356610756780738659040526 4\,k^{8}\,m^{12} - 3317167347898814527279409 36\,k^{9}\,m^{12} + 5074394029804576570777168\,k^{10}\,m^{12} +$$
$$93615816462038096167129 68\,k^{11}\,m^{12} + 16816957134765293809797 6\,k^{12}\,m^{12} - 1277202302242358473020 72\,k^{13}\,m^{12} -$$

$$
\begin{aligned}
&595395993917262049652 0\, k^{14}\, m^{12} + 721777486599167131712\, k^{15}\, m^{12} + 59819732963039519904\, k^{16}\, m^{12} - \\
&24880863307497664\, k^{17}\, m^{12} - 242430064891957888\, k^{18}\, m^{12} - 6930255720734720\, k^{19}\, m^{12} + 249310114955264\, k^{20}\, m^{12} + \\
&6310353403904\, k^{21}\, m^{12} - 4945481002192285425292994062 08\, m^{13} + 607884349160880052208232876230 4\, k\, m^{13} - \\
&527570668335379821039494518444 8\, k^{2}\, m^{13} - 122194037308788416932607456227 2\, k^{3}\, m^{13} + \\
&985664712790671415016142370432\, k^{4}\, m^{13} + 26259837004023353592805218544\, k^{5}\, m^{13} - \\
&438691026039955748225612858 08\, k^{6}\, m^{13} + 419633261706384453687903604 8\, k^{7}\, m^{13} + \\
&439189312625000454185043040\, k^{8}\, m^{13} - 257787154625507870132206800\, k^{9}\, m^{13} + 392399995287288753545024\, k^{10}\, m^{13} + \\
&587177020218406953182043 2\, k^{11}\, m^{13} + 134330817955374580609088\, k^{12}\, m^{13} - 619951095957900261678 08\, k^{13}\, m^{13} - \\
&305611105721919885057 6\, k^{14}\, m^{13} + 253546322475999066688\, k^{15}\, m^{13} + 225570361318345257 60\, k^{16}\, m^{13} + \\
&128194184393379456\, k^{17}\, m^{13} - 62378969237406976\, k^{18}\, m^{13} - 1530806964932608\, k^{19}\, m^{13} + \\
&36783428059136\, k^{20}\, m^{13} + 645141364736\, k^{21}\, m^{13} - 34184043684841009758632069414 4\, m^{14} + \\
&584148278467523653729716072 6144\, k\, m^{14} - 506068851499495569478108634889 6\, k^{2}\, m^{14} - \\
&104155624614084206179757191 5744\, k^{3}\, m^{14} + 852487754761308544938078373248\, k^{4}\, m^{14} + \\
&140947727967199205121421552 64\, k^{5}\, m^{14} - 34898028553291584972047092768\, k^{6}\, m^{14} + \\
&335852154750279592421932332 8\, k^{7}\, m^{14} + 371740975262987419791164224\, k^{8}\, m^{14} - 166982654771038950056785536\, k^{9}\, m^{14} - \\
&138463462484309723822864 0\, k^{10}\, m^{14} + 306564753992425446841952 0\, k^{11}\, m^{14} + 807023258453484092 18406 4\, k^{12}\, m^{14} - \\
&246843107191287062658 88\, k^{13}\, m^{14} - 1258810369325010801920\, k^{14}\, m^{14} + 696914950720776381 44\, k^{15}\, m^{14} + \\
&665050958089913331 2\, k^{16}\, m^{14} + 61476527487482624\, k^{17}\, m^{14} - 11738262762164224\, k^{18}\, m^{14} - \\
&230817197129728\, k^{19}\, m^{14} + 3293221617664\, k^{20}\, m^{14} + 30006575104\, k^{21}\, m^{14} - 1820590389946229359324551678 72\, m^{15} + \\
&492739819231309375916186780 4288\, k\, m^{15} - 426998725007533963758498860352 0\, k^{2}\, m^{15} - \\
&776567885846955416851665433024\, k^{3}\, m^{15} + 643996472813392777640367172992\, k^{4}\, m^{15} + \\
&557328860773209030318186240 0\, k^{5}\, m^{15} - 23965708333599194609406837568\, k^{6}\, m^{15} + \\
&223963201694158487935923040 0\, k^{7}\, m^{15} + 255009750931856201624230464\, k^{8}\, m^{15} - 906505819128442761056394 88\, k^{9}\, m^{15} - \\
&139367148682534811592774 4\, k^{10}\, m^{15} + 133108541508067853141459 2\, k^{11}\, m^{15} + 380491338142097397541 76\, k^{12}\, m^{15} - \\
&800378156768890119590 4\, k^{13}\, m^{15} - 413818661138221988352\, k^{14}\, m^{15} + 146080808341060313 60\, k^{15}\, m^{15} + \\
&1500642715116878080\, k^{16}\, m^{15} + 15996944643215360\, k^{17}\, m^{15} - 1523330705121280\, k^{18}\, m^{15} - 21246530584576\, k^{19}\, m^{15} + \\
&135029587968\, k^{20}\, m^{15} - 59234170981310287428348247680\, m^{16} + 365695173785096079578217485337 6\, k\, m^{16} - \\
&317569076818935163039568685504 0\, k^{2}\, m^{16} - 507119164637533871361710892160\, k^{3}\, m^{16} + \\
&425491153453892173748904302592\, k^{4}\, m^{16} + 110700554475154175550324019 2\, k^{5}\, m^{16} - \\
&142170356250059703797865021 44\, k^{6}\, m^{16} + 125719852565491147536950873 6\, k^{7}\, m^{16} + \\
&144424806309782901476208256\, k^{8}\, m^{16} - 413177262178407650776984326 4\, k^{9}\, m^{16} - 84405753335648637854809 6\, k^{10}\, m^{16} + \\
&478766956528732793102720\, k^{11}\, m^{16} + 142794189468500634001 92\, k^{12}\, m^{16} - 208885596855681174348 8\, k^{13}\, m^{16} - \\
&107228449811448140032\, k^{14}\, m^{16} + 2236272582972550144\, k^{15}\, m^{16} + 250232435575547392\, k^{16}\, m^{16} + \\
&2560524861706240\, k^{17}\, m^{16} - 121798563758080\, k^{18}\, m^{16} - 900197253120\, k^{19}\, m^{16} + 97220250993983357357978895 36\, m^{17} + \\
&239126077703047535164882493 1072\, k\, m^{17} - 208418211258362016733513480166 4\, k^{2}\, m^{17} - \\
&290131579743515872600011408128\, k^{3}\, m^{17} + 245944813633359772991239433728\, k^{4}\, m^{17} - \\
&481324310956764324837842944\, k^{5}\, m^{17} - 728133771155684121062560384 0\, k^{6}\, m^{17} + 596918822961389208391399936\, k^{7}\, m^{17} + \\
&680949640977816735908249 60\, k^{8}\, m^{17} - 157907488977574718021872 64\, k^{9}\, m^{17} - 377064994702352446460416\, k^{10}\, m^{17} + \\
&141621928273555295580672\, k^{11}\, m^{17} + 4267402387589104675328\, k^{12}\, m^{17} - 431141915965402216960\, k^{13}\, m^{17} - \\
&21434743108614539264\, k^{14}\, m^{17} + 230912197579317760\, k^{15}\, m^{17} + 29053711684749312\, k^{16}\, m^{17} + \\
&2381124882104 32\, k^{17}\, m^{17} - 4519740375040\, k^{18}\, m^{17} + 331991809093035714527493672 96\, m^{18} + \\
&137830627730568123686155947264 0\, k\, m^{18} - 120731216894860727306598472089 6\, k^{2}\, m^{18} - \\
&145315727855141158848338227712\, k^{3}\, m^{18} + 124282654243832027198815112192\, k^{4}\, m^{18} - \\
&658248854692914227573686272\, k^{5}\, m^{18} - 321358503575984696125226854 4\, k^{6}\, m^{18} + 239979471798704052811684352\, k^{7}\, m^{18} + \\
&267920550784843152905492 48\, k^{8}\, m^{18} - 503847411649361497266227 2\, k^{9}\, m^{18} - 130922647546599914069504\, k^{10}\, m^{18} + \\
&340637303552521190398 24\, k^{11}\, m^{18} + 1005845372564172867584\, k^{12}\, m^{18} - 68553830314672727552\, k^{13}\, m^{18} - \\
&3190901803493820416\, k^{14}\, m^{18} + 13377889652649984\, k^{15}\, m^{18} + 2096956051931136\, k^{16}\, m^{18} + \\
&9928425537536\, k^{17}\, m^{18} + 3082824495088622877584433232 8\, m^{19} + 69998917831830015707068269465 6\, k\, m^{19} - \\
&616889995963543525926461475840\, k^{2}\, m^{19} - 6360406570364765774439026688 0\, k^{3}\, m^{19} + \\
&548067646143721835342145341 44\, k^{4}\, m^{19} - 418765216652431246987023360\, k^{5}\, m^{19} - \\
&121818561159629726038833766 4\, k^{6}\, m^{19} + 815293657976702280132823 04\, k^{7}\, m^{19} + 877860544279378054785843 2\, k^{8}\, m^{19} - \\
&133204788105630429664563 2\, k^{9}\, m^{19} - 358322342377438034124 80\, k^{10}\, m^{19} + 655049951187773161574 4\, k^{11}\, m^{19} + \\
&183271749808552976384\, k^{12}\, m^{19} - 8066510780686626 56\, k^{13}\, m^{19} - 333197001679781888\, k^{14}\, m^{19} + \\
&165098168846336\, k^{15}\, m^{19} + 70830520532992\, k^{16}\, m^{19} + 2003426163292884929435934720 0\, m^{20} + \\
&312808990083866519196414191616\, k\, m^{20} - 27760419586174520074348571443 2\, k^{2}\, m^{20} - \\
&242571865900138339026010275 84\, k^{3}\, m^{20} + 210296134316191623947576893 44\, k^{4}\, m^{20} - \\
&192900506742116237980485632\, k^{5}\, m^{20} - 39465844830663116013416038 4\, k^{6}\, m^{20} + 232912179468484332218757 12\, k^{7}\, m^{20} + \\
&238079818615221636376166 4\, k^{8}\, m^{20} - 288404635420757153232896\, k^{9}\, m^{20} - 770227737724808127283 2\, k^{10}\, m^{20} + \\
&982232143724233521152\, k^{11}\, m^{20} + 24937590208950966272\, k^{12}\, m^{20} - 657769992572639232\, k^{13}\, m^{20} - \\
&21780519741843456\, k^{14}\, m^{20} - 17029669076992\, k^{15}\, m^{20} + 1032964412032747653814458777 6\, m^{21} + \\
&122714638995761764684425105408\, k\, m^{21} - 109744172242336177496405250048\, k^{2}\, m^{21} - \\
&802693344323661582040545822 4\, k^{3}\, m^{21} + 699118619425678223194074316 8\, k^{4}\, m^{21} - 701952544982789295559352 32\, k^{5}\, m^{21} - \\
&108507653890420615176585216\, k^{6}\, m^{21} + 554960016349703820157337 6\, k^{7}\, m^{21} + 528818799447446030458880\, k^{8}\, m^{21} - \\
&502710479326636674621 44\, k^{9}\, m^{21} - 12782727556735958384 64\, k^{10}\, m^{21} + 110542743863800483840\, k^{11}\, m^{21} +
\end{aligned}
$$

$$2387674003646840832\,k^{12}\,m^{21} - 32917744568090624\,k^{13}\,m^{21} - 670776591355904\,k^{14}\,m^{21} +$$
$$439984116758608301594139033 6\,m^{22} + 42116461081959814973775323136\,k\,m^{22} - 37976383393705821850635288576\,k^2\,m^{22} -$$
$$22915162690358505669585879 04\,k^3\,m^{22} + 20019683166189715513115443 20\,k^4\,m^{22} -$$
$$20747496153098175859351552\,k^5\,m^{22} - 25074473808651959791968256\,k^6\,m^{22} + 10895554081694691083223 04\,k^7\,m^{22} +$$
$$9464969843806650880819 2\,k^8\,m^{22} - 6879449003258982318080\,k^9\,m^{22} - 158483705330039316480\,k^{10}\,m^{22} +$$
$$87726438168576655 36\,k^{11}\,m^{22} + 143589219510009856\,k^{12}\,m^{22} - 752044945924096\,k^{13}\,m^{22} +$$
$$1572464910826098349556203008\,m^{23} + 12586367835872310161319542784\,k\,m^{23} - 1144770029870606296677153177 6\,k^2\,m^{23} -$$
$$5600883212826768234433740 80\,k^3\,m^{23} + 4899608810875159952894033 92\,k^4\,m^{23} - 5016377883270259486785536\,k^5\,m^{23} -$$
$$4806089960103294289592320\,k^6\,m^{23} + 1731993414515518969774 08\,k^7\,m^{23} + 1332163346933585691744\,k^8\,m^{23} -$$
$$7114239509719342612 48\,k^9\,m^{23} - 13844425986798567424\,k^{10}\,m^{23} + 437107503980331008\,k^{11}\,m^{23} +$$
$$4082168555880448\,k^{12}\,m^{23} + 4737830740223986868443545 60\,m^{24} + 3254824846851736916227424256\,k\,m^{24} -$$
$$2986982441085652080662151168\,k^2\,m^{24} - 1160376915369841244556492 80\,k^3\,m^{24} + 1014294209576395330845409 28\,k^4\,m^{24} -$$
$$9881978085386464503398 40\,k^5\,m^{24} - 7502265301947193471795 20\,k^6\,m^{24} + 2173137725173084471296 0\,k^7\,m^{24} +$$
$$1420110732917535571968\,k^8\,m^{24} - 52240982517515714560\,k^9\,m^{24} - 761041226583080960\,k^{10}\,m^{24} +$$
$$10267531718197248\,k^{11}\,m^{24} + 1201592244250337510227968 00\,m^{25} + 7224018339020263329624883 20\,k\,m^{25} -$$
$$6690583242336148108370903 04\,k^2\,m^{25} - 2010902466808759692853248 0\,k^3\,m^{25} + 1751662906260131581251764 8\,k^4\,m^{25} -$$
$$1565934923412264788951 04\,k^5\,m^{25} - 9292383010491663207628 8\,k^6\,m^{25} + 2070940682785392033792\,k^7\,m^{25} +$$
$$1078085454802881085 44\,k^8\,m^{25} - 2426125960335458304\,k^9\,m^{25} - 19821713375625216\,k^{10}\,m^{25} +$$
$$2548764151192081050579763 2\,m^{26} + 13614440392927036098360115 2\,k\,m^{26} - 1272708970190108644910039 04\,k^2\,m^{26} -$$
$$2863424988113067047780352\,k^3\,m^{26} + 2476450382375174787563520\,k^4\,m^{26} - 1951574782655522852044 8\,k^5\,m^{26} -$$
$$8784675752492766855168\,k^6\,m^{26} + 140818923988605468672\,k^7\,m^{26} + 5194404878948499456\,k^8\,m^{26} -$$
$$53534775652909056\,k^9\,m^{26} + 4471205493167514577797120\,m^{27} + 2147984437790853746786304 0\,k\,m^{27} -$$
$$2026975639484288019018547 2\,k^2\,m^{27} - 3268675452163093073756 16\,k^3\,m^{27} + 2791100532443406037155 84\,k^4\,m^{27} -$$
$$1844151671929471500288\,k^5\,m^{27} - 595327918912767000576\,k^6\,m^{27} + 6084333311942983680\,k^7\,m^{27} +$$
$$1194155765929082 88\,k^8\,m^{27} + 6377367629577282901770 24\,m^{28} + 2783203048129845926559744\,k\,m^{28} -$$
$$2651440406975931554463744\,k^2\,m^{28} - 2886630807708866130739 2\,k^3\,m^{28} + 2411124118205186388787 2\,k^4\,m^{28} -$$
$$1242900761161000550 40\,k^5\,m^{28} - 25751798158305263616\,k^6\,m^{28} + 125485156572069888\,k^7\,m^{28} +$$
$$7211410676794244726784 0\,m^{29} + 2883279849309836387430 4\,k\,m^{29} - 2773123670642249925918 72\,k^2\,m^{29} -$$
$$1867012941204522270720\,k^3\,m^{29} + 1498822727527034781696\,k^4\,m^{29} - 5325245038920204288\,k^5\,m^{29} -$$
$$534112775917535232\,k^6\,m^{29} + 6221687943922616107008\,m^{30} + 2295446207962194483609 6\,k\,m^{30} -$$
$$2229057135767136396902 4\,k^2\,m^{30} - 805510736429438730 24\,k^3\,m^{30} + 59675155425870741504\,k^4\,m^{30} -$$
$$1090336916924006 40\,k^5\,m^{30} + 384774425994357374976\,m^{31} + 1318134854721132822528\,k\,m^{31} -$$
$$1292459914054003064832\,k^2\,m^{31} - 1916523552681492480\,k^3\,m^{31} + 1142758643683295232\,k^4\,m^{31} +$$
$$1519075086935733043 2\,m^{32} + 4858246375725629440\,k\,m^{32} - 48104362009222447104\,k^2\,m^{32} -$$
$$1411653324452659 2\,k^3\,m^{32} + 287619503792062464\,m^{33} + 862858511376187392\,k\,m^{33} - 862858511376187392\,k^2\,m^{33})$$

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